A felsőfokú oktatás minőségének és hozzáférhetőségének együttes javítása a Pannon Egyetemen

EFOP-3.4.3-16-2016-00009

SZÉCHENYI 2020

MATHEMATICAL FOUNDATIONS OF ECONOMICS

PROBLEM BOOK

EXERCISES AND SOLUTIONS

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2020.

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Introduction

Mathematical Foundations of Economics is the fundamental mathematical course for students learning at the Faculty of Business and Econimics at University of Pannonia. This book contains exercises closely related to this course and the minimum theoretical knowledge needed to solve these problems. The notations of the listed definitions and theorems are the ones used by *Dr. Győri István, Dr. Pituk Mihály: Kalkulus informatikusoknak I.* textbook published by University of Pannonia, so this book is excellent for acquiring the theoretical knowledge required to complete the course. In English, we recommend *Trench, William F.: Introduction to Real Analysis.*

We have devoted separate chapters to each topic. Each chapter has an introduction, in which some fundamental definitions and propositions are prepared. There are a number of solved examples and many exercises. The problems we solve have been carefully selected to illustrate common situations, techniques and tricks to deal with these. In chapter *Solutions* we give detailed solutions of the exercises.

The book is suitable for students who wish to solve basic mathematical analysis problems. Although we recommend this book mainly for self-sudying, it can be really useful for students who participate in classes and use the benefits of them.

Veszprém, 2020.

1 *Composite Functions*

We can combine two given functions by composing one into the other. The result of this combination - if it exists- is a *new function* called composite function. In this section we study these functions.

Definition 1.1

Given two functions

$$f : B \to C$$

and

$$g : A \to B$$

the *composite function* $f \circ g$ (also called *the composition of f and g*) is defined by

$$(f \circ g)(x) = f(g(x)),$$

for x in the domain of g such that $g(x)$ is in the domain of f; that is,

$$dom(f \circ g) = \{x \in dom(g) \mid g(x) \in dom(f)\}.$$

$f(g(x))$ is read as "f of g of x". □

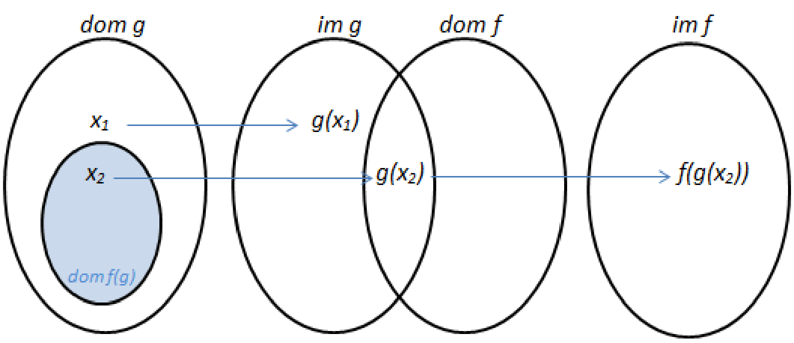


Figure 1.1: function $f \circ g$.

1.1 *Step-by-Step Examples*

Now we give a step-by-step solution to a problem. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 1.1
Composite Function

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f(x) &= 3x + 1, \quad x \in [0,7], \\ g(x) &= x^2 - 9, \quad x \in [0,5]. \end{aligned}$$

SOLUTION

First, we determine the domain of the composite function. From the definition, the domain of $f \circ g$ contain all of x in the domain of g such that $g(x)$ is in the domain of f; that is

$$dom\,(f \circ g) = \left\{x \in [0,5] \mid \left(x^2 - 9\right) \in [0,7]\right\}.$$

So we have to solve the system of inequalities

$$0 \le x^2 - 9 \le 7$$

for

$$0 \le x \le 5.$$

To determine the domain, consider

$$0 \le x^2 - 9 \le 7.$$

From this we get

$$\begin{aligned} 9 &\le x^2 \le 16, \\ 3 &\le |x| \le 4, \end{aligned}$$

so

$$3 \le -x \le 4 \text{ or } 3 \le x \le 4,$$

and from this

$$-4 \le x \le -3 \text{ or } 3 \le x \le 4.$$

Comparing this with condition $0 \leq x \leq 5$, we get

$$dom\,(f \circ g) = [3, 4]\,.$$

The composite function $f \circ g$ is formed when $g\,(x)$ is substituted for x in $f\,(x)$. More precisely,

$$\begin{aligned} f\,(x) &= 3x + 1, \\ g\,(x) &= x^2 - 9, \end{aligned}$$

$$f\,(g\,(x)) = f\left(x^2 - 9\right) = 3\left(x^2 - 9\right) + 1 = 3x^2 - 26.$$

So we get

$$(f \circ g)\,(x) = 3x^2 - 26, \quad x \in [3, 4]\,.$$

1.2 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 1.1

Find the composite function $f \circ g$, if it exists.

1.

$$\begin{aligned} f(x) &= x^2 - 9, \quad x \in [0,5], \\ g(x) &= 3x + 1, \quad x \in [0,7]. \end{aligned}$$

See Solution 8.1.1

2.

$$\begin{aligned} f(x) &= 3x^2 + 5, \quad x \in [1,125], \\ g(x) &= 2x + 3, \quad x \in [0,100]. \end{aligned}$$

See Solution 8.1.2

3.

$$\begin{aligned} f(x) &= 8 - \sin(x), \quad x \in [1,125], \\ g(x) &= 5^x, \quad x \in [-1,10]. \end{aligned}$$

See Solution 8.1.3

4.

$$\begin{aligned} f(x) &= \sin^3(x), \quad x \in [1,25], \\ g(x) &= x^2, \quad x \in [0,3]. \end{aligned}$$

See Solution 8.1.4

5.

$$\begin{aligned} f(x) &= \frac{\cos(x)}{5} - x, \quad x \in [1,4], \\ g(x) &= \log_2(x), \quad x \in [1,10]. \end{aligned}$$

See Solution 8.1.5

6.

$$\begin{aligned} f(x) &= 6 - 9\cos(x), \quad x \in [0,15], \\ g(x) &= x^2 + 2x, \quad x \in [1,10]. \end{aligned}$$

See Solution 8.1.6

7.

$$\begin{aligned} f(x) &= x^2 - 6x - 83, \quad x \in [4,7], \\ g(x) &= \sqrt{x} + 3, \quad x \in [0,23]. \end{aligned}$$

See Solution 8.1.7

8.

$$\begin{aligned} f(x) &= \sqrt{4x^2+2}+3x, \quad x \in [4,7], \\ g(x) &= \ln(x)+3, \quad x \in [0,23]. \end{aligned}$$

See Solution 8.1.8

9.

$$\begin{aligned} f(x) &= \sqrt{x+3}, \quad x \in [3,4], \\ g(x) &= \cos(x)+3, \quad x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]. \end{aligned}$$

See Solution 8.1.9

10.

$$\begin{aligned} f(x) &= \sqrt{1-x}, \quad x \leq 1, \\ g(x) &= x^2, \quad x \in \mathbb{R}. \end{aligned}$$

See Solution 8.1.10

11.

$$\begin{aligned} f(x) &= \sqrt{1-x}, \quad x \leq 1, \\ g(x) &= x^2, \quad x \in [2,4]. \end{aligned}$$

See Solution 8.1.11

2 The Inverse Function

In some cases, there is a function which does the "reverse" of a given function. This function -if it exists- is the inverse function.

Definition 2.1

A function f is called a *one-to-one function* if no two x_1, x_2 elements of *dom* (f) have the same image; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2,$$

or

$$x_1 = x_2 \qquad \text{whenever } f(x_1) = f(x_2). \qquad \square$$

The property one-to-one is also called injective. A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Definition 2.2

Let f be a one-to-one function with domain A and range B. Then its *inverse function* f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B. $\square$

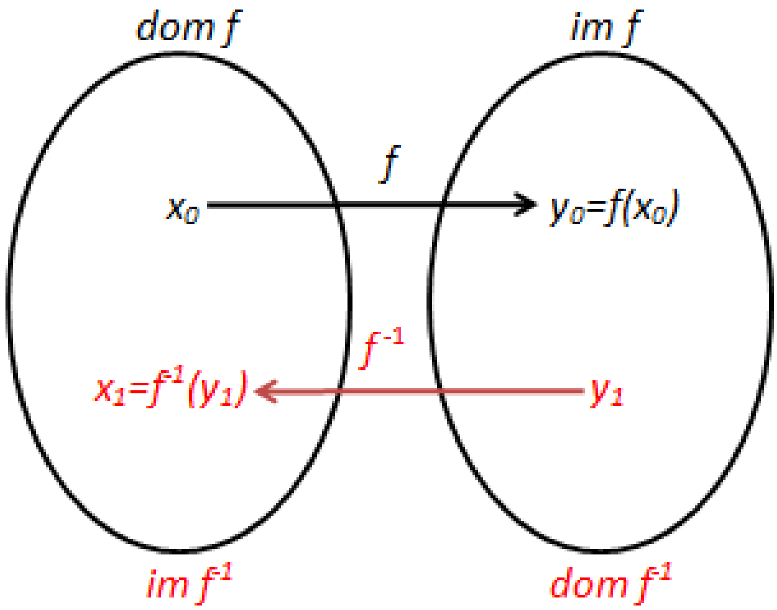


Figure 2.1: Inverse function f^{-1}.

Thus if f has an inverse, $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for all x in the domain of f and all y in the range of f.

Finding an Inverse Function

Follow these steps to find the inverse of a function f.

1. Verify that f is a one-to-one function.
   Let $f(x_1) = f(x_2)$ and examine whether it implies that $x_1 = x_2$.
2. Determine the domain of function f^{-1}.
3. Determine $f^{-1}(x)$
   (a) Write $y = f(x)$.
   (b) Solve this equation for x in terms of y (if possible).
   (c) Interchange x and y.

   The resulting equation is $y = f^{-1}(x)$.

There is an interesting relationship between the graph of a function and the graph of its inverse. The graph of the inverse is a reflection of the actual function about the line $y = x$.

2.1 *Step-by-Step Examples*

Now we give a step-by-step solution to two problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 2.1
Inverse Function

Find the inverse function of the given f function, if it exists.

$$f(x) = 1 - x^2, \quad x \in (-\infty, 0).$$

SOLUTION
First, we verify that f is a one-to-one function, i. e. if the function has the same value at two points, then the points must be the same. So let x_1, x_2 be any two values of $dom(f)$, suppose that $f(x_1) = f(x_2)$ and examine that it implies that $x_1 = x_2$. Namely,

$$1 - x_1^2 = 1 - x_2^2 \stackrel{?}{\Longrightarrow} x_1 = x_2,$$

for any $x_1, x_2 \in (-\infty, 0)$. As

$$1 - x_1^2 = 1 - x_2^2,$$

we get

$$x_1^2 = x_2^2,$$

so

$$|x_1| = |x_2|.$$

But $x_1, x_2 \in (-\infty, 0)$, so from the definition of absolute value we get

$$-x_1 = -x_2,$$

so

$$x_1 = x_2.$$

This implies that the inverse of f exists.
Now we determine the domain of function f^{-1}. As $dom\left(f^{-1}\right) = im(f)$, we determine the range of function f. As $dom(f) = (-\infty, 0)$, we have

$$x < 0.$$

From this

$$x^2 > 0,$$

so

$$-x^2 < 0,$$

and this follows

$$1 - x^2 < 1.$$

Thus,

$$(y =) f(x) = 1 - x^2 < 1.$$

So $dom\left(f^{-1}\right) = im(f) = (-\infty, 1)$.
Finally, we solve

$$y = f(x).$$

So consider

$$y = 1 - x^2,$$

where $x < 0$ and $y < 1$ and solve for "$x =$". As

$$x^2 = 1 - y,$$

we get

$$|x| = \sqrt{1 - y}.$$

By figuring out the domain and range of the inverse $(y < 1$ and $x < 0)$, we find

$$-x = \sqrt{1 - y},$$

thus

$$x = -\sqrt{1 - y}.$$

So the inverse of $f(x) = 1 - x^2,\ x \in (-\infty, 0)$ exists and

$$f^{-1}(x) = -\sqrt{1 - x}, \quad x \in (-\infty, 1).$$

SOLVED EXAMPLE 2.2
Inverse Function

Find the inverse function of the given f function, if it exists.

$$f(x) = x^2 - 2x, \quad x \in (-\infty, 0).$$

SOLUTION
To determine the inverse of function f first we have to rewrite $f(x)$. To solve this problem we write

$$f(x) = x^2 - 2x = (x-1)^2 - 1.$$

First, we verify that f is a one-to-one function, i. e. if the function has the same value at two points, then the points must be the same. So let x_1, x_2 be any two values of $dom(f)$, suppose that $f(x_1) = f(x_2)$ and examine that it implies that $x_1 = x_2$. Namely,

$$(x_1 - 1)^2 - 1 = (x_2 - 1)^2 - 1 \overset{?}{\Longrightarrow} x_1 = x_2,$$

for any $x_1, x_2 \in (-\infty, 0)$. As

$$(x_1 - 1)^2 - 1 = (x_2 - 1)^2 - 1,$$

we get

$$|x_1 - 1| = |x_2 - 1|.$$

but $x_1, x_2 \in (-\infty, 0)$, so from the definition of absolute value we get

$$1 - x_1 = 1 - x_2,$$

this follows

$$x_1 = x_2,$$

so the inverse of function f exists.
Now we determine the domain of function f^{-1}. As $dom(f^{-1}) = im(f)$, we determine the range of function f $(im(f))$. As $dom(f) = (-\infty, 0)$, we have

$$x < 0,$$

so we find

$$x - 1 < -1$$

thus

$$(x-1)^2 > 1,$$

and

$$(x-1)^2 - 1 > 0.$$

So

$$(y =) f(x) = (x-1)^2 - 1 > 0.$$

Thus $dom\left(f^{-1}\right) = im(f) = (0, \infty)$.
Finally, we solve

$$y = f(x),$$

namely consider

$$y = (x-1)^2 - 1$$

where $x < 0$ and $y > 0$ and solve for "$x =$". As

$$y + 1 = (x-1)^2,$$

we get

$$\sqrt{y+1} = |x-1|.$$

from $x < 0$ and $y > 0$ we find

$$\sqrt{y+1} = 1 - x,$$

so

$$x = 1 - \sqrt{y+1}.$$

So the inverse of $f(x) = x^2 - 2x,\ x \in (-\infty, 0)$ exists and

$$f^{-1}(x) = 1 - \sqrt{y+1}, \quad x \in (0, \infty).$$

SOLVED EXAMPLE 2.3
Inverse Function

Find the inverse function of the given f function, if it exists.

$$f(x) = \frac{2}{3 - \sqrt{x-2}}.$$

SOLUTION

Clearly

$$dom(f) = \{x \in \mathbb{R} : 2 \leq x,\ x \neq 11\}.$$

1. Before inverting f we most check *whether* f is invertible, i.e. it is *injective*. This means that, from $f(x_1) = f(x_2)$ we have to deduce $x_1 = x_2$.

$$f(x_1) = \frac{2}{3 - \sqrt{x_1 - 2}} = \frac{2}{3 - \sqrt{x_2 - 2}} = f(x_2) \quad / : 2$$

and take reciprocial. So, we have

$$\begin{array}{rcl} 3 - \sqrt{x_1 - 2} & = & 3 - \sqrt{x_2 - 2} \\ & \Downarrow & \\ -\sqrt{x_1 - 2} & = & -\sqrt{x_2 - 2} \\ & \Downarrow & \\ x_1 - 2 & = & x_2 - 2 \\ & \Downarrow & \\ x_1 & = & x_2, \end{array}$$

so f is *injective*!

Let us advice to the Reader to change the letters x and y at the end of the solution (after step 6)).

2. To find the expression for $f^{-1}(y)$, we have to solve the equality $y = f(x)$ to the unknown x (and consider y as a parameter).

$$y = \frac{2}{3 - \sqrt{x-2}} \quad / : 2$$

and take reciprocial, so we have

$$\begin{aligned}
\frac{2}{y} &= 3-\sqrt{x-2} \\
&\Downarrow \\
\frac{2}{y}-3 &= -\sqrt{x-2} \quad \text{/ square} \\
&\Downarrow \\
\left(\frac{2}{y}-3\right)^2 &= x-2 \\
&\Downarrow \\
\left(\frac{2}{y}-3\right)^2+2 &= x,
\end{aligned}$$

so we have the inverse formula

$$x=\left(\frac{2}{y}-3\right)^2+2=f^{-1}(y)\,.$$

3. For the *first glance*, from the above expression we might think that

$$dom\left(f^{-1}\right)=\{y\in\mathbb{R}:y\neq 0\}\,.$$

   However it turns out to be **BAD !**

4. Try to make the *sketch* of the graph of f :

   $f(2)=\dfrac{2}{3}$,

   f is strictly increasing and positive for $2\leq x<11$,

   $\lim\limits_{x\to 11-} f(x)=+\infty, \qquad \lim\limits_{x\to 11+} f(x)=-\infty,$

   f is strictly increasing also for $11<x$ and is negative,

   $\lim\limits_{x\to -\infty} f(x)=-0$ (see in blue in the Figure below).

   It is well known, that the graph of the inverse function f^{-1} is the reflection of the graph of f to the straight line $y=x$ (see in **green** in the Figure).

   The Figure also shows the graph of the expression (2.3) for $y\neq 0$ in red . We *must* observe that this red curve contains the reflection of the blue one (this is GOOD) **and something else** (this is BAD). In other words: $dom\left(f^{-1}\right)$ is *less* than $\mathbb{R}\setminus\{0\}$, i.e. $dom\left(f^{-1}\right)\subsetneqq\{y\in\mathbb{R}:y\neq 0\}$.

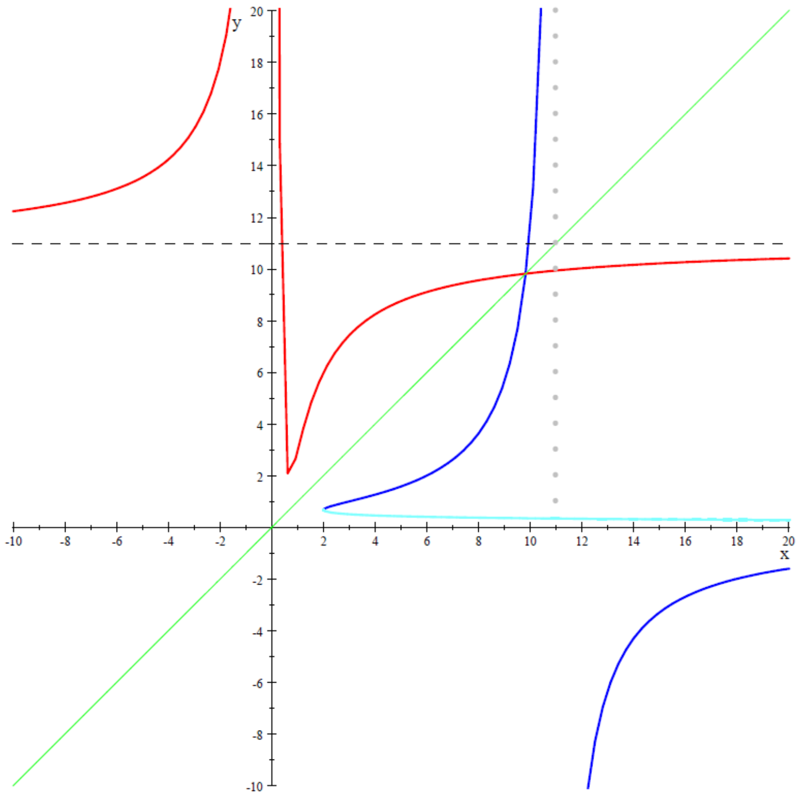


Figure: *The functions*

$$f(x) = \frac{2}{3 - \sqrt{x-2}}, \quad g(x) = \frac{2}{3 + \sqrt{x-2}},$$

and $y = x$, $f^{-1}(x) = g^{-1}(x) = \left(\frac{2}{x} - 3\right)^2 + 2.$

What went wrong in step 3.?

5. We have to check again the calculations in step 2.!

When we *squared* the expressions

$$\frac{2}{y} - 3 = -\sqrt{x-2} \quad / \text{ square},$$

we lost the *sign* of them: squaring makes everything positive. So, we have to find out the sign of $\frac{2}{y} - 3$ to find restrictions for y, i.e. to find *dom* $\left(f^{-1}\right)$.

How to make out the sign of $\frac{2}{y} - 3$? Observe, that

$$\frac{2}{y} - 3 = -\sqrt{x-2}$$

implies the *equality of the signs* of its two sides: we must have

$$\frac{2}{y} - 3 \leq 0$$

since $-\sqrt{x-2} \leq 0$. Solving this inequality (your homework) we get $y < 0$ or $\frac{2}{3} \leq y$, so

$$dom\left(f^{-1}\right) = \left\{y \in \mathbb{R} : y < 0 \quad \text{or} \quad \frac{2}{3} \leq y\right\} .$$

6. Looking to the Figure above, we may check that

$$dom\left(f^{-1}\right) = \left\{y \in \mathbb{R} : y < 0 \quad \text{or} \quad \frac{2}{3} \leq y\right\} .$$

   is correct: it *excludes* the middle part of the red graph, for

$$0 \leq y < \frac{2}{3},$$

   and the remainig two *other* parts are really the reflections of the blue original function $f(x)$.

7. As we adviced, at the very end is safe to change x and y to get the usual form of the inverse function:

$$y = f^{-1}(x) = \left(\frac{2}{x} - 3\right)^2 + 2$$

   and

$$dom\left(f^{-1}\right) = \left\{x \in \mathbb{R} : x < 0 \quad \text{or} \quad \frac{2}{3} \leq x\right\}.$$

Remark 2.1.1 *1. $dom\left(f^{-1}\right)$ can be also obtained from the result in step 4. by a thoroughful investigation, using $dom\left(f^{-1}\right) = im(f)$.*

2. What on earth the omitted middle part of the red graph (for $0 \leq y < \frac{2}{3}$) could be? Reflecting it back to the green line "$y = x$" we get the (monotone increasing) light blue graph.

After a while we can solve the puzzle: this light blue function must be

$$g(x) = \frac{2}{3 + \sqrt{x-2}} .$$

It is your homework to find $g^{-1}(y)$ and $dom\left(g^{-1}\right)$, with calculations, similar to steps 1. through 7., to get

$$x = \left(\frac{2}{y} - 3\right)^2 + 2 = g^{-1}(y),$$

yes: the formulas of $f^{-1}(y)$ and $g^{-1}(y)$ are the same, but $dom\left(g^{-1}\right)$ is different as

$$dom\left(g^{-1}\right) = \left\{y \in \mathbb{R} : 0 < y \leq \frac{2}{3}\right\}.$$

The explanation of the connection of f^{-1} and g^{-1} is the following. The only difference between

f and g ($-$ and $+$ in the denumerator) disappears when squaring and so makes

$$x = \left(\frac{2}{y} - 3\right)^2 + 2 = f^{-1}(y),$$

and

$$x = \left(\frac{2}{y} - 3\right)^2 + 2 = g^{-1}(y)$$

identical, while the sign restriction in

$$\frac{2}{y} - 3 = -\sqrt{x-2}$$

changes to $\frac{2}{y} - 3 \geq 0$ *and implies* $0 < y \leq \frac{2}{3}$ *for dom* $\left(g^{-1}\right)$.

3. *For most of the functions f to determine im (f) is not so easy as above in step 4., so careful investigation of step 2. (as in step 5. above) is compulsory.*

2.2 *Exercises*

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 2.1

Find the inverse function of the given f function, if it exists.

1.
$$f(x) = (x-2)^2, \quad x \in [1,3].$$
See Solution 8.2.1

2.
$$f(x) = 5x+1, \quad x \in [1,5],$$
See Solution 8.2.2

3.
$$f(x) = x^2+2x, \quad x \in [-1,2],$$
See Solution 8.2.3

4.
$$f(x) = 5^x+1, \quad x \in [-1,1],$$
See Solution 8.2.4

5.
$$f(x) = 1-\log_3(x), \quad x \in [1,27],$$
See Solution 8.2.5

6.
$$f(x) = \sqrt{x-1}+5, \quad x \in [1,37],$$
See Solution 8.2.6

7.
$$f(x) = x^2-4x+3, \quad x < 0,$$
See Solution 8.2.7

8.
$$f(x) = \sqrt{x}-3, \quad x \in [4,16],$$
See Solution 8.2.8

9.
$$f(x) = \log_{\frac{1}{2}}(x), \quad x \in \left[\frac{1}{2},4\right],$$
See Solution 8.2.9

10.
$$f(x) = \ln(x)+4, \quad x \in \left[1,e^2\right],$$
See Solution 8.2.10

11.
$$f(x) = 3^{x-2}, \quad x \in [1,2],$$
See Solution 8.2.11

3 Sequences

3.1 The Extended Real Number System

It is convenient to adjoin to the real number system two fictitious points, $+\infty$ (or simply ∞) and $-\infty$. We define for every real a number

Notation 3.1.1

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

$$-\infty < a \quad a < \infty \quad -\infty < \infty \ .$$

If a is a real number, then

$$\begin{aligned} -(\pm\infty) &= \mp\infty, \\ +\infty + a &= a + (+\infty) = +\infty, \quad a > -\infty, \\ -\infty + a &= a + (-\infty) = -\infty, \quad a < +\infty, \\ (\pm\infty) \cdot a &= a \cdot (\pm\infty) = \pm\infty, \quad a > 0, \\ (\pm\infty) \cdot a &= a \cdot (\pm\infty) = \mp\infty, \quad a < 0, \\ \frac{a}{\pm\infty} &= 0, \quad a \in \mathbb{R}. \end{aligned}$$

The following forms are called *indeterminate forms*, and left undefined.

$$\frac{\pm\infty}{\pm\infty},$$

$$\frac{\pm\infty}{\mp\infty},$$

$$\frac{a}{0}, \quad a \in \overline{\mathbb{R}},$$

$$+\infty - \infty,$$

$$-\infty + \infty,$$

$$(\pm\infty) \cdot 0,$$

$$0 \cdot (\pm\infty)\,.$$

3.2 Convergent and Divergent Sequences

Definition 3.1

The number $a \in \mathbb{R}$ is said to be the (finite) limit of a sequence $\{a_n\}$, if for every $\varepsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the terms a_n satisfy $|a_n - a| < \varepsilon$.
When a sequence $\{a_n\}$ has limit a, we will use the notation $a_n \to a$ or $\lim_{n\to\infty} a_n = a$. $\square$

We say

$$a_n \to a$$

if and only if

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N} : \forall n \geq n_0 \implies |a_n - a| < \varepsilon.$$

The number n_0 is called treshold, since for indexes $n \geq n_0$ the numbers a_n behave "better" than for $n < n_0$.

Definition 3.2

If a sequence has a finite limit, we say that the sequence is convergent. Any non-convergent sequence is called divergent. $\square$.

The following statements are equivalent.

Definition 3.3

A sequence $\{a_n\}$ in $\mathbb{R}$ is said to converge to $a \in \mathbb{R}$ if for every $\varepsilon > 0$ for all but a finite number of terms of n $|a_n - a| < \varepsilon$. $\square$

Definition 3.4

A sequence $\{a_n\}$ in $\mathbb{R}$ is said to converge to $a \in \mathbb{R}$ if for every neighborhood of a for all but a finite number of terms of n the terms a_n belong to the neighborhood. $\square$

Definition 3.5

We say that $\{a_n\}$ diverges to infinity and we write,

$$a_n \to \infty,$$

whenever, for all $c \in \mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $a_n > c$. □

We say

$$a_n \to \infty$$

if and only if

$$\forall c \in \mathbb{R}\ \exists n_0 \in \mathbb{N} : \forall n \geq n_0 \implies a_n > c.$$

Definition 3.6

We say that $\{a_n\}$ tends to $-\infty$ and we write,

$$a_n \to -\infty$$

whenever, for all $c \in \mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $a_n < c$. □

We say

$$a_n \to -\infty$$

if and only if

$$\forall c \in \mathbb{R}\ \exists n_0 \in \mathbb{N} : \forall n \geq n_0 \implies a_n < c.$$

Definition 3.7

If $\{n_k\}_{k=0}^{\infty}$ is strictly increasing sequence in $\mathbb{N}$ we call sequence $\{a_{n_k}\}_{k=0}^{\infty}$ the subsequence of $\{a_n\}_{n=0}^{\infty}$. □

Remark 3.2.1 *Clearly any subsequence is a subset of the original sequence.*

Theorem 3.1

Any sequence a_n has a limit if and only if all of its subsequences a_{n_k} are convergent to the same limit $a \in \overline{\mathbb{R}}$, and in this case a_n has also the limit a .

$$\lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n = a. \quad \square$$

Theorem 3.2: Limits of Special Sequences

a. The limit of the constant sequence

$$c \to c,$$

b.

$$\frac{1}{n} \to 0,$$

c. Geometric sequences

$$q^n \to \begin{cases} 0, & |q| < 1, \\ 1, & q = 1, \\ \infty, & q > 1, \\ \text{divergent if } q \le -1. \end{cases}$$

d.

$$\sqrt[n]{a} \to 1, \quad a > 0,$$

e.

$$\sqrt[n]{n} \to 1,$$

f.

$$\left(1 + \frac{1}{n}\right)^n \to e,$$

g.

$$\left(1 + \frac{a}{n}\right)^n \to e^a, \quad a \in \mathbb{R}. \qquad \square$$

Remark 3.2.2 *The limit e is a complicated but fixed real number, approximately* $e \approx 2.71828\ldots$. *Some scientists call it Eulerian number (Leonhard Euler, Swiss mathematician, 1707–1783).*

3.2.1 *Algebraic Properties*

The following algebraic properties of limits and divergent limits are often used in practice.

Theorem 3.3

If $a_n \to a$ and $b_n \to b$, where $a, b \in \mathbb{R}$, then

$$\begin{aligned} a_n \pm b_n &\to a \pm b, \\ a_n \cdot b_n &\to a \cdot b, \end{aligned}$$

if $b_n, b \neq 0$ for every n, then

$$\frac{a_n}{b_n} \to \frac{a}{b}. \qquad \square$$

Theorem 3.4

If $a_n \to \infty$, then

$$-a_n \to -\infty. \qquad \square$$

Theorem 3.5

If $a_n \to \infty$, then

$$\frac{1}{a_n} \to 0. \qquad \square$$

Theorem 3.6

If $a_n \to 0$ and $a_n > 0$ for all but a finite number of terms of n, then

$$\frac{1}{a_n} \to \infty, \qquad \square$$

Theorem 3.7

If $a_n \to 0$ and $a_n < 0$ for all but a finite number of terms of n, then

$$\frac{1}{a_n} \to -\infty. \qquad \square$$

Theorem 3.8

Every bounded, monotone sequence converges to a real number. $\square$

Theorem 3.9

If $a_n \to a$ and $\{b_n\}$ is bounded, then $a_n b_n \to 0$. $\square$

Theorem 3.10

If $a_n \to a$, $b_n \to b$, where $a, b \in \mathbb{R}$ and $a_n \leq b_n$ for all but a finite number of terms of n, then $a \leq b$. $\square$

Theorem 3.11

If $a_n \to \infty$ $(b_n \to -\infty)$ and $a_n \leq b_n$ for all but a finite number of terms of n, then $b_n \to \infty$ $(a_n \to -\infty)$. $\square$

Theorem 3.12

If $a_n \to a$ and $a > 0$, then $\sqrt{a_n} \to \sqrt{a}$. $\square$

Theorem 3.13: The Squeeze Theorem

If $a_n \to a$, $c_n \to a$, where $a \in \mathbb{R}$ and $a_n \leq b_n \leq c_n$ for all but a finite number of terms of n, then $b_n \to a$. $\square$

3.3 Sequences with Limit "$\frac{\infty}{\infty}$"

In this section we give solutions to sequences with limit $\frac{\infty}{\infty}$.

3.4 Step-by-Step Examples

Now we give a step-by-step solution to some common problems. At the end of this section there are more exercises for practice.

3.4.1 Rational Fraction Sequences

We already know that $\frac{1}{n} \to 0$. From this with the properties of limits we can find the limit of any sequences of racional fractions.

SOLVED EXAMPLE 3.1
Sequences

Find the limit of the following sequence.

$$a_n = \frac{5n+1}{n^2+n}.$$

SOLUTION
As $n \to \infty$, we have

$$5n+1 \to \infty$$

and

$$n^2+n \to \infty.$$

So preliminary manipulations are necessary before applying the properties of limits. First, we identify the largest power of n in the denominator (and yes, we only consider the denominator for this) and then we factor this out of both the numerator and denominator. This gives,

$$a_n = \frac{5n^1+1}{1n^2+n} = \frac{n^2\left(\frac{5}{n}+\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n}\right)} = \frac{\frac{5}{n}+\frac{1}{n^2}}{1+\frac{1}{n}} \to \frac{0+0}{1+0} = 0.$$

SOLVED EXAMPLE 3.2
Sequences

Find the limit of the following sequence.

$$b_n = \frac{5n^2 - 1}{n^2 + n}.$$

SOLUTION

As $n \to \infty$, we have

$$b_n = \frac{5n^2 - 1}{n^2 + n} \to \frac{\infty}{\infty},$$

again, which is an indeterminate form. Now the largest power of n in the denominator is n^2, so we factor this out of both the numerator and denominator. This gives,

$$b_n = \frac{5n^2 - 1}{1n^2 + n} = \frac{n^2\left(5 - \frac{1}{n^2}\right)}{n^2\left(1 + \frac{1}{n}\right)} = \frac{5 - \frac{1}{n^2}}{1 + \frac{1}{n}} \to \frac{5 - 0}{1 + 0} = \mathbf{5}.$$

SOLVED EXAMPLE 3.3
Sequences

Find the limit of the following sequence.

$$c_n = \frac{3n^3 - 1}{2n^2 + n}.$$

SOLUTION

As $n \to \infty$, we have

$$b_n = \frac{3n^3 - 1}{2n^2 + n} \to \frac{\infty}{\infty},$$

again, which is an indeterminate form. Now the largest power of n in the denominator is n^2, so we factor this out of both the numerator and denominator. This gives,

$$c_n = \frac{3n^3 - 1}{2n^2 + n} = \frac{n^2\left(3n - \frac{1}{n^2}\right)}{n^2\left(2 + \frac{1}{n}\right)} = \frac{3n - \frac{1}{n^2}}{2 + \frac{1}{n}} \to \frac{\infty - 0}{2 + 0} = \infty.$$

SOLVED EXAMPLE 3.4
Sequences

Find the limit of the following sequence.

$$d_n = \frac{8n^3 + 4}{1 - 2n^2 + n}.$$

SOLUTION
Now the largest power of n in the denominator is n^2, so we factor this out of both the numerator and denominator. This gives,

$$d_n = \frac{8n^3 + 4}{1 - 2n^2 + n} = \frac{n^2\left(8n + \frac{4}{n^2}\right)}{n^2\left(\frac{1}{n^2} - 2 + \frac{1}{n}\right)} = \frac{8n + \frac{4}{n^2}}{\frac{1}{n^2} - 2 + \frac{1}{n}} \to \frac{\infty + 0}{0 - 2 + 0} = \frac{\infty}{-2} = -\infty.$$

In the following example we use the same process as above, although the sequence is not a rational fraction sequence.

SOLVED EXAMPLE 3.5
Sequences

Find the limit of the following sequence.

$$a_n = \frac{5\sqrt{n} + 1}{\sqrt{n^2 + n} + n}.$$

SOLUTION
As $n \to \infty$, we have

$$b_n = \frac{5\sqrt{n} + 1}{\sqrt{n^2 + n} + n} \to \frac{\infty}{\infty},$$

again, which is an indeterminate form. Now the largest power of n in the denominator is $n = \sqrt{n^2}$, so we factor this out of both the numerator and denominator.

This gives,

$$a_n = \frac{5\sqrt{n} + 1}{\sqrt{n^2 + n} + n} = \frac{n\left(\frac{5}{\sqrt{n}} + \frac{1}{n}\right)}{n\left(\frac{\sqrt{n^2 + n}}{n} + 1\right)} = \frac{\frac{5}{\sqrt{n}} + \frac{1}{n}}{\frac{\sqrt{n^2 + n}}{n} + 1}.$$

As
$$\frac{\sqrt{n^2+n}}{n} = \sqrt{\frac{n^2+n}{n^2}},$$
and from the eariler results we find
$$\frac{n^2+n}{n^2} \to 1,$$
so
$$\sqrt{\frac{n^2+n}{n^2}} \to \sqrt{1} = 1,$$
which implies
$$a_n = \frac{5\sqrt{n}+1}{\sqrt{n^2+n}+n} = \frac{\frac{5}{\sqrt{n}}+\frac{1}{n}}{\sqrt{\frac{n^2+n}{n^2}}+1} \to \frac{0+0}{1+1} = 0.$$

3.4.2 *Geometric Sequences*

Now we turn our attention to the geometric sequences. A sequence $\{a_n\}$ in which the ratio of a_{n+1} and a_n is the same for all $n \in \mathbb{N}$ is called a geometric sequence. For simplicity, we use the general form $a_n = q^n$ of geometric sequences, where $n \in \mathbb{N}$ and $q \in \mathbb{R}$. The convergence properties of geometric sequences are listed in *Theorem* 3.2.

SOLVED EXAMPLE 3.6
Sequences

Find the limit of the following sequence.
$$a_n = \frac{3 \cdot 5^n + 12 \cdot 4^n}{2 \cdot 5^n + 3^n + 2}.$$

SOLUTION
$$b_n = \frac{3 \cdot 5^n + 12 \cdot 4^n}{2 \cdot 5^n + 3^n + 2} \to \frac{\infty}{\infty},$$
again, which is an indeterminate form. Now the dominant term in the denominator is 5^n, so we factor this out of both the numerator and denominator. This gives,

$$a_n = \frac{3 \cdot 5^n + 12 \cdot 4^n + 1}{2 \cdot 5^n + 3^n + 2} = \frac{5^n \left(3 + 12\left(\frac{4}{5}\right)^n + \left(\frac{1}{5}\right)^n\right)}{5^n \left(2 + \left(\frac{3}{5}\right)^n + 2\left(\frac{1}{5}\right)^n\right)} = \frac{3 + 12\left(\frac{4}{5}\right)^n + \left(\frac{1}{5}\right)^n}{2 + \left(\frac{3}{5}\right)^n + 2\left(\frac{1}{5}\right)^n}$$

Use the fact that

$$\left(\frac{4}{5}\right)^n \to 0,$$

similalry,

$$\left(\frac{1}{5}\right)^n \to 0,$$

and

$$\left(\frac{3}{5}\right)^n \to 0,$$

we can conclude that

$$a_n = \frac{3 \cdot 5^n + 12 \cdot 4^n + 1}{2 \cdot 5^n + 3^n + 2} \to \frac{3 + 0 + 0}{2 + 0 + 0} = \frac{3}{2}.$$

3.5 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

3.5.1 Rational Fraction Sequences

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 3.1: Rational Fraction Sequences

Find the limit of the following sequences.

1. $$a_n = \frac{2016n^{32} - 5n^6 + 1}{n^{32} + n^5 - 76},$$

See Solution 8.3.1

2. $$a_n = \frac{n^{2017} + 8n^5 + 1}{n^{62} + n^7 - 98},$$

See Solution 8.3.2

3. $$a_n = \frac{7 - 54\sqrt{n^4 + 1}}{n^2 + \sqrt{n - 9}},$$

See Solution 8.3.3

4. $$a_n = \frac{\sqrt[3]{5n + 1} - 6}{n^2 + \sqrt[3]{n} - 8},$$

See Solution 8.3.4

5. $$a_n = \frac{87n^{13} + 42}{1 - 2n^{25} + 87n},$$

See Solution 8.3.5

Exercises 3.2

Find the limit of the following sequences.

1. $$a_n = \frac{\sqrt[3]{n + 1} + \sqrt{n} - 6}{\sqrt{n^2 - 8} + \sqrt[3]{n}},$$

See Solution 8.3.6

2. $$a_n = \frac{1 - n}{\sqrt{n^2 + 1} + \sqrt{n^2 + n}},$$

See Solution 8.3.7

3.
$$a_n = \frac{\sqrt{5n+1}-6}{n^2+\sqrt[3]{n}-7},$$

See Solution 8.3.8

4.
$$a_n = \frac{65\sqrt[3]{n}-6}{13\sqrt[3]{n}-8},$$

See Solution 8.3.9

5.
$$a_n = \frac{-1}{\sqrt{n}+\sqrt{n+1}}.$$

See Solution 8.3.10

3.5.2 *Geometric Sequences*

Exercises 3.3: Geometric Sequences

Find the limit of the following sequences.

1.
$$a_n = \frac{(-1)^n+1}{5^n+3^n+2},$$

See Solution 8.3.11

2.
$$a_n = \frac{3\cdot 5^n+(-4)^n+12}{2\cdot 5^n+3^n+2},$$

See Solution 8.3.12

3.
$$a_n = \frac{5^{n+1}+2\cdot 3^n}{2\cdot 5^n+2^n+9},$$

See Solution 8.3.13

4.
$$a_n = \frac{3\cdot 7^n+12\cdot 4^n}{9\cdot 4^n+3^n+29},$$

See Solution 8.3.14

5.
$$a_n = \frac{3\cdot 8^{n-1}+12\cdot 4^n-1}{2\cdot 7^n+(-3)^n+2},$$

See Solution 8.3.15

6.
$$a_n = \frac{3\cdot 5^n+12\cdot 4^n}{2\cdot(-9)^n+3^n+2},$$

See Solution 8.3.16

7.
$$a_n = \frac{7\cdot(-5)^n+92\cdot 7^n}{2\cdot 5^n+(-8)^n+2},$$

See Solution 8.3.17

8.
$$a_n = \frac{(-1)^n+9\cdot 4^n}{3^n+2},$$

See Solution 8.3.18

9.
$$a_n = \frac{(-1)^n + 9 \cdot 4^n}{3^n + 2 \cdot 4^n},$$

See Solution 8.3.19

10.
$$a_n = \frac{(-1)^{n+1} + 5 \cdot 3^n}{3^{n-2} + 9}.$$

See Solution 8.3.20

3.6 Sequences with Limit "$\infty - \infty$"

We already know how to deal with limit $\frac{\infty}{\infty}$. In this section we give solutions to sequences with limit "$\infty - \infty$".

3.7 Step-by-Step Examples

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

3.7.1 Difference of Terms of Different Order

In the following limits, we identify the dominant term and find the limit by factoring it out. Note that factoring works thanks to the fact that dominant term is always unique here.

SOLVED EXAMPLE 3.7
Sequences

Find the limit of the following sequence.

$$a_n = n^2 - n.$$

SOLUTION
As n approaches infinity, then n to the power 2 can only get larger. So, we have

$$a_n = n^2 - n \to \infty - \infty,$$

which is an indeterminate form. Without more work there is simply no way to know what will be $\infty - \infty$. First, we identify the largest power of n in the polynomial and we factor this out of the whole polynomial as follows,

$$a_n = n^2 - n = n^2 \left(1 - \frac{1}{n}\right) \to \infty \cdot 1 = \infty.$$

SOLVED EXAMPLE 3.8
Sequences

Find the limit of the following sequence.

$$b_n = 4^n - 3^n.$$

SOLUTION

As $n \to \infty$, we find

$$b_n = 4^n - 3^n \to \infty - \infty$$

again. As 4^n is the dominant term in the polynomial, we factor this out of the whole polynomial as follows,

$$b_n = 4^n - 3^n = 4^n \left(1 - \left(\frac{3}{4}\right)^n\right) \to \infty \cdot 1 = \infty.$$

3.7.2 *Difference of Terms of Equal Order*

As we will see the "factor the dominant terms out of the whole polynomial" does not work here, because this would result $(\pm\infty) \cdot 0$, which would be an indeterminate form. Now we multiply and divide the difference by the same expression, but with plus, and use the formula

$$(a - b)(a + b) = a^2 - b^2.$$

The above formula can be applied to substractions of square roots, read carefully.

SOLVED EXAMPLE 3.9
Sequences

Find the limit of the following sequence.

$$a_n = \sqrt{n} - \sqrt{n+1}.$$

SOLUTION

As $n \to \infty$, we find

$$b_n = \sqrt{n} - \sqrt{n+1} \to \infty - \infty$$

again.

So multiply and divide the difference by $\left(\sqrt{n}+\sqrt{n+1}\right)$.

$$a_n = \sqrt{n}-\sqrt{n+1} = \left(\sqrt{n}-\sqrt{n+1}\right)\cdot\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}} =$$

$$= \frac{\left(\sqrt{n}\right)^2-\left(\sqrt{n+1}\right)^2}{\sqrt{n}+\sqrt{n+1}} = \frac{n-(n+1)}{\sqrt{n}+\sqrt{n+1}} = \frac{-1}{\sqrt{n}+\sqrt{n+1}} \to 0.$$

The essence of the applied method is to cancel the square roots and so to make possible the substraction.

SOLVED EXAMPLE 3.10
Sequences

Find the limit of the following sequence.

$$b_n = \sqrt{n^2+1} - \sqrt{n^2+n}.$$

SOLUTION

As $n \to \infty$, we find

$$b_n = \sqrt{n^2+1} - \sqrt{n^2+n} \to \infty - \infty$$

again. So multiply and divide the difference by $\left(\sqrt{n^2+1} + \sqrt{n^2+n}\right)$ and use the earlier results of the previous section.

$$b_n = \sqrt{n^2+1} - \sqrt{n^2+n} = \left(\sqrt{n^2+1} - \sqrt{n^2+n}\right) \cdot \frac{\sqrt{n^2+1} + \sqrt{n^2+n}}{\sqrt{n^2+1} + \sqrt{n^2+n}} =$$

$$= \frac{\left(\sqrt{n^2+1}\right)^2 - \left(\sqrt{n^2+n}\right)^2}{\sqrt{n^2+1} + \sqrt{n^2+n}} = \frac{n^2+1-(n^2+n)}{\sqrt{n^2+1} + \sqrt{n^2+n}} = \frac{1-n}{\sqrt{n^2+1} + \sqrt{n^2+n}} =$$

$$= \frac{n\left(\frac{1}{n}-1\right)}{n\left(\frac{\sqrt{n^2+1}}{n} + \frac{\sqrt{n^2+n}}{n}\right)} = \frac{n\left(\frac{1}{n}-1\right)}{n\left(\sqrt{\frac{n^2+1}{n^2}} + \sqrt{\frac{n^2+n}{n^2}}\right)} = \frac{\frac{1}{n}-1}{\sqrt{1+\frac{1}{n^2}} + \sqrt{1+\frac{1}{n}}} \to -\frac{1}{2}.$$

3.8 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 3.4

Find the limit of the following sequences.

1.
$$a_n = 12 \cdot 4^n - 3 \cdot 7^n + 3,$$
See Solution 8.3.21

2.
$$a_n = 26n^{312} - 59n^6 + 31,$$
See Solution 8.3.22

3.
$$a_n = n^{32} - 9n^5 - 796,$$
See Solution 8.3.23

4.
$$a_n = 7 \cdot (-5)^n + 92 \cdot 7^n,$$
See Solution 8.3.24

5.
$$a_n = \sqrt{n^3 + n} - \sqrt{n^3 + 2},$$
See Solution 8.3.25

6.
$$a_n = \sqrt{n^2 + 1} - n,$$
See Solution 8.3.26

7.
$$a_n = \sqrt{n^2 + n - 1} - n,$$
See Solution 8.3.27

8.
$$a_n = \sqrt{n^4 + 1} - n^2,$$
See Solution 8.3.28

9.
$$a_n = \sqrt{n^4 + n - 1} - n^2,$$
See Solution 8.3.29

10.
$$a_n = \sqrt{4^n + 1} - 2^n,$$
See Solution 8.3.30

11.
$$a_n = \sqrt{2^n + 1} - \sqrt{2^n + 3},$$
See Solution 8.3.31

12.
$$a_n = \frac{1}{\sqrt{9^n+1} - \sqrt{9^n-2}},$$
See Solution 8.3.32

13.
$$a_n = \frac{1}{\sqrt{n^4+1} - \sqrt{n^4-2}},$$
See Solution 8.3.33

14.
$$a_n = \frac{1}{\sqrt{n^4+n} - \sqrt{n^4-2}}.$$
See Solution 8.3.34

3.9 *Application of The Squeeze Theorem*

In the following examples, we show how *Theorem 3.13* can be used to prove convergence of a sequence. First we establish a lemma - a direct consequence of Definition 3.1, in fact - which will be useful.

Lemma 3.1

If $a_n \to 1$, then there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2} \leq a_n \leq \frac{3}{2}$$

for every $n \geq n_0$. □

3.10 *Step-by-Step Examples*

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 3.11
Sequences

Find the limit of the following sequence.

$$a_n = \sqrt[n]{3n^3 + 5n^2 - 3n + 2}.$$

SOLUTION
We use the following known results on limit of sequences.

$$\sqrt[n]{a} \to 1, \quad a > 0$$

and

$$\sqrt[n]{n} \to 1.$$

As

$$\mathbf{3n}^3 + 5n^2 - 3n + 2 = \mathbf{3n}^3 \cdot \left(1 + \frac{5}{n} - \frac{3}{n^2} + \frac{2}{n^3}\right)$$

and

$$1 + \frac{5}{n} - \frac{3}{n^2} + \frac{2}{n^3} \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$.

$$\frac{1}{2} \leq 1 + \frac{5}{n} - \frac{3}{n^2} + \frac{2}{n^3} \leq \frac{3}{2}$$

So, if $n \geq n_0$, then

$$\mathbf{3n}^3 \cdot \frac{1}{2} \leq 3n^3 + 5n^2 - 3n + 2 \leq \mathbf{3n}^3 \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{\mathbf{3n}^3 \cdot \frac{1}{2}} \leq \sqrt[n]{3n^3 + 5n^2 - 3n + 2} \leq \sqrt[n]{\mathbf{3n}^3 \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. This follows that for $n \geq n_0$, we have

$$\sqrt[n]{\frac{3}{2}} \cdot \left(\sqrt[n]{n}\right)^3 = \sqrt[n]{3n^3 \cdot \frac{1}{2}} \leq \sqrt[n]{3n^3 + 5n^2 - 3n + 2} \leq \sqrt[n]{3n^3 \cdot \frac{3}{2}} = \left(\sqrt[n]{n}\right)^3 \cdot \sqrt[n]{\frac{9}{2}}.$$

As

$$\sqrt[n]{\frac{3}{2}} \left(\sqrt[n]{n}\right)^3 \to 1$$

and

$$\left(\sqrt[n]{n}\right)^3 \sqrt[n]{\frac{9}{2}} \to 1,$$

from the Squezze Theorem, we get

$$a_n = \sqrt[n]{3n^3 + 5n^2 - 3n + 2} \to 1.$$

SOLVED EXAMPLE 3.12

Sequences

Find the limit of the following sequence.

$$b_n = \sqrt[n]{3 \cdot 4^n + 5 \cdot 3^n - 3 \cdot 2^n + 2}$$

SOLUTION

As

$$\mathbf{3} \cdot \mathbf{4}^n + 5 \cdot 3^n - 3 \cdot 2^n + 2 = \mathbf{3} \cdot \mathbf{4}^n \left(1 + \frac{5}{3} \cdot \left(\frac{3}{4}\right)^n - \frac{3}{3} \cdot \left(\frac{2}{4}\right)^n + \frac{2}{3}\left(\frac{1}{4}\right)^n\right)$$

and

$$1 + \frac{5}{3} \cdot \left(\frac{3}{4}\right)^n - \frac{3}{3} \cdot \left(\frac{2}{4}\right)^n + \frac{2}{3}\left(\frac{1}{4}\right)^n \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2} \le 1 + \frac{5}{3} \cdot \left(\frac{3}{4}\right)^n - \frac{3}{3} \cdot \left(\frac{2}{4}\right)^n + \frac{2}{3}\left(\frac{1}{4}\right)^n \le \frac{3}{2}$$

for every $n \ge n_0$. So, if $n \ge n_0$, then

$$\mathbf{3} \cdot \mathbf{4}^n \cdot \frac{1}{2} \le 3 \cdot 4^n + 5 \cdot 3^n - 3 \cdot 2^n + 2 \le \mathbf{3} \cdot \mathbf{4}^n \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{\mathbf{3} \cdot \mathbf{4}^n \cdot \frac{1}{2}} \le \sqrt[n]{3 \cdot 4^n + 5 \cdot 3^n - 3 \cdot 2^n + 2} \le \sqrt[n]{\mathbf{3} \cdot \mathbf{4}^n \cdot \frac{3}{2}},$$

whenever $n \ge n_0$.

As

$$\sqrt[n]{\frac{3}{2}} \cdot 4 = \sqrt[n]{\frac{3}{2}} \sqrt[n]{4^n} = \sqrt[n]{\mathbf{3} \cdot \mathbf{4}^n \cdot \frac{1}{2}}$$

and

$$\sqrt[n]{\mathbf{3} \cdot \mathbf{4}^n \cdot \frac{3}{2}} = \sqrt[n]{4^n} \sqrt[n]{\frac{9}{2}} = 4 \cdot \sqrt[n]{\frac{9}{2}}.$$

for $n \ge n_0$, we have

$$\sqrt[n]{\frac{3}{2}} \cdot 4 \le \sqrt[n]{3n^3 + 5n^2 - 3n + 2} \le 4 \cdot \sqrt[n]{\frac{9}{2}}.$$

As

$$\sqrt[n]{\frac{3}{2}} \to 1$$

and

$$\sqrt[n]{\frac{9}{2}} \to 1,$$

from the Squezze Theorem, we get

$$a_n = \sqrt[n]{3 \cdot 4^n + 5 \cdot 3^n - 3 \cdot 2^n + 2} \to 4.$$

SOLVED EXAMPLE 3.13
Sequences

Find the limit of the following sequence.

$$c_n = \sqrt[n]{3n - 2\cos(n)}.$$

SOLUTION
Using

$$-1 \le \cos(n) \le 1,$$

we get, that

$$-1 \le -\cos(n) \le 1,$$

$$-2 \le -2\cos(n) \le 2,$$

$$3n - 2 \le 3n - 2\cos(n) \le 3n + 1.$$

As $n \ge 2$ integer,

$$3n - 2n \le 3n - 2$$

and

$$3n + 1 \le 3n + n.$$

This follows

$$\sqrt[n]{n} \le \sqrt[n]{3n - 2\cos(n)} \le \sqrt[n]{4n} = \sqrt[n]{4}\sqrt[n]{n},$$

so from the Squezze Theorem, we get

$$c_n = \sqrt[n]{3n - 2\cos(n)} \to 1.$$

3.11 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 3.5

Find the limit of the following sequences.

1.
$$a_n = \sqrt[n]{n^2 + n + 2},$$
See Solution 8.3.35

2.
$$a_n = \sqrt[n]{n^4 - n^3 - n - 2},$$
See Solution 8.3.36

3.
$$a_n = \sqrt[n]{n^5 + n^4 - 2n^3 + 32},$$
See Solution 8.3.37

4.
$$a_n = \sqrt[n]{n^2 + 5\sqrt{n} + 9},$$
See Solution 8.3.38

5.
$$a_n = \sqrt[n]{7 \cdot 8^n + 3 \cdot 5^n + 9},$$
See Solution 8.3.39

6.
$$a_n = \sqrt[n]{2 \cdot 9^n + 3 \cdot 6^n + 9 \cdot 2^n},$$
See Solution 8.3.40

7.
$$a_n = \sqrt[n]{7 \cdot 5^n - 2 \cdot 3^n - 89},$$
See Solution 8.3.41

8.
$$a_n = \sqrt[n]{7 \cdot 10^n - 31 \cdot 6^n + 72},$$
See Solution 8.3.42

9.
$$a_n = \sqrt[n]{7n + 5\sin(n)},$$
See Solution 8.3.43

10.
$$a_n = \sqrt[n]{7n^2 + 2\sin(n)}.$$
See Solution 8.3.44

3.12 The Sequence $\left(1+\frac{1}{n}\right)^n$

First we establish the following theorem.

Theorem 3.14

If

$$\lim_{n\to\infty} a_n = \infty,$$

then

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e. \qquad \square$$

3.13 Step-by-Step Examples

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 3.14

Sequences

Find the limit of the following sequence.

$$a_n = \left(\frac{n+5}{n}\right)^n.$$

SOLUTION

As

$$\left(1+\frac{a}{n}\right)^n \to e^a, \quad a \in \mathbb{R}$$

after some basic manipulations, we have

$$a_n = \left(\frac{n+5}{n}\right)^n = \left(1+\frac{5}{n}\right)^n \to e^5.$$

SOLVED EXAMPLE 3.15

Sequences

Find the limit of the following sequence.

$$b_n = \left(\frac{4n+5}{4n-2}\right)^n.$$

SOLUTION

As

$$\left(1+\frac{a}{n}\right)^n \to e^a, \quad a \in \mathbb{R}$$

after some basic manipulations, we have

$$b_n = \left(\frac{4n+5}{4n-2}\right)^n = \left(\frac{4n\left(1+\frac{5}{4n}\right)}{4n\left(1-\frac{2}{4n}\right)}\right)^n = \frac{\left(1+\frac{5}{4n}\right)^n}{\left(1-\frac{2}{4n}\right)^n} = \frac{\left(1+\frac{\frac{5}{4}}{n}\right)^n}{\left(1+\frac{-\frac{2}{4}}{n}\right)^n} \to \frac{e^{\frac{5}{4}}}{e^{\frac{-2}{4}}}.$$

SOLVED EXAMPLE 3.16

Sequences

Find the limit of the following sequence.

$$c_n = \left(\frac{3n^2+5}{3n^2-2}\right)^{n^2}.$$

SOLUTION

Let $m = n^2$. So

$$\left(\frac{3n^2+5}{3n^2-2}\right)^{n^2} = \left(\frac{3m+5}{3m-2}\right)^m.$$

As $n \to \infty$, we have $m \to \infty$. After some basic manipulations, we have

$$\left(\frac{3m+5}{3m-2}\right)^m = \left(\frac{3m\left(1+\frac{5}{3m}\right)}{3m\left(1-\frac{2}{3m}\right)}\right)^m = \frac{\left(1+\frac{5}{3m}\right)^m}{\left(1-\frac{2}{3m}\right)^m} = \frac{\left(1+\frac{\frac{5}{3}}{m}\right)^m}{\left(1+\frac{-\frac{2}{3}}{m}\right)^m} \to \frac{e^{\frac{5}{3}}}{e^{\frac{-2}{3}}},$$

thus

$$c_n = \left(\frac{3n^2+5}{3n^2-2}\right)^{n^2} \to \frac{e^{\frac{5}{3}}}{e^{\frac{-2}{3}}}.$$

SOLVED EXAMPLE 3.17
Sequences

Find the limit of the following sequence.

$$d_n = \left(1 + \frac{1}{2n+1}\right)^n.$$

SOLUTION

After some basic manipulations, we have

$$d_n = \left(1 + \frac{1}{2n+1}\right)^n = \left(\left(1 + \frac{1}{2n+1}\right)^{\frac{2n+1}{2n+1}}\right)^n = \left(\left(1 + \frac{1}{2n+1}\right)^{2n+1}\right)^{\frac{n}{2n+1}}.$$

As $(2n+1) \to \infty$, using Theorem 3.14, we find

$$\left(1 + \frac{1}{2n+1}\right)^{2n+1} \to e,$$

whenever $n \to \infty$. Furthermore,

$$\frac{n}{2n+1} \to \frac{1}{2},$$

whenever $n \to \infty$. So

$$d_n = \left(1 + \frac{1}{2n+1}\right)^n \to e^{\frac{1}{2}}.$$

3.14 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 3.6

Find the limit of the following sequences.

1. $$a_n = \left(\frac{n+9}{n-4}\right)^n,$$
See Solution 8.3.45

2. $$a_n = \left(\frac{9n+5}{9n-3}\right)^n,$$
See Solution 8.3.46

3. $$a_n = \left(1+\frac{1}{9n}\right)^{3n-1},$$
See Solution 8.3.47

4. $$a_n = \left(\frac{n^3+9}{n^3+14}\right)^{n^3},$$
See Solution 8.3.48

5. $$a_n = \left(1+\frac{1}{n^2}\right)^{3n^2-1},$$
See Solution 8.3.49

6. $$a_n = \left(\frac{3n^2+9}{3n^2+14}\right)^{n^2},$$
See Solution 8.3.50

7. $$a_n = \left(1-\frac{21}{6n-3}\right)^n,$$
See Solution 8.3.51

8. $$a_n = \left(1+\frac{1}{5n+1}\right)^{1-3n^2},$$
See Solution 8.3.52

9. $$a_n = \left(1+\frac{1}{5n^2+1}\right)^{1-3n},$$
See Solution 8.3.53

10. $$a_n = \left(\frac{4n-5}{5n+2}\right)^n.$$
See Solution 8.3.54

3.15 Finding the Limit of a Sequence with Definition

In the previous sections we computed the limits of sequences using the properties of limits.

Now we show some examples how the limit of a sequence can be determined using *Definition* 3.1, *Definition* 3.5 or *Definition* 3.6.

3.16 Step-by-Step Examples

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 3.18
Sequences

Use *Definition* 3.1 to prove that

$$\frac{4n-5}{3n+2} \to \frac{4}{3}.$$

SOLUTION
Let

$$a_n = \frac{4n-5}{3n+2},$$

and

$$a = \frac{4}{3}.$$

We want to show that

$$\lim_{n\to\infty} \frac{4n-5}{3n+2} = \frac{4}{3}.$$

From *Definition* 3.1, given any $\varepsilon > 0$, we want to make

$$\left| \frac{4n-5}{3n+2} - \frac{4}{3} \right| < \varepsilon.$$

Thus we have to find n_0 $(n_0 = n_0(\varepsilon))$, such that if $n \geq n_0$, then

$$\left| \frac{4n-5}{3n+2} - \frac{4}{3} \right| < \varepsilon.$$

holds. As

$$\left| \frac{4n-5}{3n+2} - \frac{4}{3} \right| < \varepsilon$$

after some equivalent (!) manipulations, we have

$$\begin{aligned}
\left|\frac{4n-5}{3n+2}-\frac{4}{3}\right| &< \varepsilon \\
&\Updownarrow \\
\left|\frac{3(4n-5)-4(3n+2)}{3(3n+2)}\right| &< \varepsilon \\
&\Updownarrow \\
\left|\frac{12n-15-12n-8}{3(3n+2)}\right| &< \varepsilon \\
&\Updownarrow \\
\left|\frac{-23}{3(3n+2)}\right| &< \varepsilon \\
&\Updownarrow \qquad \text{as } n>0 \\
\frac{23}{3(3n+2)} &< \varepsilon \\
&\Updownarrow \qquad \text{as } \varepsilon>0 \\
\frac{23}{3\varepsilon} &< 3n+2 \\
&\Updownarrow \\
\frac{\frac{23}{3\varepsilon}-2}{3} &< n.
\end{aligned}$$

If $\varepsilon \geq \frac{23}{6}$, then the left side of the last inequality is nonpositive, thus it holds for any $n \in \mathbb{N}$. Thus we have

$$n_0 = \begin{cases} 0, & \text{if } \varepsilon \geq \frac{23}{6}, \\ \left[\frac{\frac{23}{3\varepsilon}-2}{3}\right]+1, & \text{if } 0<\varepsilon<\frac{23}{6} \end{cases}.$$

Hence $n \geq n_0$ implies

$$\left|\frac{4n-5}{3n+2}-\frac{4}{3}\right| < \varepsilon,$$

by *Definition 3.1*, this proves that

$$\lim_{n\to\infty}\frac{4n-5}{3n+2}=\frac{4}{3}.$$

SOLVED EXAMPLE 3.19
Sequences

Use *Definition* 3.5 to prove that

$$n^2 - 1 \to \infty.$$

SOLUTION
Let

$$a_n = n^2 - 1.$$

By *Definition* 3.5, we need to show that, given any $c \in \mathbb{R}$ there exists an n_0 $(n_0 = n_0(c))$, such that if $n \geq n_0$, then

$$n^2 - 1 > c$$

holds. As

$$n^2 - 1 > c,$$

we have

$$n^2 > c + 1,$$

If $c < 1$, then the right side is negative, so for any $n \in \mathbb{N}$ the inequality holds. Suppose that $c \geq 1$. Then

$$n^2 > c + 1$$

follows

$$n > \sqrt{c+1}.$$

Thus

$$n_0 = \begin{cases} 0, & \text{if } c < 1, \\ \left[\sqrt{c+1}\right] + 1, & \text{if } c \geq 1 \end{cases},$$

and $n \geq n_0$ implies

$$n^2 - 1 > c,$$

and by definition, this proves that

$$n^2 - 1 \to \infty.$$

SOLVED EXAMPLE 3.20
Sequences

Use *Definition* 3.6 to prove that

$$1 - n^2 \to -\infty.$$

SOLUTION

Let $c \in \mathbb{R}$ given and

$$a_n = 1 - n^2.$$

By *Definition* 3.6, we need to show that, given any $c \in \mathbb{R}$ there exists an n_0 $(n_0 = n_0(c))$, such that if $n \geq n_0$, then

$$1 - n^2 < c$$

holds. As

$$1 - n^2 < c,$$

we find

$$1 - c < n^2,$$

If $c > 1$, then the left side is negative, so for any $n \in \mathbb{N}$ the inequality holds. Suppose that $c \leq 1$. Then

$$\sqrt{1-c} < n.$$

Thus

$$n_0 = \begin{cases} 0, & \text{if } c > 1, \\ \left[\sqrt{1-c}\right] + 1, & \text{if } c \leq 1 \end{cases}.$$

and $n \geq n_0$ implies

$$1 - n^2 < c$$

and by definition, this proves that

$$1 - n^2 \to -\infty.$$

3.17 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 3.7

Use *Definition* 3.1 to prove that

1.

$$a_n = \frac{5n-4}{2n+3} \to \frac{5}{2},$$

See Solution 8.3.55

2.

$$a_n = \frac{6n+1}{9n-2} \to \frac{2}{3},$$

See Solution 8.3.56

3.

$$a_n = \left(\frac{1}{2}\right)^n \to 0,$$

See Solution 8.3.57

4.

$$a_n = \left(\frac{-1}{5}\right)^n \to 0,$$

See Solution 8.3.58

5.

$$a_n = \frac{1}{n^2+1} \to 0.$$

See Solution 8.3.59

Exercises 3.8

Use *Definition* 3.5 to prove that

1.

$$a_n = 2^n \to \infty,$$

See Solution 8.3.60

2.

$$a_n = \ln(n) \to \infty,$$

See Solution 8.3.61

3.

$$a_n = n^2 + 2n + 1 \to \infty.$$

See Solution 8.3.62

Exercises 3.9

Use *Definition* 3.6 to prove that

1.

$$a_n = 1 - \lg(n) \to -\infty,$$

See Solution 8.3.63

2.

$$a_n = 1 - 3^n \to -\infty.$$

See Solution 8.3.64

3.18 Divergent Sequences

In this section we turn our attention to the divergence of a sequence. (Recall, that non-convergent sequences are called divergent.) Definition 3.7 and Theorem 3.1 are the keys to our solutions.

3.19 Step-by-Step Examples

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 3.21
Sequences

Prove that the sequence is divergent.

$$a_n = \frac{(-1)^n n + 3}{2n + 1}$$

SOLUTION
Consider the "odd" and "even" subsequences of a_n. If

$$n_{k_1} = 2k, \qquad k \in \mathbb{N},$$

then

$$a_{2k} = \frac{(-1)^{2k}(2k) + 3}{2(2k) + 1}.$$

As $(-1)^{2k} = 1$, we have

$$a_{2k} = \frac{2k + 3}{2(2k) + 1} = \frac{2k + 3}{4k + 1} \to \frac{2}{4},$$

whenever $k \to \infty$. Now let

$$n_{k_2} = 2k + 1,$$

where $k \in \mathbb{N}$, so we have

$$a_{2k+1} = \frac{(-1)^{2k+1}(2k + 1) + 3}{2(2k + 1) + 1}.$$

As $(-1)^{2k+1} = -1$, we find

$$a_{2k+1} = \frac{-(2k + 1) + 3}{2(2k + 1) + 1} = \frac{-2k + 2}{4k + 3} \to \frac{-2}{4}.$$

whenever $k \to \infty$.

We can conclude, that

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = \frac{(-1)^n n + 3}{2n+1}$$

has no limit at all.

SOLVED EXAMPLE 3.22

Sequences

Prove that the sequence is divergent.

$$a_n = (-1)^n \frac{4n+2}{3n+5},$$

SOLUTION

Although we can use the same method as above, we give another solution to this example. No matter n is odd or even, we have

$$\frac{4n+2}{3n+5} \to \frac{4}{3}$$

holds. From this we get for $n = 2k$ that $(-1)^{2k} = 1 \to 1$, so

$$a_{2k} \to 1 \cdot \frac{4}{3} = \frac{4}{3}$$

whenever $k \to \infty$.

Now let $n = 2k+1$ then $(-1)^{2k+1} = -1 \to -1$, so

$$a_{2k+1} \to -1 \cdot \frac{4}{3} = -\frac{4}{3}$$

whenever $k \to \infty$. This follows

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = (-1)^n \frac{4n+2}{3n+5}$$

has no limit at all.

3.20 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 3.10

Prove that the sequence is divergent.

1. $$a_n = \frac{(-1)^n n^2}{n^2+7},$$

See Solution 8.3.65

2. $$a_n = \frac{1+(-1)^n n^2}{n+1},$$

See Solution 8.3.66

3. $$a_n = (-1)^n \frac{5n+3}{6n+1},$$

See Solution 8.3.67

4. $$a_n = \frac{n}{n^{(-1)^n}+2},$$

See Solution 8.3.68

5. $$a_n = \frac{5^n\left(1+(-1)^n\right)+3^n}{5^n+2^n},$$

See Solution 8.3.69

6. $$a_n = (n+1)^{(-1)^n},$$

See Solution 8.3.70

7. $$a_n = (-1)^n \frac{6^n+1}{2\cdot 6^n-1},$$

See Solution 8.3.71

8. $$a_n = (-1)^n \sqrt{n+1}-\sqrt{n},$$

See Solution 8.3.72

9. $$a_n = (-1)^n \sqrt{n^2+1}-n,$$

See Solution 8.3.73

10. $$a_n = (-1)^n \left(\sqrt{n^2+n}-n\right).$$

See Solution 8.3.74

4 Limit and Continuity of One Variable Real Functions

Calculating $f(a)$ for any (or some) $a \in dom(f)$ with a pocket calculator can *not* reveal the important properties of the function f . In many cases $a \notin dom(f)$, i.e. $f(a)$ can *not* be computed at all, or we are interested in $f(a)$ for *large* a, which are beyond the capacities of calculators. In these cases we have to use mathematical tools: investigating the values of $f(x)$ for values x "close to a", we can deduce the possible value of the missing $f(a)$ or "$f(\pm\infty)$", more exactly than predicting!

Another motivation of the following methods is the *continuity* of the function f : can its graph be drawn with a single curve or it has breaks, separate lines are needed? Think a little bit on this question: when drawing, the horizontal coordinate is x, the vertical is $y = f(x)$, so a fine (little) movement of x does imply also a fine movement of y ? In other words, does "x is close to a" imply "$f(x)$ is close to $f(a)$" ?

Since we already have fairly good experience in *sequences* and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, first we give the shorter definitions using sequences, and after the exact (longer) definition of "close to".

In many cases we can deal with inputs $x < a$ or $a < x$ *only*, these problems are called "*one sided*", but first we consider "*two sided*" limits.

4.1 *Two Sided Limits*

Definition 4.1: Neighbourhood of Point a

For $a \in \mathbb{R}$ an open interval of the form $(a-\delta, a+\delta)$ is called a **neighbourhood** of a for any positive δ. □

Definition 4.2: Deleted Neighbourhood of Point a

For $a \in \mathbb{R}$ a set of the form $(a-\delta, a+\delta) \setminus \{a\}$ is called a **deleted neighbourhood** of a for any positive δ. □

Definition 4.3: Limit of Function

Any $b \in \overline{\mathbb{R}}$ is called the **limit of** the function f **at** the "point" $a \in \overline{\mathbb{R}}$ if $dom\,(f)$ contains a deleted neighbourhood of a *and for any* sequence $\{x_n\}_{n=0}^{\infty}$ such that $\forall n \;\; x_n \in dom\,(f)$ and $x_n \to a$ we have the sequence $\{y_n\}_{n=0}^{\infty}$ where $y_n = f\,(x_n)$ converges to b. □.

Notation 4.1.1

$$\lim_{x\to a} f\,(x) = b\,.$$

Let us emphasize that $f\,(a)$ is irrelevant when computing $\lim\limits_{x\to a} f\,(x)$, even when $a \notin dom\,(f)$ is allowed.

For example for every sequence $x_n \to 2$, $x_n \neq 0$, we have $y_n = \dfrac{1}{x_n}$ converges to $\dfrac{1}{2}$ (i.e.$f\,(x) = \dfrac{1}{x}$, $a = 2$ and $b = \dfrac{1}{2}$), so we can write $\lim\limits_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

The Reader is asked to think on the statement " $\lim\limits_{x\to 0} \dfrac{x}{1-e^{1/x}} = 0$ " (i.e. $f\,(x) = \dfrac{x}{1-e^{1/x}}$, $a = 0$ and $b = 0$, see also the solution of Exercise 4.5 6.

Now we turn to the definitions using the phrase "close to". Since "close" to $\pm\infty$ (infinites) are quite another than to real (finite) numbers (e.g. $\sqrt{2}$), we need separate defintions for $\pm\infty$ and $a \in \mathbb{R}$.

Definition 4.4: Finite Limit at Finite Point

Let $a \in \mathbb{R}$, suppose $dom\,(f)$ contains a deleted neighbourhood of a .
Then f has a finite limit at a if there is a number $b \in \mathbb{R}$ such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\text{if } |x-a| < \delta \quad \text{then} \quad |f(x) - b| < \varepsilon\,. \quad \square$$

Notation 4.1.2

$$\lim_{x \to a} f(x) = b\,.$$

Definition 4.5: Infinite Limit at Finite Point

Let $a \in \mathbb{R}$, suppose $dom\,(f)$ contains a deleted neighbourhood of a .
Then f has a finite limit at a

(i) $+\infty$ if for any $p \in \mathbb{R}$ there is a $\delta > 0$ such that

$$\text{if } |x-a| < \delta \quad \text{then} \quad f(x) > p\,.$$

(ii) -∞ if for any $p \in \mathbb{R}$ there is a $\delta > 0$ such that

$$\text{if } |x-a| < \delta \quad \text{then} \quad f(x) < p\,. \quad \square$$

Notation 4.1.3

$$\lim_{x \to a} f(x) = +\infty$$

and

$$\lim_{x \to a} f(x) = -\infty\,.$$

Remark 4.1.1 *The red and blue colors are essentials above: i) and ii) are differ only in these "minor" symbols.*

Definition 4.6: Finite Limits at Infinity

Function f has **finite** limit

(i1) **at** $+\infty$, if there is a number $b \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a number $K \in \mathbb{R}$ such that

$$\text{if } K < x \quad \text{then} \quad |f(x) - b| < \varepsilon\,,$$

(i2) **at** $-\infty$, if there is a number $b \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a number $K \in \mathbb{R}$ such that

$$\text{if } x < K \quad \text{then} \quad |f(x) - b| < \varepsilon\,. \quad \square$$

Notation 4.1.4

$$\lim_{x \to +\infty} f(x) = b$$

and

$$\lim_{x \to -\infty} f(x) = b\,.$$

Clearly *(i1)* is interesting for *large positive* $K \to +\infty$), while in *(i2)* we deal with *large negative* $K \to -\infty$. Similar remarks yield for the forthcoming definitions, too.

Definition 4.7: Infinite Limits at Infinity

Function f has limit

(ii1) $+\infty$ **at** $+\infty$, if for any number $p \in \mathbb{R}$ there is a number $K \in \mathbb{R}$ such that

$$\text{if } K < x \quad \text{then} \quad p < f(x) \ ,$$

(ii2) $+\infty$ **at** $-\infty$, if for any number $p \in \mathbb{R}$ there is a number $K \in \mathbb{R}$ such that

$$\text{if } x < K \quad \text{then} \quad p < f(x) \ ,$$

(iii1) $-\infty$ **at** $+\infty$, if for any number $p \in \mathbb{R}$ there is a number $K \in \mathbb{R}$ such that

$$\text{if } K < x \quad \text{then} \quad p > f(x) \ ,$$

(iii2) $-\infty$ **at** $-\infty$, if for any number $p \in \mathbb{R}$ there is a number $K \in \mathbb{R}$ such that

$$\text{if } K > x \quad \text{then} \quad p > f(x)$$

Notation 4.1.5

$$\lim_{x\to+\infty} f(x) = +\infty$$

and

$$\lim_{x\to-\infty} f(x) = +\infty \, .$$

Notation 4.1.6

$$\lim_{x\to+\infty} f(x) = -\infty$$

and

$$\lim_{x\to-\infty} f(x) = -\infty \, .$$

Remark 4.1.2 *Clearly, the numbers p we are interested in (ii1) and (ii2) are large positive ($p \to +\infty$), while in (iii1) és (iii2) large negative ($p \to -\infty$).*

Observe, that the colors are important in the above definition: red **<** *signs correspond to red* **+** *ones, and similarly blue* **>** *signs to blue* **-** *.*

Make drafts of function graphs for the different cases ($x \to +\infty$, $x \to a$, $x \to -\infty$) for better understanding the definitions and the role of the letters b , ε , K , p .

4.2 *One Sided Limits*

In many cases, when computing $\lim\limits_{x\to a} f(x)$, $f(x)$ can be computed for $x < a$ or for $a < x$ only, or the limit can be different for "below" and "above" a , these tasks are called also "half-sided" ones.

Typical situations are the limits of type " $\frac{c}{0}$ " ($c \neq 0$), i.e. when the limit of the *numerator* is far from 0 but the limit of the *denomirator* is 0 . Since fractions "big/small" have *huge values*, the *sign* of these huge numbers are important ("running either to $-\infty$ or to $+\infty$"), which sign depend on the sign of the *denomirator*. This question in most of the cases can be decided by investigating the question "$x < a$" or "$a < x$".

These cases are discussed in this subsection, detailed explanation can be found in Solved Examples 4.11 and 4.12.

We provide the definitions using sequences and $b \in \overline{\mathbb{R}}$, the "$\varepsilon - \delta$" -definitions can be invented by anyone, or can be found in many books.

Notation 4.2.1

$$\lim_{x\to a-0} f(x) = b$$

and

$$\lim_{x\to a+0} f(x) = b\,.$$

or shortly

$$\lim_{x\to a-} f(x) = b$$

and

$$\lim_{x\to a+} f(x) = b\,.$$

Definition 4.8

Consider function $f : \mathbb{R} \to \mathbb{R}$. We say function f has

(i) left sided limit $b \in \overline{\mathbb{R}}$ at the point $a \in (-\infty, \infty]$ if $dom(f)$ contains a deleted left sided neighbourhood of a and for *any* sequence $\{x_n\}_{n=0}^{\infty}$ in this neighbourhood, i.e. $x_n \in dom(f)\ \forall n$ and $x_n < a$ we have the sequence $y_n := f(x_n)$ has limit b .

(ii) right sided limit $b \in \overline{\mathbb{R}}$ at the point $a \in [-\infty, \infty)$ if $dom(f)$ contains a deleted right sided neighbourhood of a and for *any* sequence $\{x_n\}_{n=0}^{\infty}$ in this neighbourhood, i.e. $x_n \in dom(f)\ \forall n$ and $a < x_n$ we have the sequence $y_n := f(x_n)$ has limit b. □

Let us emphasize, that **the symbols** "$x \to a - 0$" and "$x \to a + 0$" mean $x < a$ and $x < a$, and are originated from "$x \approx a + 0$" and "$x \approx a - 0$" , respectively.

An easy but important connection among half-sided and two.sided limit is the following.

Notation 4.2.2 *If our calculations show $f(x) \le b\ \forall x$ or $b \le f(x)\ \forall x$ for any limit $\lim_{x\to a}$ or $\lim_{x\to a-}$ or $\lim_{x\to a+}$, we can denote this fact by*

$$\lim f(x) = b- \quad \textit{and} \quad \lim f(x) = b+$$

Theorem 4.1

Any function f has a *two-sided* **limit** (either finite or infinite) at the "point" $a \in \overline{\mathbb{R}}$ if and only if *both* $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ do exist, and moreover

$$\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = \lim_{x\to a} f(x)\,. \qquad \square$$

4.3 Methods for calculating

At this point we finished listing the definitions, now we start to make *limit* calculations.

Theorem 4.2: Rules for Limit Calculation

For any $a \in \overline{\mathbb{R}}$, assuming that the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do exist, we have

$$\lim_{x \to a} \left(f(x) \pm g(x) \right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \,,$$

$$\lim_{x \to a} \left(f(x) \cdot g(x) \right) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \,,$$

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \text{ when } \lim_{x \to a} g(x) \neq 0 \,. \qquad \square$$

Remark 4.3.1 *Recall, that the operations $+, -, \cdot$ and $:$ in $\overline{\mathbb{R}}$ were discussed in Section 3.1.*

Remark 4.3.2 *When calculating $\lim_{x \to a} f(x)$ for $a = \pm\infty$ we can use the* same *methods learned for calculating the limits of sequenes, taking care both of the possibilities "$n \to +\infty$" and "$n \to -\infty$" (i.e. the sign of n can be negative, too).*

The following special limits are often used in problem solving, but recall the methods and formulae learned in Chapter 3.

Theorem 4.3: Famous Limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \,, \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \,, \qquad \lim_{x \to 0} \frac{\ln(x+1)}{x} = 1 \,,$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \,, \qquad \lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \,, \qquad \lim_{x \to 0} x \cdot \ln(x) = 0 \,,$$

$$\lim_{x \to 0+} \frac{1}{x} = +\infty \,, \qquad \lim_{x \to 0-} \frac{1}{x} = -\infty \,, \qquad \lim_{x \to 0} \frac{1}{x} \text{ does not exist.} \qquad \square$$

4.4 *Step-by-Step Examples*

Now we give step-by-step solutions to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 4.1
Function Limit

Calculate

$$\lim_{x\to-\infty}\frac{x^2-2x+1}{x^2-1}\ .$$

SOLUTION

The solution method is the same as of the problem

$$\lim_{n\to-\infty}\frac{n^2-2n+1}{n^2-1}\ .$$

The fact $x \to -\infty$ causes both the numerator and the denomirator to tend to $\pm\infty$, so we extract the main term of the *denomirator*, i.e. x^2 from both of the numerator and the denomirator, and simplify after:

$$\lim_{x\to-\infty}\frac{x^2-2x+1}{x^2-1}=\lim_{x\to-\infty}\frac{x^2\left(1-\frac{2}{x}+\frac{1}{x^2}\right)}{x^2\left(1-\frac{1}{x^2}\right)}=\lim_{x\to-\infty}\frac{1-\frac{2}{x}+\frac{1}{x^2}}{1-\frac{1}{x^2}}=1\ .$$

SOLVED EXAMPLE 4.2
Function Limit

Calculate

$$\lim_{x\to\infty}\left(\sqrt{x^2+x+4}-\sqrt{x^2-x+2}\right).$$

SOLUTION

Square roots can not be substracted, so first we have to eliminate them by using the following identity.

$$(a-b)(a+b)=a^2-b^2$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+x+4}-\sqrt{x^2-x+2}\right)=$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+x+4}-\sqrt{x^2-x+2}\right)\left(\sqrt{x^2+x+4}+\sqrt{x^2-x+2}\right)}{\sqrt{x^2+x+4}+\sqrt{x^2-x+2}}=$$

$$=\lim_{x\to\infty}\frac{(x^2+x+4)-(x^2-x+2)}{\sqrt{x^2+x+4}+\sqrt{x^2-x+2}}=\lim_{x\to\infty}\frac{2x+2}{\sqrt{x^2+x+4}+\sqrt{x^2-x+2}}.$$

This latter limit is of form "$\frac{\infty}{\infty}$" , so we simplify it as in the previous example, by $\sqrt{x^2}=x$:

$$\lim_{x\to\infty}\frac{2x+2}{\sqrt{x^2+x+4}+\sqrt{x^2-x+2}}$$

$$=\lim_{x\to\infty}\frac{x\left(2+\frac{2}{x}\right)}{\sqrt{x^2}\left(\sqrt{1+\frac{1}{x}+\frac{4}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{2}{x^2}}\right)}=$$

$$=\lim_{x\to\infty}\frac{2+\frac{2}{x}}{\sqrt{1+\frac{1}{x}+\frac{4}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{2}{x^2}}}=\frac{2}{1+1}=2$$

In the following examples $x\to x_0$ but $x_0\neq\pm\infty$, so we have to learn new methods for limit calculations.

First we mention an identity which is useful when dealing quadratic polinomials.

Theorem 4.4

Any quadratic polinomial $ax^2 + bx + c$, $a \neq 0$, which has roots x_1 and x_2 ($x_1 = x_2$ is allowed) can be decomposed as

$$ax^2 + bx + c = a(x - x_1)(x - x_2) \ . \quad \square$$

The following method and examples are for problems

$$\lim_{x \to x_0} \frac{p(x)}{q(x)} \quad ,$$

where $p(x)$ and $q(x)$ are polynomials and

$$p(x_0) = q(x_0) = 0,$$

i.e. this is a special problem of type "$\frac{0}{0}$".

Definition 4.9

Fractions of two polinomials are called **rational (integer) functions**. $\square$

Let us emphasize that limits *not* of type "$\frac{0}{0}$" or *not* rational (integer) functions need another methods, which we introduce in later examples.

SOLVED EXAMPLE 4.3
Function Limit

Calculate

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^2 - 1} \ .$$

SOLUTION

The limits of both the numerator and the denomirator are 0 ("type $\frac{0}{0}$"). Since we need the limit of a rational function, we can extract the term $(x - x_0)$ from both the numerator and the denominator (since they are polynomials), and then simplify the fraction with this term.

The numerator $x^2 - 2x + 1$ has roots $x_{1,2} = \frac{2 \pm \sqrt{2^2 - 4}}{2} = 1$, the roots of the denominator are $x_{1,2} = \pm 1$, and $x_0 = 1 = x_1$, so we have

$$\lim_{x\to 1} \frac{x^2 - 2x + 1}{x^2 - 1} = \lim_{x\to 1} \frac{(x-1)(x-1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x-1}{x+1} = \frac{0}{2} = 0 .$$

SOLVED EXAMPLE 4.4
Function Limit

Calculate

$$\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} .$$

SOLUTION

We eliminate the root by extending the fraction as usual

$$\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2} = \lim_{x\to 0} \frac{4 + x - 4}{x\left(\sqrt{4+x} + 2\right)} =$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{4+x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4} .$$

In the following examples and exercises we (can) use the "*Famous limits*" listed in Theorem 4.3.

SOLVED EXAMPLE 4.5
Function Limit

Calculate

$$\lim_{x\to 0} \frac{\sin(5x)}{x}.$$

SOLUTION

First, this problem of type "$\frac{0}{0}$". Second, it reminds us to the famous limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, so we try to transform our actual problem to the famous one. However, the variable x plays different role in these limits, so we advice to rewrite the famous one to $\lim_{y\to 0} \frac{\sin(y)}{y} = 1$. Now we can try to use the substitution $y = 5x$ and solve our problem as

$$\lim_{x\to 0} \frac{\sin(5x)}{x} = \lim_{x\to 0} \frac{\sin(5x)}{5x} \cdot 5 = \lim_{y\to 0} \frac{\sin(y)}{y} \cdot 5 = 1 \cdot 5 = 5 .$$

Do not forget to check, that $x \to 0 \iff y \to 0$.

SOLVED EXAMPLE 4.6
Function Limit

Calculate
$$\lim_{x\to 1}\frac{\sin(x-1)}{x-1}.$$

SOLUTION

Now use the substitution $y = x - 1$ (and check $x \to 1 \Longleftrightarrow y \to 0$) :
$$\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = \lim_{y\to 0}\frac{\sin(y)}{y} = 1\,.$$

SOLVED EXAMPLE 4.7
Function Limit

Calculate
$$\lim_{x\to 0}\frac{\sin(x^2+x)}{x^2+2x}.$$

SOLUTION

In order to substitute $y = x^2 + x$ we write
$$\lim_{x\to 0}\frac{\sin(x^2+x)}{x^2+2x} = \lim_{x\to 0}\frac{\sin(x^2+x)}{x^2+x}\cdot\frac{x^2+x}{x^2+2x} =$$
$$= \lim_{y\to 0}\frac{\sin(y)}{y}\cdot\lim_{x\to 0}\frac{x+1}{x+2} = 1\cdot\frac{1}{2} = \frac{1}{2}\,,$$
and do not forget to check $x \to 0 \Longleftrightarrow y \to 0$.

SOLVED EXAMPLE 4.8
Function Limit

Calculate
$$\lim_{x\to 0}\frac{\sin(5x)}{\sin(2x)}.$$

SOLUTION

As
$$\lim_{x\to 0}\frac{\sin(5x)}{\sin(2x)} = \lim_{x\to 0}\frac{\sin(5x)}{5x}\cdot\frac{2x}{\sin(2x)}\cdot\frac{5}{2}\,,$$
now we use substitutions $y_1 = 5x$ and $y_2 = 2x$.
$$\lim_{x\to 0}\frac{\sin(5x)}{5x}\cdot\frac{2x}{\sin(2x)}\cdot\frac{5}{2} = \lim_{y_1\to 0}\frac{\sin(y_1)}{y_1}\cdot\lim_{y_2\to 0}\frac{y_2}{\sin(y_2)}\cdot\frac{5}{2} = 1\cdot 1\cdot\frac{5}{2} = \frac{5}{2}\,.$$

SOLVED EXAMPLE 4.9
Function Limit

Calculate
$$\lim_{x \to 0} \frac{\ln(1+4x)}{\sin(2x)}.$$

SOLUTION
As
$$\lim_{x \to 0} \frac{\ln(1+4x)}{\sin(2x)} = \lim_{x \to 0} \frac{\ln(1+4x)}{4x} \cdot \frac{2}{\sin(2x)} \cdot \frac{4}{2}$$
we use
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$
and
$$\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1,$$
with substitutions $y_1 = 4x$ and $y_2 = 2x$. This follows
$$\begin{aligned} \lim_{x \to 0} \frac{\ln(1+4x)}{4x} \cdot \frac{2}{\sin(2x)} \cdot \frac{4}{2} &= \lim_{x \to 0} \frac{\ln(1+4x)}{4x} \cdot \lim_{x \to 0} \frac{2}{\sin(2x)} \cdot \frac{4}{2} = \\ &= \lim_{y_1 \to 0} \frac{\ln(1+y_1)}{y_1} \cdot \lim_{y_2 \to 0} \frac{2}{\sin(y_2)} \cdot \frac{4}{2} = \\ &= 1 \cdot 1 \cdot \frac{4}{2} = 2. \end{aligned}$$

SOLVED EXAMPLE 4.10
Function Limit

Calculate

$$\lim_{x \to 0} \frac{1 - \cos(x)}{\sin^2(x)}.$$

SOLUTION

As

$$\lim_{x \to 0} \frac{1 - \cos(x)}{\sin^2(x)} = \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \frac{x^2}{\sin^2(x)} = \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \left(\frac{x}{\sin(x)}\right)^2,$$

we use

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

and

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

So we get

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \left(\frac{x}{\sin(x)}\right)^2 = \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \lim_{x \to 0} \left(\frac{x}{\sin(x)}\right)^2 =$$

$$= \frac{1}{2} \cdot 1^2 = \frac{1}{2}.$$

As mentioned in the introduction, in many cases we have to deal with left- and right- hand limits separately.

SOLVED EXAMPLE 4.11
Function Limit

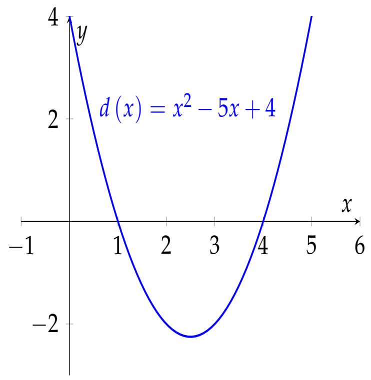
Calculate

$$\lim_{x \to 1} \frac{x^2 + 5x}{x^2 - 5x + 4}.$$

SOLUTION

Let us denote the *numerator* by $e(x) := x^2 + 5x$ and the *numerator* by $d(x) := x^2 - 5x + 4$ for short. Their limits are $\lim_{x \to 1} e(x) = 6$ and $\lim_{x \to 1} d(x) = 0$, so the problem is of type "$\frac{c}{0}$" ($c \neq 0$), more exactly the 0 in this fraction makes our head-ache.

For the following you are adviced to make a draft of $d(x)$, especially for the values and their *sign (+/-)* when x is close to 1.



Since $c = 6 = e(1) = \lim_{x\to 1} e(x)$, we can conclude that $e(x)$ is close to 6, especially $e(x)$ is always *positive*, for *any* x which is close to 1. Recall, that " $x \to 1$ " means "x is close to 1" but does *not* mean "$x = 1$", i.e. we are adviced to assume "$x \neq 1$". These good news says, that there is no 0 in the denomirator, in fact, "only" $d(x)$ is extremaly small. Imagine: $\dfrac{e(x)}{d(x)}$ means: dividing "almost 6" by "almost 0" = "very small" number, we receive an extremaly "huge" number. *However*, numbers close to 0 may be both negative and positive, their reciprocials also + or -, but *huge* + numbers tend to $+\infty$ and huge - numbers tend to $-\infty$. And this is a huge difference: $+\infty$ and $-\infty$ are very far from each other!

This is why we have to investigate the *sign* (+/-) of $e(x)$ and $d(x)$!

Concerning $d(x)$, it is a parabola meeting the x axe at point $x = 1$, having before positive and after negative values.

Now we can start to solve the problem.

$$\lim_{x\to 1} \frac{e(x)}{d(x)} = ?$$

The left-side limit says " $x \to 1-$ " which means $x < 1$, Figure clearly shows that $d(x) > 0$ for these x . So we can solve the left-side limit:

$$\lim_{x\to 1-} \frac{x^2+5x}{x^2-5x+4} = \text{"}\frac{6}{0+}\text{"} = +\infty \,.$$

Similarly, " $x \to 1+$ " means $x > 1$ and Figure shows $d(x) < 0$ for these x , so the right-side limit is

$$\lim_{x\to 1+} \frac{x^2+5x}{x^2-5x+4} = \text{"}\frac{6}{0-}\text{"} = -\infty \,.$$

SOLVED EXAMPLE 4.12
Function Limit

Calculate

$$\lim_{x\to 3} 2^{\frac{1}{3-x}} .$$

SOLUTION

First we calculate the limits of the exponent: $x \to 3-$ means $x < 3$ and $x \to 3+$ means $3 < x$, so

$$\lim_{x\to 3-} \frac{1}{3-x} = {}''\frac{1}{0+}{}'' = +\infty , \qquad \lim_{x\to 3+} \frac{1}{3-x} = {}''\frac{1}{0-}{}'' = -\infty .$$

Since we also know

$$\lim_{k\to -\infty} 2^k = 0 \quad \text{and} \quad \lim_{k\to +\infty} 2^k = +\infty ,$$

we conclude that

$$\lim_{x\to 3-} 2^{\frac{1}{3-x}} = {}''2^{+\infty}{}'' = +\infty \quad \text{and} \quad \lim_{x\to 3+} \frac{1}{3-x} = {}''2^{-\infty}{}'' = 0 .$$

4.5 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

4.5.1 Limits at Infinity

Exercises 4.1: Limits at Infinity

Calculate the following limits.

1. $$\lim_{x\to-\infty}\frac{x^3-4x^2+1}{x^3-x^2-1},$$

See Solution 8.4.1

2. $$\lim_{x\to\infty}\frac{x^3+3x}{x^2-x+1},$$

See Solution 8.4.2

3. $$\lim_{x\longrightarrow\infty}\frac{x^5+x^4}{3x^6+x^2+1},$$

See Solution 8.4.3

4. $$\lim_{x\to\infty}\frac{\sqrt{x^2+2x}+1}{x^2-1},$$

See Solution 8.4.4

5. $$\lim_{x\to\infty}\frac{2x+1}{\sqrt[5]{x^2}-1},$$

See Solution 8.4.5

6. $$\lim_{x\to\infty}\frac{\sqrt{x}+1}{\sqrt{x}-1},$$

See Solution 8.4.6

7. $$\lim_{x\to\infty}\left(\sqrt{x+4}-\sqrt{x+2}\right),$$

See Solution 8.4.7

8. $$\lim_{x\to\infty}x\left(\sqrt{x+1}-\sqrt{x}\right),$$

See Solution 8.4.8

9. $$\lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right),$$

See Solution 8.4.9

10. $$\lim_{x\to\infty}\left(\sqrt{x^2+4}-(x+2)\right),$$

See Solution 8.4.10

11.

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2-1}-x}.$$

See Solution 8.4.11

4.5.2 *Limits at Finite Point*

Exercises 4.2: Limits at Finite Point

Calculate the limits. Please check in all cases if the problem is of type "$\frac{0}{0}$" or not!

1.

$$\lim_{x\to -1} \frac{x^2+2x+1}{x^2-1},$$

See Solution 8.4.12

2.

$$\lim_{x\to 3} \frac{x^2-x-6}{x^2-5x+6},$$

See Solution 8.4.13

3.

$$\lim_{x\to -2} \frac{2-x-x^2}{x^2+3x+2},$$

See Solution 8.4.14

4.

$$\lim_{x\to 4} \frac{x^2-5x+4}{x^2-6x+1},$$

See Solution 8.4.15

5.

$$\lim_{x\to 1} \frac{x^3-x}{x^2+2x-3},$$

See Solution 8.4.16

6.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-\sqrt{1-x}}{x},$$

See Solution 8.4.17

7.

$$\lim_{x\to 0} \frac{\sqrt{x^2+x+1}-1}{x},$$

See Solution 8.4.18

8.

$$\lim_{x\to 1} \frac{x^2-x}{\sqrt{x}-1},$$

See Solution 8.4.19

9.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x^2-1},$$

See Solution 8.4.20

10.

$$\lim_{x\to 0} \frac{x}{\sqrt{9+x}-3},$$

See Solution 8.4.21

11.

$$\lim_{x\longrightarrow 1} \frac{1-x^3}{1-x},$$

See Solution 8.4.22

12.

$$\lim_{x\to 0} \frac{\sqrt[3]{x+1}-1}{x}.$$

See Solution 8.4.23

4.5.3 *Famous Limits I.*

Exercises 4.3: Famous Limits I.

Calculate the following limits.

1.

$$\lim_{x\to 0} \frac{\sin(2x)}{x},$$

See Solution 8.4.24

2.

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2},$$

See Solution 8.4.25

3.

$$\lim_{x\to 0} \frac{\sin^2(2x)}{x},$$

See Solution 8.4.26

4.

$$\lim_{x\to 0+} \frac{\sin(\sqrt{x})}{\sqrt{x}},$$

See Solution 8.4.27

5.

$$\lim_{x\to -2} \frac{\sin(x+2)}{x+2},$$

See Solution 8.4.28

6.

$$\lim_{x\to -2} \frac{\sin(x^2-4)}{x+2},$$

See Solution 8.4.29

7.

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(7x)},$$

See Solution 8.4.30

8.

$$\lim_{x \to -1} \frac{\sin(4x+4)}{x^2+x},$$

See Solution 8.4.31

9.

$$\lim_{x \to 0} \frac{\sin(x^2+x)}{x},$$

See Solution 8.4.32

10.

$$\lim_{x \to 0} \frac{\sin(2x)}{\tan(5x)}.$$

See Solution 8.4.33

4.5.4 *Famous Limits II.*

Exercises 4.4: Famous Limits II.

Calculate the following limits.

1.

$$\lim_{x \to 0} \frac{\sin(2x) - 2x}{x},$$

See Solution 8.4.34

2.

$$\lim_{x \to 0} \frac{\sin(3x)}{\ln(1+5x)},$$

See Solution 8.4.35

3.

$$\lim_{x \to 0} \frac{\ln(1+3x)}{2\sin(x)},$$

See Solution 8.4.36

4.

$$\lim_{x \to 0} \frac{e^{2x} - 1}{\sin(5x)},$$

See Solution 8.4.37

5.

$$\lim_{x \to 0} \frac{\tan(x)}{e^x - 1},$$

See Solution 8.4.38

6.

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x \cdot \sin(x)},$$

See Solution 8.4.39

7.

$$\lim_{x \to 0} \frac{e^{x^2} - \cos(x)}{x^2},$$

See Solution 8.4.40

8.

$$\lim_{x \to 0} \cot(x) \cdot \ln(1+2x),$$

See Solution 8.4.41

9.

$$\lim_{x\to 0}\frac{2\cos(3x)-2+9x^2}{2x},$$

See Solution 8.4.42

10.

$$\lim_{x\to 0}\frac{1-\cos(x)}{\ln(x+1)-(x+1)^2+1}.$$

See Solution 8.4.43

4.5.5 *Function Limits* $"\frac{c}{0}"$ $(c \neq 0)$

Exercises 4.5: Function Limit $"\frac{c}{0}"$ $(c \neq 0)$

Calculate the following limits.

1.

$$\lim_{x\to -1}\frac{x^2-2x+1}{x^2-1},$$

See Solution 8.4.44

2.

$$\lim_{x\to 4}\frac{x^2-6x+1}{x^2-5x+4},$$

See Solution 8.4.45

3.

$$\lim_{x\to -1}\frac{\sin(4x+2)}{x^2+x},$$

See Solution 8.4.46

4.

$$\lim_{x\to \frac{\pi}{2}}\frac{\ln(1-\sin(x))}{\frac{\pi}{2}-x},$$

See Solution 8.4.47

5.

$$\lim_{x\to a} e^{\frac{x}{1-x}}=\lim_{x\to a}\exp\left(\frac{x}{1-x}\right), \quad \text{where } a=0,1,\pm\infty,$$

See Solution 8.4.48

6.

$$\lim_{x\to a}\frac{x}{1-e^{1/x}}, \quad \text{where } a=0,1,\pm\infty.$$

See Solution 8.4.49

4.6 *Continuity of Functions*

Having all neccessary definitions, now we can discuss the problem of continuity of functions' graphs: what happens when moving our pencil (eyes) horizontally (x) and vertically (y) at the same time. Consider for example the (usual) graph of the function $f(x) = x^2$, i.e. $y = x^2$. Why can you draw this curve with *smoothly* moving your pencil, for example between 0 and 2, i.e. for $0 \leq x \leq 2$? Why are *not* any bump (distortion, jump) anywhere, for example at $x_0 = 1.5$?

Look: when you consider points with x -coordinates approaching x_0 , the y -coordinates of these points became more and more closer to $y_0 = 2.25$. This value is exactly 1.5^2, i.e. $f(1.5)$. *This* coincidence ensures the *smoothness* of the move of our pencil, i.e. of the (graph of the) function $f(x)$.

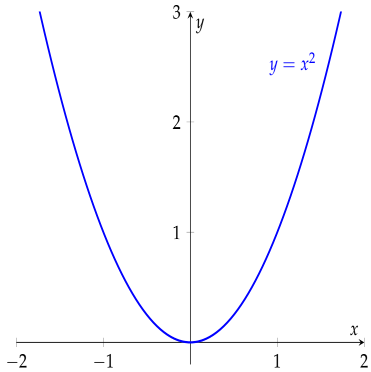


Figure 4.1: Graph of function x^2.

We hope, that the explanations mentioned so far make the following definitions clear.

Definition 4.10

The function $f : \mathbb{R} \to \mathbb{R}$ is (two sided) **continuous** at the point $a \in dom(f)$ if

$$\lim_{x \to a} f(x) = f(a) \; . \qquad \square$$

Remark 4.6.1 *Let us emphasize that a and $b := \lim_{x\to a} f(x)$ are finite real numbers (not $\pm\infty$), and moreover $a \in dom(f)$ is also required (which was unimportant for $\lim_{x\to a} f(x)$).*

Half-sided continuity can be defined similarly:

Definition 4.11

The function $f : \mathbb{R} \to \mathbb{R}$ is **continuous from left** / **from right** at the point $a \in dom(f)$ if

$$\lim_{x \to a-} f(x) = f(a) \quad / \quad \lim_{x \to a+} f(x) = f(a). \qquad \square$$

Clearly we have

Theorem 4.5

The function $f : \mathbb{R} \to \mathbb{R}$ is (two sided) **continuous** at the point $a \in dom\,(f)$ if and only if

$$\lim_{x \to a+} f\,(x) = \lim_{x \to a-} f\,(x) = f\,(a)\ . \qquad \square$$

However, neither a function nor its graph mean a single point, rather an interval of (infinite) points.

Definition 4.12

For any interval $I \subseteq dom\,(f)$ the function $f : \mathbb{R} \to \mathbb{R}$ is **continuous on the interval** I if f is continuous at *each* point $a \in I$. $\square$

We assure our Readers that in practice we only have to calculate limits in some cases only, since (by Theorems) all base and composite functions **are** continuous at most of the points of their *dom* .

In many practical applications *one* formula is not enough for a phenomenon, especially in extremal circumstances. For example the physical properties of engines must be described in low and in high temperatures, or interest dependencies in stock exchange are different under low and high economical restrictions, etc. Such phenomena can not be described by one formula, in mathematics we use the following notation.

Definition 4.13

For given functions $g\,(x)$ and $h\,(x)$ and $a \in \mathbb{R}$ the definition of $f\,(x)$ as

$$f\,(x) = \begin{cases} g\,(x)\,, & \text{if } x \le a \\ h\,(x)\,, & \text{if } x > a \end{cases}\ .$$

has the following meaning: for the phenomenon (function) $f\,(x)$ we have to use the formula $g\,(x)$ for $x \le a$ and $h\,(x)$ for $a < x$. Clearly the continuity of $f\,(x)$ is crucial at the joint point a , which is our task in the subsequent part of this Section.

Similar notations are in use, too, which can be understood easily, e.g.

$$f(x) = \begin{cases} g(x), & \text{if } x < a \\ b, & \text{if } x = a \\ h(x), & \text{if } x > a \end{cases} . \qquad \square$$

4.7 *Step-by-Step Examples*

Now we give step-by-step solutions to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 4.13
Function Continuity

Check the continuity of the function below at the point $a = 0$.

$$f(x) = \begin{cases} e^{2x} + 8, & \text{if } x \leq 0 \\ \frac{x}{\sqrt{x+25} - 5}, & \text{if } x > 0 \end{cases}.$$

SOLUTION

Function f is continuous at the point $a = 0$ just in the case

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0) \quad .$$

From the definition of function f, we have

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} e^{2x} + 8 = e^0 + 8 = f(0) = 9$$

(i.e. $f(x)$ is continuous from left), so the only task left is to calculate

$$\lim_{x \to 0+} f(x)$$

and check $= 9$ (from right). From the definition of function f, we have

$$\begin{aligned} \lim_{x \to 0+} f(x) &= \lim_{x \to 0+} \frac{x}{\sqrt{x+25} - 5} = \\ &= \lim_{x \to 0+} \frac{x}{\sqrt{x+25} - 5} \cdot \frac{\sqrt{x+25} + 5}{\sqrt{x+25} + 5} = \\ &= \lim_{x \to 0+} \frac{x\left(\sqrt{x+25} + 5\right)}{x + 25 - 25} = \\ &= \lim_{x \to 0+} \left(\sqrt{x+25} + 5\right) = 10 \neq 9 = f(0). \end{aligned}$$

Though the right hand limit does exist, but it is different from $f(0)$, so $f(x)$ is not continuous from right at the point $a = 0$. See Figure 4.7.

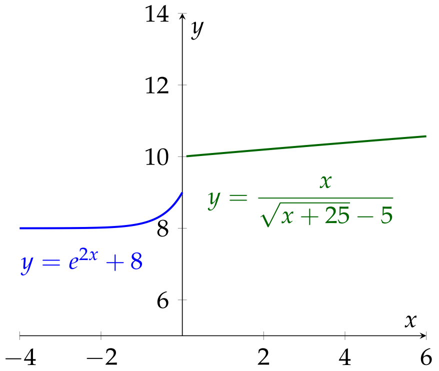


Figure 4.2: Graph of function f.

The above computations show the break of function f at point $a = 0$, which break can not be eliminated.

SOLVED EXAMPLE 4.14
Function Continuity

Check the continuity of function below at point $a = 0$.

$$f(x) = \begin{cases} \dfrac{\sqrt{x+2} - \sqrt{2-x}}{x}, & \text{if } x \in [-2, 2] \setminus \{0\} \\ \dfrac{\sqrt{2}}{2}, & \text{if } x = 0 \end{cases} .$$

SOLUTION

$$\begin{aligned} \lim_{x \to 0} f(x) &= \lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2-x}}{x} = \\ &= \lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2-x}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2-x}}{\sqrt{x+2} + \sqrt{2-x}} = \\ &= \lim_{x \to 0} \frac{x+2-(2-x)}{x\left(\sqrt{x+2} + \sqrt{2-x}\right)} = \lim_{x \to 0} \frac{2x}{x\left(\sqrt{x+2} + \sqrt{2-x}\right)} = \\ &= \lim_{x \to 0} \frac{2}{\sqrt{x+2} + \sqrt{2-x}} = \frac{2}{2\sqrt{2}} . \end{aligned}$$

Since

$$\lim_{x \to 0} f(x) = \frac{\sqrt{2}}{2} = f(0) ,$$

we must say that $f(x)$ is continuous at the point $a = 0$.

4.8 *Exercises*

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 4.6

Decide the continuity of the functions below at the point where the two formulae ($g(x)$ and $h(x)$) are joined.

1.
$$f(x) = \begin{cases} \dfrac{\sin(8x)}{\sin(4x)}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases},$$

See Solution 8.4.50

2.
$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x^2 - 2x - 3}, & \text{if } x \neq 3 \\ \dfrac{5}{4}, & \text{if } x = 3 \end{cases},$$

See Solution 8.4.51

3.
$$f(x) = \begin{cases} 2^{x-1}, & \text{if } x \leq 0 \\ \dfrac{\sqrt{x+1} - 1}{x}, & \text{if } x > 0 \end{cases},$$

See Solution 8.4.52

4.
$$f(x) = \begin{cases} \dfrac{x^2 - x}{2 - 2x}, & \text{if } x < 1 \\ \log_{\frac{1}{2}}(2^x + 1), & \text{if } x \geq 1 \end{cases}.$$

See Solution 8.4.53

5 *Derivatives of Real Functions*

When investigating functions, e.g. increasing or decreasing the **tangent line** (meeting smoothly the function curve or graph) is a good approximation of the function at the point $P_0(x_0, y_0)$ where $y_0 = f(x_0)$ and can help us. However, to calculate the **slope** $m = \tan(\alpha)$ of the tangent line $y = m \cdot x + b$ is not so obvious, so we have to calculate **secant lines** of the function: a straight line meeting the function curve at (least) two points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ where $y_1 = f(x_1)$.

We omit now the exact *geometrical* definition of the *tangent line*.

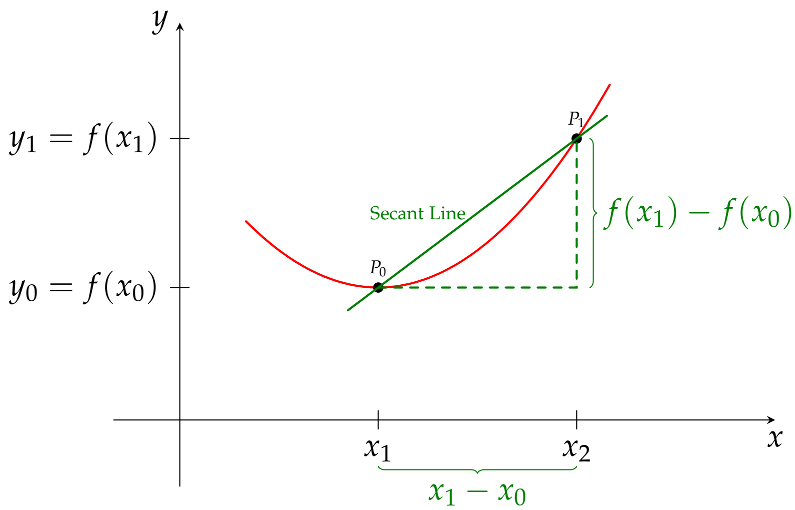


Figure 5.1: A secant line of a curve.

5.1 *Basic Definitions and Theorems*

It is well known from secondary schools, that the secant line crossing the two points P_0 and P_1 has the equality

$$\begin{aligned} y &= y_0 + \frac{y_1 - y_0}{x_1 - x_0} \cdot (x - x_0) = \\ &= \frac{y_1 - y_0}{x_1 - x_0} \cdot x + \left(y_0 - x_0 \cdot \frac{y_0 - y_1}{x_0 - x_1} \right). \end{aligned}$$

One can except that the *slopes* of the *secant lines* approximate the slope of the *tangent line*, i.e.

$$m = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0}, \tag{5.1.1}$$

as can be seen *here*.

The next definition is a slight variation of the above definition of secant line.

In fact (5.1.2) is a fraction, both the enumerator and the denominator are *differences*.

Definition 5.1: Derivative of a Function at a Point

Let f be a continuous function and let $x_0 \in dom\,(f)$ an inner point.

(i) For each $x \in dom\,(f)$, $x \neq x_0$ the fraction

$$\frac{f(x) - f(x_0)}{x - x_0} \tag{5.1.2}$$

is called **difference fraction**.

(ii) If (5.1.2) has a finite limit when $x \to x_0$, then this limit is denoted by $f'(x_0)$ (f **prime**):

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{5.1.3}$$

and is called the **differential fraction** or the **derivative** of the function f at the point x_0. □

Remark 5.1.1 *Clearly the limit (5.1.3) is convergent only for continuous functions, so f ultimately must be continuous at x_0 .*

The difference fraction (5.1.2) is the slope of the secant lines at the fixed point $P_0(x_0, y_0)$ and the "moving" point $P(x, y)$, while the differential fraction (5.1.3) is of the tangent line at the fixed point $P_0(x_0, y_0)$, of course $y_0 = f(x_0)$ and $y = f(x)$.

Some other experts and many (modern) computer programs still use the old fashioned notations

$$\frac{d}{dx} f(x) \quad or \quad \frac{df}{dx}(x)$$

for the derivative $f'(x_0)$, so we are adviced to know these notations.

Though (5.1.3) is defined only for a fixed (inner) point $x_0 \in dom\,(f)$, and $f'(x_0)$ is a single real number, we can extend for other points as in the next definition.

Definition 5.2: Derivative Function

Let f be a continuous function and let $H \subseteq dom\,(f)$ be the set *(possibly empty)* of inner points x_0 for which (5.1.3) is convergent. Then we can define the **derivative function of** f as: $f' : H \to \mathbb{R}$ and for each $x_0 \in H$ we set $f'(x_0) :=$ the limit in (5.1.3). □

Please have in mind that f' is *another* function, related to f.

The word "*to derive*" means *to deduce, give reasoning, to trace the source, ...*", and really f' is deduced from f. Moreover, we will deduce many properties of f *from* f'!

In German: *Ableitung, ableiten* with the same literary meaning.

Higher order derivatives $f''(x)$, $f'''(x)$, ... , $f^{(n)}(x)$, ... (n is any natural number) of a function $f(x)$ can "easily" be defined: $f''(x) := (f'(x))'$, $f'''(x) := (f''(x))' = \left((f'(x))'\right)'$, ... Let us emphasize, that the number n in the exponent $f^{(n)}(x)$ must be in brackets to denote the n -th derivative, otherwise $f^n(x) := f(x) \cdot ... \cdot f(x) = (f(x))^n$ is the usual n -th **power of** f (w.r.t.[1]) the multiplication "·").

[1] **w.r.t.** = with respect to

5.2 *Differentiation Rules*

Since all complicated functions (expressions) are build up from basic functions, basic operations (+,-,·,/) and compositions, using the following Theorems we can derive *all* functions!

Assuming f and g have the derivative functions f' and g' and $c \in \mathbb{R}$ is any fixed number, then the following equalities hold.

Theorem 5.1: Sum and Difference Rule

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)\,. \quad \square$$

Theorem 5.2: Constant Multiple Rule

$$(c \cdot f(x))' = c \cdot f'(x)\,. \quad \square$$

Theorem 5.3: Product Rule

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)\,. \quad \square$$

Theorem 5.4: Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}\,. \quad \square$$

Theorem 5.5: Chain Rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$[f(g(h(x)))]' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) . \quad \square$$

Remark 5.2.1 *Theorem 5.5 is often called "**onion-** , **cabbage-** or **chain rule**" since "first we derive the outer part (the inner part remains), then (·) we derive the next part (inner part remains), " , or, another explanation: the multipliers in the theorem with the symbol · look like a chain.*

When calculating derivative functions, using the above theorems and omitting the limit (5.1.3), is called **formal derivative calculus**. Deriving is one of the most fundamental calculation method in Mathematical Analysis (calculus), it can be considered as the 5^{th} (or 6^{th}) basic operation.

You must be an expert in deriving any function!

In the forthcoming Sections we learn many applications of derivatives of functions.

5.3 *Table of Derivatives*

How to calculate the derivative function f' ? The limit (5.1.3) is usually hard. Fortunately we have a lot of theorems making *much easier* calculating derivative functions.

Please say many thanks to the mathematicians!

Theorem 5.6

All the basic functions can be derived ((5.1.3) exist) at almost all inner points of *dom* (f) (with few exceptions), these derivative functions can be found in the next Table. □

$f(x)$	$f'(x)$
c	0
x^α	$\alpha x^{\alpha-1}$
e^x	e^x
a^x	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

$f(x)$	$f'(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\cot(x)\,dx$	$-\frac{1}{\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\text{arccot}(x)$	$-\frac{1}{1+x^2}$

$$c, \alpha \in \mathbb{R}, \qquad a \in (0,\infty) \setminus \{1\}.$$

5.4 *Step-by-Step Examples*

Now we give a step-by-step solution to two problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 5.1
Derivation

Derivate the following function.

$$f(x) = 3x^4 + \frac{2}{x^5} - \sqrt[3]{x}$$

SOLUTION

First, we use Theorem 5.1.

$$f'(x) = \left(3x^4 + \frac{2}{x^5} - \sqrt[3]{x}\right)' = \left(3x^4\right)' + \left(\frac{2}{x^5}\right)' - (\sqrt[3]{x})'.$$

Using Theorem 5.2 and some basic mathematics, all the derivations can be evaluated using 5.3 Table of Derivatives, that is

$$\left(3x^4\right)' = 3 \cdot 4x^3,$$

$$\left(\frac{2}{x^5}\right)' = \left(2x^{-5}\right)' = 2 \cdot (-5)x^{-6} = \frac{-10}{x^6},$$

and

$$(\sqrt[3]{x})' = \left(x^{\left(\frac{1}{3}\right)}\right)' = \frac{1}{3}x^{\left(\frac{-2}{3}\right)} = \frac{1}{3}\frac{1}{\sqrt[3]{x^2}}.$$

This follows

$$\left(3x^4 + \frac{2}{x^5} - \sqrt[3]{x}\right)' = 3 \cdot 4x^3 + \frac{-10}{x^6} - \frac{1}{3}\frac{1}{\sqrt[3]{x^2}}.$$

SOLVED EXAMPLE 5.2
Derivation

Derivate the following function.

$$f(x) = \sin(x)e^x + \frac{\ln(x)}{x^4 - 5x}$$

SOLUTION

First, we use Theorem 5.1, that is

$$\left(\sin(x)e^x + \frac{\ln(x)}{x^4 - 5x}\right)' = (\sin(x)e^x)' + \left(\frac{\ln(x)}{x^4 - 5x}\right)'.$$

Using Theorem 5.3 and 5.4, all the derivations can be evaluated using 5.3 Table of Derivatives. As

$$(\sin(x)e^x)' = (\sin(x))' e^x + \sin(x)\,(e^x)' = \cos(x)e^x + \sin(x)e^x,$$

and

$$\left(\frac{\ln(x)}{x^4 - 5x}\right)' = \frac{(\ln(x))'\,(x^4 - 5x) - \ln(x)\,(x^4 - 5x)'}{(x^4 - 5x)^2} = \frac{\frac{1}{x}(x^4 - 5x) - \ln(x)\,(4x^3 - 5)'}{(x^4 - 5x)^2},$$

the result is

$$\left(\sin(x)e^x + \frac{\ln(x)}{x^4 - 5x}\right)' = \cos(x)e^x + \sin(x)e^x + \frac{\frac{1}{x}(x^4 - 5x) - \ln(x)\,(4x^3 - 5)'}{(x^4 - 5x)^2}.$$

SOLVED EXAMPLE 5.3
Derivation

Derivate the following function.

$$f(x) = \arctan\left(\frac{1}{x}\right)$$

SOLUTION

Now, we use Theorem 5.5), that is

$$f'(x) = \left(\arctan\left(\frac{1}{x}\right)\right)' = \arctan'\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)'.$$

As

$$(\arctan(x))' = \frac{1}{1 + x^2},$$

and

$$\left(\arctan\left(\frac{1}{x}\right)\right)' = \frac{1}{1+\left(\frac{1}{x}\right)^2},$$

this with

$$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -x^{-2} = \frac{-1}{x^2}.$$

follows that

$$f'(x) = \left(\arctan\left(\frac{1}{x}\right)\right)' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\frac{-1}{x^2}$$

is the final result.

5.5 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 5.1

Derivate the following functions.

1.
$$F(x) = \sqrt{x + \sqrt{x}},$$
See Solution 8.5.1

2.
$$F(x) = \frac{\ln(2x - 4x^3)}{\sqrt[3]{4x + 1}},$$
See Solution 8.5.2

3.
$$F(x) = \frac{\ln(x^2 + 2x)}{\sin(e^x)},$$
See Solution 8.5.3

4.
$$F(x) = \sin^2(x) \tan(x^3 - 5x),$$
See Solution 8.5.4

5.
$$F(x) = \frac{e^{2-3x^4}}{\sqrt[4]{\cot(x)}},$$
See Solution 8.5.5

6.
$$F(x) = \frac{\tan(e^x)}{\ln(x^2)},$$
See Solution 8.5.6

7.
$$F(x) = \cos(x^3 - 2x^2) \ln(\sin(x)),$$
See Solution 8.5.7

8.
$$F(x) = \frac{3^{5x+2}}{\ln(x^2 + x)},$$
See Solution 8.5.8

9.
$$F(x) = e^{3\ln(x)} \cot(x^3 - 5x),$$
See Solution 8.5.9

10.
$$F(x) = (x^2 + 1)^{1974} \cos(x^5 - 3x^2).$$
See Solution 8.5.10

5.6 *Application I. - The Tangent Line*

Now we give the equation for the tangent line, (meeting smoothly the function curve). So, we are given a continuous function $f(x)$ and a point $P_0(x_0, y_0)$ on the function curve where $x_0 \in dom(f)$ an inner point and $y_0 = f(x_0)$, and we look for the equation $y = mx + b$ for the tangent line.

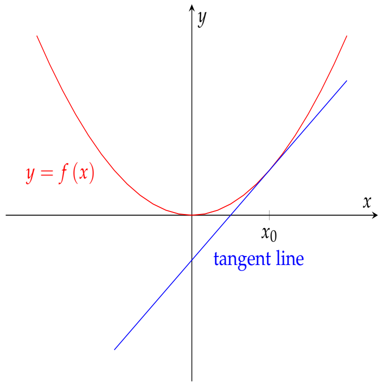


Figure 5.2: Tangent line to function f at the point x_0.

Definition 5.3: The Tangent Line

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at the inner point x_0, than

$$y = f'(x_0)(x - x_0) + f(x_0)$$

line called **tangent line** to the function f at the point x_0. □

5.7 *Step-by-Step Example*

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 5.4
The tangent line

Write the equation of the tangent line of $x_0 = 1$ to the graph of the following function.

$$f(x) = e^{-3x}.$$

SOLUTION First we need to compute a following derivative function

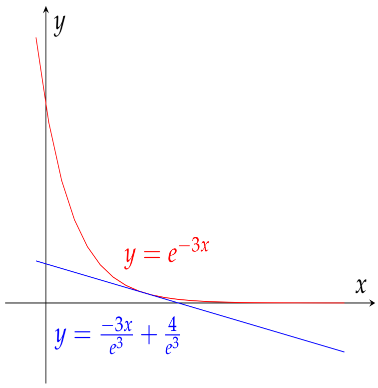
$$f'(x) = \left(e^{-3x}\right)' = -3e^{-3x}.$$

Then we need to compute a following values:

$$\begin{aligned} f(x_0) &= f(1) = e^{-3} = \frac{1}{e^3} \\ f'(x_0) &= f'(1) = -3e^{-3} = \frac{-3}{e^3} \end{aligned}$$

Finally write the equation of the tangent line.

$$\begin{aligned} y &= \frac{-3}{e^3}(x-1)+\frac{1}{e^3} \\ y &= \frac{-3x}{e^3}+\frac{4}{e^3} \end{aligned}$$



5.8 *Exercises*

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 5.2

Write the equation of the tangent line to the graph of the following function at that x_0 point.

1.
$$f(x) = 4x - \frac{1}{x^2}, \qquad x_0 = 4,$$
See Solution 8.5.11

2.
$$f(x) = x\ln(x), \qquad x_0 = e,$$
See Solution 8.5.12

3.
$$f(x) = \sqrt{x+1}, \qquad x_0 = 3,$$
See Solution 8.5.13

4.
$$f(x) = \frac{x+2}{x-3}, \qquad x_0 = 2,$$
See Solution 8.5.14

5.
$$f(x) = 2x - \frac{1}{x+1}, \qquad x_0 = 0.$$
See Solution 8.5.15

5.9 *Application II. - Extremal Values of Functions and Monotonicity*

First, we have to define *precisely* what we are looking for.

Definition 5.4

A function f is said to be

(i) **monotone increasing / decreasing on the** nonempty **interval** (a, b) iff for every $x_1, x_2 \in (a, b)$, $x_1 < x_2$ we have

$$f(x_1) \leq f(x_2) \quad / \quad f(x_1) \geq f(x_2)$$

(ii) **strictly monotone increasing / decreasing on the** nonempty **interval** (a, b) iff for every $x_1, x_2 \in (a, b)$, $x_1 < x_2$ we have

$$f(x_1) < f(x_2) \quad / \quad f(x_1) > f(x_2) .$$ □

Remark 5.9.1 *Remark, that ("pure") monotone increasing/decreasing functions can be constant, i.e. horizontal graph on any subinterval of* (a, b)*, see for example*

$$f(x) = \frac{1}{2}(|x+1| - |x-1|)$$

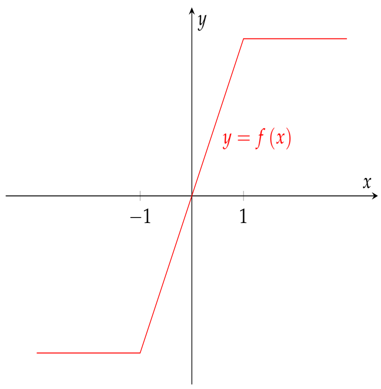


Figure 5.3: The graph of function f.

So, better titles would be **monotone not decreasing / not increasing**. Clearly *strictly* monotone functions must *not* have horizontal parts.

Remark 5.9.2 *"iff" is a short form of "if and only if"*

Now, how to decide from a complicated function on which intervals it is monotone increasing or decreasing? Even using modern HD computer graphics, the answer is still difficult, see for example

$$g(x) = x^3 + 7.5x^2 + 18x, \quad x \in \mathbb{R}$$

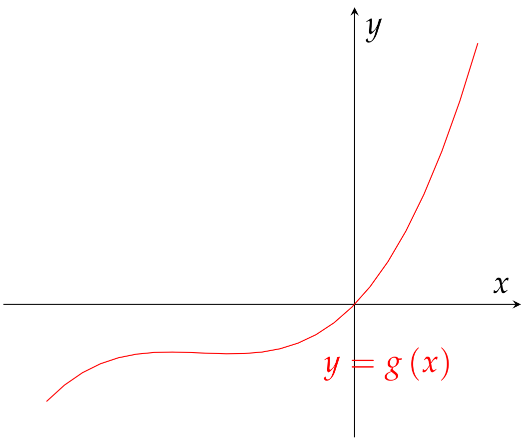


Figure 5.4: Graph of function g.

Despite this function has a relatively simple furmula, it is hard (or impossible) to find *exactly* where it changes from increasing to decreasing or vica versa.

Theorem 5.7

(i) If the continuous function f is monotone increasing / decreasing on the nonempty interval (a, b) *then* for its derivative $f'(x)$ we have

$0 \leq f'(x)$ / $f'(x) \leq 0$ on the whole interval (a, b).

(ii) If for the derivative $f'(x)$ we have on the whole interval (a, b)

$0 < f'(x)$ / $f'(x) < 0$, *then* (the original) function f is monotone increasing / decreasing on the nonempty interval (a, b). □

Sorry, we are not allowed to say "if and only". For godness sake we use only *(ii)* in the practice.

Though this theorem requires a precise proof, it fits to our intuition well. First, monotone increasing / decreasing functions have tangent lines with positive / negative slopes. Second, looking at the definition (5.1.3) of the derivative we must observe that the enumerator *and* the denominator must have the same / opposite signs for increasing / decreasing functions.

In practice the extremal (minimal and maximal) values of several variable functions are very important.

Definition 5.5: Local Maximal Value

Function $f : \mathbb{R} \to \mathbb{R}$ has a **local maximal value** $y_0 = f(x_0)$ at the fixed **place** (point) $x_0 \in dom(f)$ *iff* there is a neighborhood around x_0 (for some $\varepsilon > 0$) such that $f(x_0)$ is **greater** than $f(x)$ *for any* x from this neighborhood, that is

$$f(x_0) \geq f(x) \iff f(x_0) \text{ is } \textbf{maximal}$$

for any $x : x_0 - \varepsilon < x < x_0 + \varepsilon$.

Definition 5.6: Local Minimal Value

Function $f : \mathbb{R} \to \mathbb{R}$ has a **local minimal value** $y_0 = f(x_0)$ at the fixed **place** (point) $x_0 \in dom(f)$ *iff* there is a neighborhood around x_0 (for some $\varepsilon > 0$) such that $f(x_0)$ is **smaller** than $f(x)$ *for any* x from this neighborhood, that is

$$f(x_0) \leq f(x) \iff f(x_0) \text{ is } \textbf{minimal}$$

for any $x : x_0 - \varepsilon < x < x_0 + \varepsilon$.

Definition 5.7: Global Maximal Value

Function $f : \mathbb{R} \to \mathbb{R}$ has a **global maximal value** $y_0 = f(x_0)$ at the fixed **place** (point) $x_0 \in dom(f)$ *iff* the neighborhood $(x_0 - \varepsilon < x < x_0 + \varepsilon)$ above can be replaced to the whole $dom(f)$, that is

$$f(x_0) \geq f(x) \iff f(x_0) \text{ is } \textbf{global maximal}$$

for any $x \in dom(f)$.

Definition 5.8: Global Minimal Value

Function $f : \mathbb{R} \to \mathbb{R}$ has a **global minimal value** $y_0 = f(x_0)$ at the fixed **place** (point) $x_0 \in dom(f)$ *iff* the neighborhood $(x_0 - \varepsilon < x < x_0 + \varepsilon)$ above can be replaced to the whole $dom(f)$, that is

$$f(x_0) \leq f(x) \iff f(x_0) \text{ is } \textbf{global minimal}$$

for any $x \in dom(f)$.

Any maximal / minimal *value* $f(x_0)$ is called shortly a **maximum** / **minimum** of f, the respective plurals are **maxima** and **minima**. The collective word for maximum and minimum is **extremal value** or **extremum**, the plural form is **extrema.** In each case x_0 is the **place** or **spot** of the extremum.

The following theorems and methods tell us how to find the spots and the values of the *local* extrema. After, *global* extrema can be found by considering the largest local maximum / smallest local minimum and checking the functions's values at the endpoints of $dom(f)$.

Theorem 5.8

If the function $f : \mathbb{R} \to \mathbb{R}$ has a *local* extremum at the point $x_0 \in dom(f)$ *then* the derivative of f at x_0 must be zero, that is

$$f'(x_0) = 0\,. \quad \square$$

Definition 5.9: Critical Point

The points $x_0 \in dom(f)$ satisfying the above equality

$$f'(x_0) = 0$$

are called **stationary** or **critical points** or **places**. □

Remark 5.9.3 *i) The equality in Theorem 5.8 should be seem obvious: the tangent lines at maxima and minima are horizontal, so their slope are 0.*

ii) Let us emphasize, that the requirement in Theorem 5.8 is neccessary only but not sufficient, i.e. from

$$f'(x_0) = 0$$

we should not deduce that the function f would have any extremum. Later theorems and explanations help us to resolve this situation.

For deciding if stationary points are local extrema, there are methods using second derivatives $f''(x)$, but they are not 100% safe. So we suggest the following method.

Theorem 5.9

If $x_0 \in dom(f)$ is a *stationary point* of f *and* the derivative $f'(x)$ **changes its sign** at x_0 , *then* the (original) function **has** a local extremum at x_0 .

In more detail: *if* there is an ε such that

i) $f'(x)$ is **positive** for $x_0 - \varepsilon < x < x_0$ and **negative** for $x_0 < x < x_0 + \varepsilon$ *then* f has a local **maximum** x_0 ,

ii) $f'(x)$ is **negative** for $x_0 - \varepsilon < x < x_0$ and **positive** for $x_0 < x < x_0 + \varepsilon$ *then* f has a local **minimum** x_0 . □

5.10 *Step-by-Step Example*

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 5.5
Monotonity

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the function.

$$f(x) = x^3 + 2x^2 - x - 2, \quad x \in \mathbb{R}$$

SOLUTION First, we have to calculate the following limits.

$$\lim_{x \to -\infty} x^3 + 2x^2 - x - 2 = -\infty$$
$$\lim_{x \to \infty} x^3 + 2x^2 - x - 2 = \infty$$

This follows that there is neither a global maximal nor a global minimal value, just local. Now, we determine the derivate function.

$$f'(x) = \left(x^3 + 2x^2 - x - 2\right)' = 3x^2 + 4x - 1$$

Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0.$$

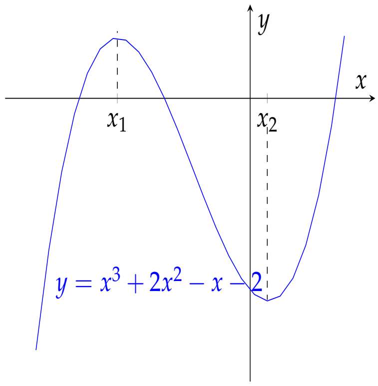
$$f'(x) = 3x^2 + 4x - 1 = 0$$
$$\Downarrow$$
$$x_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \cdot 3 \cdot (-1)}}{6} = \frac{-4 \pm \sqrt{28}}{6}$$
$$x_1 = -\frac{2}{3} + \frac{\sqrt{7}}{3} \approx -1,549, \; x_2 = -\frac{2}{3} - \frac{\sqrt{7}}{3} \approx 0,215$$

This means that function f can have local extremal value at x_1 and x_2.

We use Theorem 5.7 to determine the intervals, where function f is increasing or decreasing. For this, we determine the intervals where the derivative function f' is negative, the intervals where it is positive and examine whether function f' changes sign at x_1 and x_2.

We use the following table to summarise our results.

	$x < x_1$	$x = x_1$	$x_1 < x < x_2$	$x = x_2$	$x_2 < x$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	max	$\searrow$	min	$\nearrow$



SOLVED EXAMPLE 5.6
Monotonity

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the function

$$f(x) = x + \frac{1}{x-1}, \quad x \in \mathbb{R}\backslash\{1\}.$$

SOLUTION

This function has one breaking point ($x = 1$), so we have to calculate the following limits.

$$\lim_{x \to -\infty} x + \frac{1}{x-1} = -\infty$$
$$\lim_{x \to 1^-} x + \frac{1}{x-1} = -\infty$$
$$\lim_{x \to 1^+} x + \frac{1}{x-1} = \infty$$
$$\lim_{x \to \infty} x + \frac{1}{x-1} = \infty$$

This follows that there is neither a global maximal nor a global minimal value, just local. Now, we determine the derivate function

$$f'(x) = \left(x + \frac{1}{x-1}\right)' = 1 - \frac{1}{(x-1)^2}.$$

Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0.$$

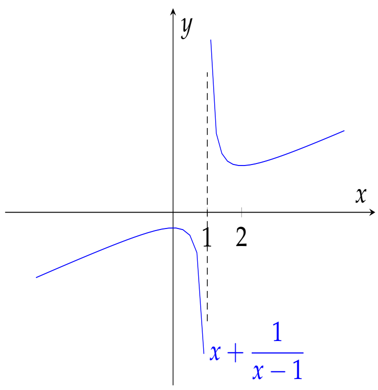
$$f'(x) = 1 - \frac{1}{(x-1)^2} = 0$$
$$\frac{(x-1)^2 - 1}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = 0$$
$$\Downarrow$$
$$x_1 = 0$$
$$x_2 = 2$$

This means that function f can have local extremal value at x_1 and x_2.

We use Theorem 5.7 to determine the intervals, where function f is increasing or decreasing. For this, we determine the intervals where the derivative function f' is negative, the intervals where it is positive and examine whether function f' changes sign at x_1 and x_2.

We use the following table to summarise our results.

	$x < 0$	$x = 0$	$0 < x < 1$	$1 < x < 2$	$x = 2$	$2 < x$
$f'(x)$	$+$	0	$-$	$-$	0	$+$
$f(x)$	$\nearrow$	max	$\searrow$	$\searrow$	min	$\nearrow$



SOLVED EXAMPLE 5.7
Monotonity

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the function.

$$f : \left[\frac{\pi}{2}; \frac{5\pi}{2}\right] \to \mathbb{R}, \quad f(x) = e^{\cos(x)}$$

SOLUTION
As function *cos* is bounded, we get

$$e^{-1} \leq e^{\cos(x)} \leq e^{1}.$$

In this case we can get eighter global or local extremal points. Firstly, we need to derivate the function.

$$f'(x) = \left(e^{\cos(x)}\right)' = -\sin(x)e^{\cos(x)}$$

Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0.$$

As for $x \in \left[\frac{\pi}{2}; \frac{5\pi}{2}\right]$, $\quad e^{\cos(x)} > 0$, we have

$$f'(x) = -\sin(x)e^{\cos(x)} = 0,$$
$$\Downarrow$$
$$\sin(x) = 0,$$
$$x_1 = \pi,$$
$$x_2 = 2\pi.$$

This means that function f can have local extremal value at x_1 and x_2.

We use Theorem 5.7 to determine the intervals, where function f is increasing or decreasing. For this, we determine the intervals where the derivative function f' is negative, the intervals where it is positive and examine whether function f' changes sign at x_1 and x_2.

We summarise our results in the following table.

	$x = \frac{\pi}{2}$	$\frac{\pi}{2} < x < \pi$	$x = \pi$	$\pi < x < 2\pi$	$x = 2\pi$	$2\pi < x < \frac{5\pi}{2}$	$x = \frac{5\pi}{2}$
$f'(x)$	$-$	$-$	0	$+$	0	$-$	$-$
$f(x)$	max	$\searrow$	min	$\nearrow$	max	$\searrow$	min

This follows, we have two maximal and two minimal values, so we have to decide, which one is global, a which one is local.

$$f(\frac{\pi}{2}) = e^{\cos\left(\frac{\pi}{2}\right)} = e^0 = 1, \quad \text{local maximal value}$$
$$f(\pi) = e^{\cos(\pi)} = e^{-1} = \frac{1}{e}, \quad \text{global minimal value}$$
$$f(2\pi) = e^{\cos(2\pi)} = e^1 = e, \quad \text{global maximal value}$$
$$f(\frac{5\pi}{2}) = e^{\cos\left(\frac{5\pi}{2}\right)} = e^0 = 1, \quad \text{local minimal value.}$$

5.11 *Exercises*

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 5.3

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the function.

1.
$$f(x) = \frac{x+1}{x-3}, \quad x \in \mathbb{R} \setminus \{3\},$$

See Solution 8.5.16

2.
$$f(x) = x^3 - 6x^2, \quad x \in [-1;2],$$

See Solution 8.5.17

3.
$$f(x) = x^5 + 5x^4, \quad x \in \mathbb{R},$$

See Solution 8.5.18

4.
$$f(x) = 3x^3 + 9x^2, \quad x \in [-1;1],$$

See Solution 8.5.19

5.
$$f(x) = 2x^3 - 3x^2 - 12x, \quad x \in [0;4],$$

See Solution 8.5.20

6.
$$f(x) = 2x^3 - 3x^2 - 120x, \quad x \in [-2;6],$$

See Solution 8.5.21

7.
$$f(x) = \frac{x^3}{3} + x^2 - 15x, \quad x \in \mathbb{R}.$$

See Solution 8.5.22

5.12 *Application III. - Convexity of Functions and Points of Inflection*

In the previous section we studied the monotonity of a function. However, there is another issue to consider when we study the shape of the graph of a function. How does it curves? Figure 5.5 shows curves of different shapes. Not only the shapes, but the *rates* of the monotonicity of these functions are changing differently. This phenomenon is investigated in this section.

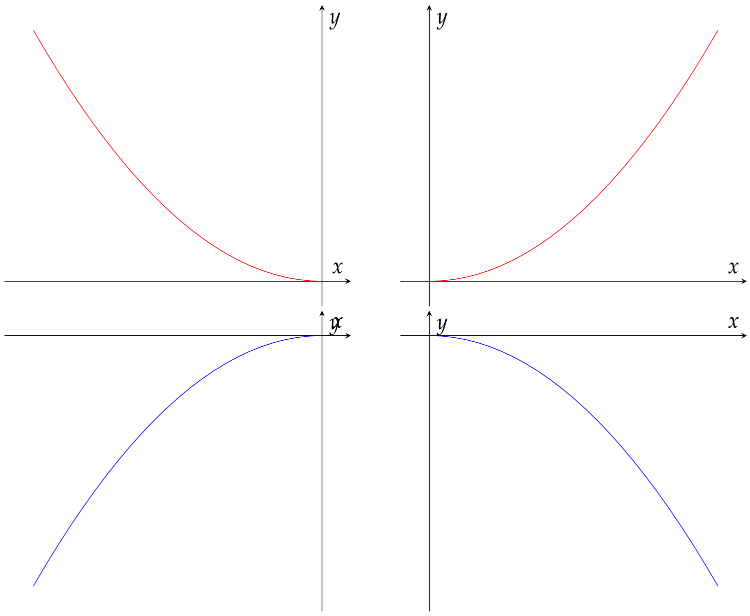


Figure 5.5: Curves of different shapes.

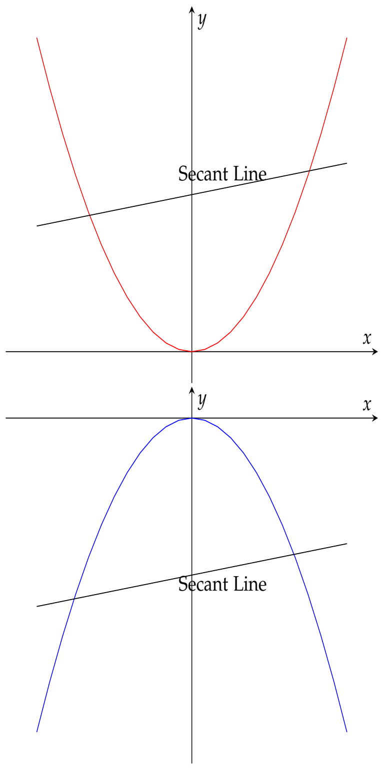


Figure 5.6: Graph of a **convex** / **concave** function.

Definition 5.10: Convex / Concave Function

A function f is called **convex** / **concave** over the interval $(a, b) \subseteq dom\,(f)$ **if** the graph of the function is under / above *all* the secant lines. □

The explanation of the words *convex* and *concave* came from geometry: if you draw any straight line *above* the graph of the function, the geometrical plane figure you get is (geometrically) *convex* or *concave*, looking from below.

How to decide exactly from function it is *convex* or *concave*?

Theorem 5.10

(i) *If* the continuous function f is *convex* / *concave* on the nonempty interval (a, b) *then* for its *second* derivative $f''(x)$ we have

$$f''(x) \quad / \quad \leq f''(x) \leq 0$$

on the whole interval (a, b).

(ii) *If* for the second derivative $f''(x)$ we have on the whole interval (a, b)

$$0 < f''(x) \quad / \quad f''(x) < 0,$$

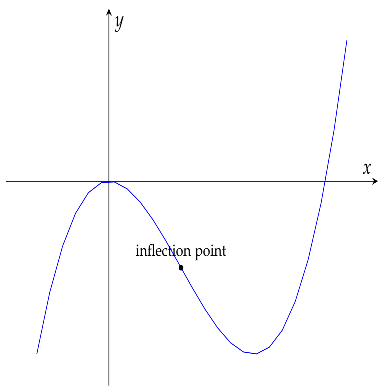
then (the original) function f is *convex* / *concave* on the interval (a, b). □

Sorry, we are not allowed to say "if and only", we use only (ii) in the practice.

Though this theorem requires a precise proof, it fits to our intuition well. Once, draw several tangent lines of the function graph and investigate the *changes* of their *slopes*. At a *convex* function these slopes are *increasing* (when moving x_0 from left to right), at a *concave* function these slopes are *decreasing*.

Second, looking to the *changes* of the *slopes*, one can imagine the unknown graph of the function.

How to find the convex and concave inervals of a function? Determine first the second derivative f'' and its roots, i.e. solve $f''(x) = 0$, and find the *sign* of $f''(x)$ between these roots (and be aware of some other complications).



"Inflection" comes from the word "flexible": the graph of f is really flexible at inflection point(s).

Definition 5.11: Inflection Point

Any point $x_0 \in dom(f)$ is called **inflection point** (or **point of inflection**) of f if f is continuous at x_0 and changes convexity at x_0. □

Theorem 5.11

Let $x_0 \in dom\,(f)$. *If* there is an ε such that the sign of f'' is different on the intervals $(x_0 - \varepsilon\,,\ x_0)$ /left/ and $(x_0\,,\ x_0 + \varepsilon)$ /right/, then f has an inflection point at x_0 . $\square$

5.13 *Step-by-Step Example*

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 5.8
Convexity

Determine all intervals where f is convex / concave and list all inflection points.

$$f\,(x) = xe^{-2x}, \quad x \in \mathbb{R}$$

SOLUTION
We use the second derivative function to determine the inflection points. For this, we calculate the first derivate function

$$f'\,(x) = \left(xe^{-2x}\right)' = e^{-2x} - 2xe^{-2x},$$

and then we give the second derivative function.

$$\begin{aligned} f''\,(x) &= \left(xe^{-2x}\right)'' = \left(e^{-2x} - 2xe^{-2x}\right)' = \\ &= -2e^{-2x} - \left(2e^{-2x} + 2xe^{-2x}\cdot(-2)\right) = 4xe^{-2x} - 4e^{-2x}. \end{aligned}$$

Now, we solve the following equation.

$$f''\,(x) = 4xe^{-2x} - 4e^{-2x} = 0$$

As $e^{-2x} > 0$, we have

$$\begin{aligned} 4xe^{-2x} - 4e^{-2x} &= 0, \\ 4e^{-2x}\,(x-1) &= 0, \end{aligned}$$

which follows

$$x - 1 = 0.$$

So the solution is

$$x = 1.$$

This means that function f can have inflection poin at $x = 1$.

We use Theorem 5.10 to determine the intervals, where function f is convex or concave. For this, we determine the intervals where the derivative function f'' is negative, the intervals where it is positive and examine whether function f'' changes sign at $x = 1$.

We can summarise our results in the following table.

$dom\,(f)$
sign of f''
convexity of f and inflection points

For function f we get the following table.

$dom\,(f) = \mathbb{R}$	$x < 1$	1	$1 < x$
f'	$-$	0	$+$
f	$\cap$	inflection point	$\cup$

SOLVED EXAMPLE 5.9
Convexity

Determine all intervals where f is convex / concave and list all inflection points.

$$f(x) = x + \frac{1}{x}, \quad x \in \mathbb{R} \setminus \{0\}$$

SOLUTION
We use the second derivative function to determine the inflection points. For this, we calculate the first derivate function

$$f'(x) = \left(x + \frac{1}{x}\right)' = 1 - x^{-2} = 1 - \frac{1}{x^2},$$

and then we give the second derivative function.

$$f''(x) = \left(x + \frac{1}{x}\right)'' = \left(1 - x^{-2}\right)' = 2x^{-3} = \frac{2}{x^3},$$

and solve the following equation.

$$f''(x) = \frac{2}{x^3} = 0$$

But this equation has no solution, which means there is no inflection point for function f.

We use Theorem 5.10 to determine the intervals, where function f is convex or concave. For this, we determine the intervals where the derivative function f'' is negative, the intervals where it is positive.

We can summarise our results in the following table.

$dom(f) = \mathbb{R} \setminus \{0\}$	$x < 0$	0	$0 < x$
f''	$-$	×	$+$
f	$\cap$	×	$\cup$

SOLVED EXAMPLE 5.10
Convexity

Determine all intervals where f is convex / concave and list all inflection points.

$$f(x) = x^3 + 2x^2 - x - 2, \quad x \in \mathbb{R}$$

SOLUTION

Firstly, we have to derivate the function twice.

$$f'(x) = \left(x^3 + 2x^2 - x - 2\right)' = 3x^2 + 4x - 1$$

$$f''(x) = \left(3x^2 + 4x - 1\right)' = 6x + 4$$

Using the second derivative function we can give the inflexion points and the intervals, where the function is convex or concave.

$$f''(x) = 6x + 4 = 0$$

$$\Downarrow$$

$$x = -\frac{2}{3}$$

$dom\,(f) = \mathbb{R}$	$x < -\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3} < x$
f''	$+$	0	$-$
f	$\cup$	inflection point	$\cap$

5.14 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 5.4

Determine all intervals where f is convex / concave and list all inflection points.

1.
$$f(x) = \frac{x+1}{x-3}, \quad x \in \mathbb{R} \setminus \{3\},$$

See Solution 8.5.23

2.
$$f(x) = \frac{1-x}{e^x}, \quad x \in \mathbb{R},$$

See Solution 8.5.24

3.
$$f(x) = \frac{x+1}{e^x}, \quad x \in \mathbb{R},$$

See Solution 8.5.25

4.
$$f(x) = \frac{x^4}{12} + \frac{x^3}{3} - 4x^2 + 6x, \quad x \in \mathbb{R},$$

See Solution 8.5.26

5.
$$f(x) = \frac{x^4}{12} + \frac{x^3}{6} - 3x^2 + 12x, \quad x \in \mathbb{R}.$$

See Solution 8.5.27

5.15 *Application IV. - L'Hospital's Rule*

In this section we examine a useful tool for evaluating limits type of $''\frac{0}{0}''$ or $''\frac{\infty}{\infty}''$.

The following result is named after *Guillaume François Antoine Marquis de* **L'Hôpital** though it was invented by his teacher, **Johann Bernoulli** (1667-1748) Swiss mathematician.

Guillaume François Antoine Marquis de **L'Hôpital** (1661-1704) French army officer, knight, amateur mathematician. His full name was: *Guillaume- François- Antoine Marquis de l'Hôpital, Marquis de Sainte- Mesme, Comte d'Entremont and Seigneur d'Ouques-la-Chaise.*

He learned this theorem from his teacher, **Johann Bernoulli** (1667-1748), a Swiss mathematician and published it in his book. See more *here*.

Theorem 5.12: L'Hospital's Rule

Consider the limit problem

$$\lim_{x\to A} \frac{f(x)}{g(x)}, \tag{5.15.1}$$

where either $A \in \mathbb{R}$ or $A = \pm\infty$. Assume further the following three conditions:

i) either $\lim_{x\to A} f(x) = \lim_{x\to A} g(x) = 0$ ("(5.15.1) is of type $\frac{0}{0}$")

or $\lim_{x\to A} f(x) = \pm\infty$ and $\lim_{x\to A} g(x) = \pm\infty$ ("(5.15.1) is of type $\frac{\infty}{\infty}$"),

ii) both $f'(x)$ and $g'(x)$ exist and $g'(x) \neq 0$ for $x \to A$,

iii) the limit $\lim_{x\to A} \frac{f'(x)}{g'(x)}$ does exist (either $\in \mathbb{R}$ or $= \pm\infty$).

Then

$$\lim_{x\to A} \frac{f(x)}{g(x)} = \lim_{x\to A} \frac{f'(x)}{g'(x)} .$$

□

5.16 *Step-by-Step Example*

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 5.11
L'Hospital's Rule

Evaluate the following limit.

$$\lim_{x\to\infty}\frac{x^2}{e^{3x}}$$

SOLUTION

As the limit is type of " $\left(\frac{\infty}{\infty}\right)$ " and there are differentiable functions in the numerator and in the demoninator, we can apply L'Hospital' rule.

$$\lim_{x\to\infty}\frac{x^2}{e^{3x}} = \lim_{x\to\infty}\frac{\left(x^2\right)'}{\left(e^{3x}\right)'} = \lim_{x\to\infty}\frac{2x}{e^{3x}\cdot 3}.$$

As the new limit is still type of " $\left(\frac{\infty}{\infty}\right)$ " and we still have differentiable functions in the numerator and in the demoninator, we can apply L'Hospital's rule again.

$$\lim_{x\to\infty}\frac{2x}{3e^{3x}} = \lim_{x\to\infty}\frac{(2x)'}{\left(3e^{3x}\right)'} = \lim_{x\to\infty}\frac{2}{3e^{3x}\cdot 3} = 0.$$

SOLVED EXAMPLE 5.12
L'Hospital's Rule

Evaluate the following limit.

$$\lim_{x\to 0}\frac{2\arctan(x)-(x+1)^2+1}{\sin^2(x)}$$

SOLUTION

First, we substitute $x=0$ to the fraction.

$$\frac{2\arctan(0)-(1)^2+1}{\sin^2(0)} = \frac{0}{0}$$

We obtain that the limit is type of " $\left(\frac{0}{0}\right)$ " and we have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule.

$$\lim_{x\to 0}\frac{2\arctan(x)-(x+1)^2+1}{\sin^2(x)}=\lim_{x\to 0}\frac{\dfrac{2}{1+x^2}-2(x+1)}{2\sin(x)\cos(x)}$$

Substitute $x=0$ again.

$$\frac{\dfrac{2}{1}-2}{2\sin(0)\cos(0)}=\frac{0}{0}.$$

We get, that the limit is type of "$\left(\frac{0}{0}\right)$" and we still have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule again.

$$\lim_{x\to 0}\frac{\dfrac{2}{1+x^2}-2(x+1)}{2\sin(x)\cos(x)}=\lim_{x\to 0}\frac{\dfrac{-4x}{(1+x^2)^2}-2}{2\left(\cos^2(x)-\sin^2(x)\right)}$$

Substiting $x=0$ again, we have

$$\frac{\dfrac{0}{(1+0^2)^2}-2}{2\left(\cos^2(0)-\sin^2(0)\right)}=-1,$$

so the result is

$$\lim_{x\to 0}\frac{2\arctan(x)-(x+1)^2+1}{\sin^2(x)}=\lim_{x\to 0}\frac{\dfrac{-4x}{(1+x^2)^2}-2}{2\left(\cos^2(x)-\sin^2(x)\right)}=-1.$$

5.17 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 5.5

Evaluate the following limit.

1. $$\lim_{x\to 0} \frac{\arctan(x) - x}{1 - \cos(x)},$$

See Solution 8.5.28

2. $$\lim_{x\to 1} \frac{\cos(x-1) + \ln(x) - x}{(x-1)^2},$$

See Solution 8.5.29

3. $$\lim_{x\to 0} \frac{\arctan(x) + 2x^2 + x}{\cos(x) - 1},$$

See Solution 8.5.30

4. $$\lim_{x\to -1^-} \frac{\tan(x+1)}{(x+1)^2},$$

See Solution 8.5.31

5. $$\lim_{x\to 2} \frac{\ln(x-1) - \sin(x-2)}{(x-2)^2}.$$

See Solution 8.5.32

6 Antiderivatives and Indefinite Integrals of Real Functions

Previously, we calculated the derivatives of many functions. Now we have the following question: If we have a function f can we find another function F with derivative f, i. e. $F' = f$? How many solutions do exist? How can we find the solutions? To answer these questions, first we introduce the basic definitions and theorems.

6.1 Basic Definitions and Theorems

Definition 6.1: Primitive Function

Let $I \subset \mathbb{R}$ be an interval and consider function $f : I \subset \mathbb{R}$. A function $F : I \subset \mathbb{R}$ is an **antiderivative or primitive function** of function f over interval I if F is differentiable and $F'(x) = f(x)$ for all $x \in I$. □

Theorem 6.1

If F is an antiderivative of function f over interval I then for each $C \in \mathbb{R}$, function $F + C$ is also an antiderivative of f over I, and every antiderivative of f over I has the form of $F + C$, where $C \in \mathbb{R}$. □

The functions $F + C$, where C is any real number is also known as the family of antiderivatives of f.

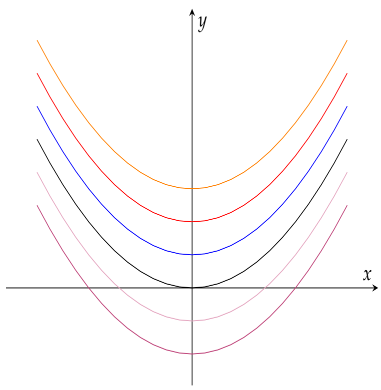


Figure 6.1: The family of antiderivatives of function $2x$.

Definition 6.2: Indefinite Integral

Let $I \subset \mathbb{R}$ be an interval and consider function $f : I \to \mathbb{R}$. The **indefinite integral** of f over interval $I \subset \mathbb{R}$ is the set of the antiderivatives of f over I (if it is not empty). It is denoted by $\int f$ or $\int f(x)\,dx$. Function f is called the **integrand**. □

Notation 6.1.1

$$\int f(x)\,dx = F(x) + C.$$

Theorem 6.2: Linearity

Let $(a, b) \subset \mathbb{R}$, and $k \in \mathbb{R}$. If F and G are the antiderivatives of functions f and g over (a, b), respectively, then kF is the antiderivative of kf and $F + G$ is the antiderivative of $f + g$ over (a, b) and

$$\int (kf) = k \int f,$$
$$\int (f+g) = \int f + \int g. \qquad \square$$

Theorem 6.3: Newton's Theorem

For any *continuous* function f a primitive function F of f (i.e. $F' = f$) *does exist.* □

Liouville[1] showed that primitive function F of "*certain*" functions f *can not be written with a formula*. The description of these "*certain*" functions is complicated, but Liouville's result says that we are unable to write F (both in paper with pencil and in computer). Many simple functions, used in engineering, economics and many other areas of our life, belong to these "*certain*" functions. For example

$$\int e^{x^2} dx, \quad \int e^{ax^n} dx, \quad \int \frac{e^{ax}}{x} dx \quad (a \neq 0, n \in \mathbb{N}, n > 1),$$

$$\int e^{-x^2} dx\,, \quad \int \frac{e^x}{x} dx, \quad \int \sin\left(x^2\right) dx, \quad \int \cos\left(x^2\right) dx,$$

$$\int \sqrt{1 - k \cdot \sin^2(x)}\, dx \quad (k \neq 1),$$

and moreover, the most important ones

[1] Joseph Liouville (1809-1882), French mathematician.

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt\,, \quad Ci(x) := \int \frac{\cos(x)}{x}\, dx,$$

$$Si(x) := \int \frac{\sin(x)}{x}\, dx, \qquad Li(x) := \int \frac{1}{\ln(x)}\, dx,$$

and many others.

To overcome this trouble, tables for the corresponding primitive functions have been constructed (the appropriate methods belong to the subject of *Numerical Analysis*).

6.2 *Table of Standard Indefinite Integrals*

The table below contains the integrals used in many problems.

$\int f(x)\, dx$	$F(x) + C$
$\int x^\alpha dx$	$\frac{x^{\alpha+1}}{\alpha+1} + C$
$\int \frac{1}{x} dx$	$\ln\|x\| + C$
$\int e^x dx$	$e^x + C$
$\int a^x dx$	$\frac{a^x}{\ln a} + C$
$\int \sin(x)\, dx$	$-\cos(x) + C$
$\int \cos(x)\, dx$	$\sin(x) + C$

$\int f(x)\, dx$	$F(x) + C$
$\int \frac{1}{\cos^2(x)} dx$	$\tan(x) + C$
$\int -\frac{1}{\sin^2 x} dx$	$\cot(x) + C$
$\int \frac{1}{\sqrt{1-x^2}} dx$	$\arcsin(x) + C$
$\int \left(-\frac{1}{\sqrt{1-x^2}}\right) dx$	$\arccos(x) + C$
$\int \frac{1}{1+x^2} dx$	$\arctan(x) + C$
$\int \left(-\frac{1}{1+x^2}\right) dx$	$\operatorname{arccot}(x) + C$

$$c, \alpha \in \mathbb{R}, \quad \alpha \neq 1, \quad a \in (0, \infty) \setminus \{1\}.$$

6.3 *Step-by-Step Examples*

Now we give a step-by-step solution to some basic problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 6.1
Indefinite Integral

Evaluate the following indefinite integral.

$$\int \left(x^2 + 5x^3 - \sqrt{x}\right) dx.$$

SOLUTION
Integration is the reverse process of differentiation. We are really just asking what we differentiated to get the given function.

We use the linear proprety of the indefinite integral (*Theorem* 6.2) and some basic mathematics.

$$\int \left(x^2 + 5x^3 - \sqrt{x}\right) dx = \int x^2 dx + 5\int x^3 dx - \int \sqrt{x} dx = \int x^2 dx + 5\int x^3 dx - \int x^{\frac{1}{2}} dx.$$

All the integrals can be evaluated using the 6.2 Table of Standard Indefinite Integrals. That is

$$\int x^2 dx = \frac{x^3}{3} + C_1,$$

$$\int x^3 dx = \frac{x^4}{4} + C_2,$$

and

$$\int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_3.$$

This follows

$$\int \left(x^2 + 5x^3 - \sqrt{x}\right) dx = \frac{x^3}{3} + C_1 + 5 \cdot \left(\frac{x^4}{4} + C_2\right) - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_3.$$

But we do not need three separate constants, so we combine them as one C. That is

$$\int \left(x^2 + 5x^3 - \sqrt{x}\right) dx = \frac{x^3}{3} + 5 \cdot \frac{x^4}{4} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C.$$

SOLVED EXAMPLE 6.2
Indefinite Integral

Evaluate the following indefinite integral.

$$\int \left(\sin(x) + \frac{3}{x} - \frac{1}{x^2} \right) dx.$$

SOLUTION

We are asked to determine all functions whose derivative is $\sin(x) + \frac{3}{x} - \frac{1}{x^2}$. We use the linear proprety of the indefinite integral *(Theorem 6.2)* again.

$$\begin{aligned} \int \left(\sin(x) + \frac{3}{x} - \frac{1}{x^2} \right) dx &= \int \sin(x)\, dx + \int \frac{3}{x} dx - \int \frac{1}{x^2} dx = \\ &= \int \sin(x)\, dx + 3 \int \frac{1}{x} dx - \int x^{-2} dx \end{aligned}$$

All the integrals can be evaluated using the 6.2 Table of Standard Indefinite Integrals. That is

$$\int \left(\sin(x) + \frac{3}{x} - \frac{1}{x^2} \right) dx = -\cos(x) + 3 \ln|x| - \frac{x^{-1}}{-1} + C.$$

6.4 *Exercises*

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 6.1

Evaluate the following indefinite integrals.

1.
$$\int \sqrt{x\sqrt{x}}dx,$$

See Solution 8.6.1

2.
$$\int \frac{x^2+5}{x}dx,$$

See Solution 8.6.2

3.
$$\int \left(2^x - \frac{5}{x^2+1}\right) dx,$$

See Solution 8.6.3

4.
$$\int e^{2x+3}dx.$$

See Solution 8.6.4

6.5 *Integration by Parts*

Theorem 6.4: Integration by Parts

Let $(a,b) \subset \mathbb{R}$. If f and g is differentiable on (a,b) and function fg' has antiderivative over (a,b) then function $f'g$ also has antiderivative over (a,b) and

$$\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx,$$

for $x \in (a,b)$. □

Integration by parts is a method for integrating special products of functions which corresponds to the product rule for derivatives.
We have to calculate function $f(x)$ and $g'(x)$. For this we determine functions $f'(x)$ and $g(x)$ and we **integrate** function $f'(x)$ and **differentiate** function $g(x)$.
The following table will be very useful.

Integration by Parts

$$\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx.$$

	Given	Calculated
Integrate	f'	f
Differentiate	g	g'

This integration technique can be very useful if a product of a transcendental function and an algebraic function is the integrand. The following table contains some common cases.

f'	g
$\sin(ax+b)$	$P(x)$
$\cos(ax+b)$	$P(x)$
e^{ax+b}	$P(x)$

where $P(x)$ is a polynomial function of x and $a, b \in \mathbb{R}$, $a \neq 0$,

and

f'	g
$P(x)$	$\ln(x)$
$\sqrt[n]{x}$	$\ln(x)$
$\dfrac{1}{x^n}$	$\ln(x)$

where $P(x)$ is a polynomial function of x and $n \geq 2$ integer.

Remark 6.5.1 *We use method shown in Solved Example 6.3 in the following cases*

$$\int P(x) \cdot \sin(ax+b)\,dx,$$

$$\int P(x) \cdot \cos(ax+b)\,dx,$$

and

$$\int P(x) \cdot e^{ax+b} dx,$$

where P is a polinomial and $a, b \in \mathbb{R}$, $a \neq 0$.

Remark 6.5.2 *We use method shown in Solved Example 6.5 in the following cases*

$$\int P(x) \cdot \log_a(bx+c)\,dx,$$

$$\int \sqrt[n]{x} \cdot \log_a(bx+c)\,dx,$$

$$\int \frac{1}{x^n} \cdot \log_a(bx+c)\,dx,$$

and

$$\int \frac{1}{\sqrt[n]{x}} \cdot \log_a(bx+c)\,dx,$$

where P is a polinomial and , $n \geq 2$ integer, $b, c \in \mathbb{R}$, $b \neq 0$, and $a \in (0, \infty) \setminus \{1\}$.

6.6 *Step-by-Step Examples*

Now we give a step-by-step solution to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 6.3
Indefinite Integral - Integration by Parts

Evaluate the following indefinite integral.

$$\int x \sin(x)\, dx.$$

SOLUTION

To use this technique we need to identify candidates for functions $f'(x)$ and $g(x)$. We wish to replace integral $\int f'g$ with another $\left(\int fg'\right)$, which can be easier to evaluate. Let

$$f'(x) = \sin(x)$$

and

$$g(x) = x.$$

We have to calculate function $f(x)$ and $g'(x)$. For this we determine functions $f'(x)$ and $g(x)$ and we **integrate** function $f'(x)$ and **differentiate** function $g(x)$.

$$f(x) = \int \sin(x)\, dx = -\cos(x),$$

and

$$g'(x) = 1.$$

or shortly

	Given	Calculated
I	$f'(x) = \sin(x)$	$f(x) = -\cos(x)$
D	$g(x) = x$	$g'(x) = 1$

from *Theorem* 6.4, we obtain

$$\int x \sin(x)\, dx = -\cos(x) \cdot x - \int \left(-\cos(x)\right) \cdot 1\, dx = -\cos(x) \cdot x + \int \cos(x)\, dx.$$

As

$$\int \cos(x)\, dx = \sin(x) + C,$$

the solution is

$$\int x \sin(x)\, dx = -\cos(x) \cdot x + \sin(x) + C.$$

SOLVED EXAMPLE 6.4

Indefinite Integral - Integration by Parts

Evaluate the following indefinite integral.

$$\int x^2 e^x dx.$$

SOLUTION

First, we need to identify candidates for functions $f'(x)$ and $g(x)$.

So let

	Given	Calculated
I	$f'(x) = e^x$	$f(x) = e^x$
D	$g(x) = x^2$	$g'(x) = 2x$

Using Theorem 6.4, we get

$$\int x^2 e^x dx = e^x \cdot x^2 - \int e^x \cdot 2x \, dx = x^2e^x - 2\int xe^x dx.$$

Note that

$$\int xe^x dx$$

is not a standard integral. We calculate this integral by repeated integration by parts. From

	Given	Calculated
I	$f'(x) = e^x$	$f(x) = e^x$
D	$g(x) = x$	$g'(x) = 1$

we obtain

$$\int x \, e^x dx = e^x \cdot x - \int e^x \cdot 1 \, dx = xe^x - \int e^x dx = xe^x - e^x.$$

Combining this with the previous result, we get

$$\int x^2e^x dx = x^2e^x - 2\left(xe^x - e^x\right) + C.$$

SOLVED EXAMPLE 6.5
Indefinite Integral - Integration by Parts

Evaluate the following indefinite integral.

$$\int x \ln(x)\, dx.$$

SOLUTION
Let

$$f'(x) = x$$

and

$$g(x) = \ln(x).$$

We have to calculate function $f(x)$ and $g'(x)$. For this we **integrate** function $f'(x)$ and **differentiate** function $g(x)$.

Using the idea of the previous example, we get

	Given	Calculated
I	$f'(x) = x$	$f(x) = \frac{x^2}{2}$
D	$g(x) = \ln(x)$	$g'(x) = \frac{1}{x}$

and from *Theorem* 6.4, we get

$$\int \ln(x)\, x\, dx = \tfrac{x^2}{2} \cdot \ln(x) - \int \tfrac{x^2}{2} \cdot \tfrac{1}{x} dx = \frac{x^2}{2} \ln(x) - \int \frac{x^2}{2} \cdot \frac{1}{x} dx.$$

Next, we must simplify $\frac{x^2}{2} \cdot \frac{1}{x}$.

That is

$$\frac{x^2}{2} \cdot \frac{1}{x} = \frac{x}{2},$$

so we get the original $\int x dx$ integral back, but with another coefficient.

Hence

$$\int x \ln(x)\, dx = \frac{x^2}{2} \ln(x) - \frac{1}{2} \int x dx.$$

This yields

$$\int x \ln(x)\, dx = \frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C.$$

6.7 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 6.2

Evaluate the following indefinite integrals.

1.
$$\int x^2 \sin(x)\, dx,$$
See Solution 8.6.5

2.
$$\int (x-1)\cos(x)\, dx,$$
See Solution 8.6.6

3.
$$\int x \cdot 2^x dx,$$
See Solution 8.6.7

4.
$$\int \ln(x)\, dx,$$
See Solution 8.6.8

5.
$$\int \sqrt{x}\ln(x)\, dx,$$
See Solution 8.6.9

6.
$$\int \frac{1}{\sqrt{x}}\ln(x)\, dx,$$
See Solution 8.6.10

7.
$$\int \left(x^2+1\right)\ln(x)\, dx,$$
See Solution 8.6.11

8.
$$\int 2x\log_2(x)\, dx,$$
See Solution 8.6.12

9.
$$\int (2x+2)\ln(x+2)\, dx,$$
See Solution 8.6.13

10.
$$\int xe^{2x+3} dx.$$
See Solution 8.6.14

6.8 Integration by Substitution

Theorem 6.5: Substitution Rule

Let g be differentiable function on $(a, b) \subset \mathbb{R}$ whose range is an interval. If F is an antiderivative of f on $g((a,b))$, then $F \circ g$ is an antiderivative of $(f \circ g) \cdot g'$ on (a, b), i. e.

$$\int f(g(x))\, g'(x)\, dx = \left[\int f(u)\, du\right]_{u=g(x)} = F(g(x)) + C,$$

for $x \in (a, b)$. □

Now, we consider the following special cases of the substitution rule.

First, let us apply the above theorem to the composite function $f(ax+b)$, i.e. when $g(x) = ax + b$.

So, consider an integral of the form

$$\int f(ax+b)\, dx$$

where $a, b \in \mathbb{R}$, $a \neq 0$. Letting

$$ax + b = y,$$

we have

$$adx = dy,$$

so

$$dx = \frac{1}{a} dy.$$

This allows us to change variable from x to y, that is

$$\int f(ax+b)\, dx = \int f(y) \frac{1}{a} dy = \frac{1}{a} \int f(y)\, dy.$$

Suppose

$$\int f(y)\, dy = F(y) + C,$$

then

$$\int f(ax+b)\, dx = \frac{1}{a} F(ax+b) + C.$$

So we get the following theorem.

Theorem 6.6: Linear Substitution

Let F be the antiderivative of f and g over (α, β), let $g(x) = ax + b$ be a linear function where $a, b \in \mathbb{R}$, $a \neq 0$ and let (γ, δ) be an interval such $g((\gamma, \delta)) \subset (\alpha, \beta)$. Then $\frac{1}{a}(F \circ g)$ is an antiderivative of $f \circ g$ over (γ, δ) and

$$\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C,$$

for $x \in (a, b)$. $\square$

Now we show another application of Theorem 6.5. Consider an integral of the form

$$\int \frac{f'(x)}{f(x)} dx$$

where f is differentiable and $f \neq 0$. Letting

$$f(x) = y,$$

we have

$$f'(x)\,dx = dy.$$

This allows us to change variable from x to y, that is

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{f(x)} f'(x)\,dx = \int \frac{1}{y} dy.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C.$$

So we get the following theorem.

Theorem 6.7

If f is differentiable on an interval $I \subset \mathbb{R}$ and $f(x) \neq 0$ for any $x \in I$. Then

$$\int \frac{f'}{f} = \ln(|f|) + C. \quad \square$$

Finally, as another application of Theorem 6.5,consider an integral of the form

$$\int f^{\alpha}(x) \cdot f'(x)\,dx,$$

where f is differentiable and $\alpha \neq -1$. Letting

$$f(x) = y,$$

we have

$$f'(x)\,dx = dy.$$

This allows us to change variable from x to y, that is

$$\int f^{\alpha}(x) \cdot f'(x)\,dx = \int y^{\alpha} dy.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int f^{\alpha}(x) \cdot f'(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C,$$

For example $\alpha = 1$, we have

$$\int f(x) \cdot f'(x)\,dx = \frac{f^{2}(x)}{2} + C.$$

So we get the following theorem.

Theorem 6.8

Let f be differentiable function on an interval $(a, b) \subset \mathbb{R}$ and suppose f is not constant on (a, b). Then for any $x \in (a, b)$ we have

$$\int f^{\alpha}(x) \cdot f'(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C,$$

where $\alpha \neq -1$ real. □

Integration by substitution is another method for integrating special products of functions. It corresponds to the chain rule for derivatives. It can be used when an integral contains a composite function and the derivative of its inner function. In the following examples we apply *Theorem 6.5*, *Theorem 6.6*, *Theorem 6.7* and *Theorem 6.8* for evaluating indefinite integrals.

6.9 *Step-by-Step Examples*

Now we give a step-by-step solutions to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 6.6
Indefinite Integral - Substitution Rule

Evaluate the following indefinite integral.

$$\int e^{\sin(x)} \cos(x)\, dx.$$

SOLUTION
The integral contains a composite function and the derivative of its inner function. We use *Theorem* 6.5 and make the substitution

$$\sin(x) = y.$$

Then

$$\cos(x)\, dx = dy.$$

This allows us to change variable from x to y, that is

$$\int e^{\sin(x)} \cos(x)\, dx = \int e^{y} dy$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int e^{\sin(x)} \cos(x)\, dx = \int e^{y} dy = e^{y} + C = e^{\sin(x)} + C.$$

SOLVED EXAMPLE 6.7
Indefinite Integral - Substitution Rule

Evaluate the following indefinite integral.

$$\int \cos(2x+3)\,dx.$$

SOLUTION
The integral contains a composite function and its inner function is a linear function. So the result can be obtained by using

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C.$$

Let

$$f(x) = \cos(x)$$

and

$$a = 2.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int \cos(x)\,dx = \sin(x)+C.$$

Combining this with the *Theorem* 6.6, the result is

$$\int \cos(2x+3)\,dx = \frac{\sin(2x+3)}{2}+C.$$

SOLVED EXAMPLE 6.8
Indefinite Integral - Substitution Rule

Evaluate the following indefinite integral.

$$\int \frac{1}{x\ln(x)}dx.$$

SOLUTION
We need to rewrite the integral

$$\int \frac{1}{x\ln(x)}dx = \int \frac{\frac{1}{x}}{\ln(x)}dx.$$

As

$$(\ln(x))' = \frac{1}{x}.$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C.$$

Let

$$f(x) = \ln(x),$$

so from *Theorem* 6.7, we get

$$\int \frac{1}{x \ln(x)} dx = \int \frac{\frac{1}{x}}{\ln(x)} dx = \ln|\ln(x)| + C.$$

SOLVED EXAMPLE 6.9

Indefinite Integral - Substitution Rule

Evaluate the following indefinite integral.

$$\int \frac{\ln(x)}{x} dx.$$

SOLUTION

As

$$\int \frac{\ln(x)}{x} dx = \int \ln(x) \frac{1}{x} dx,$$

and

$$(\ln(x))' = \frac{1}{x}.$$

the result can be obtained by using

$$\int f(x) \cdot f'(x)\, dx = \frac{f^2(x)}{2} + C.$$

Let

$$f(x) = \ln(x),$$

so from *Theorem* 6.8, we get

$$\int \frac{\ln(x)}{x} dx = \int \ln(x) \frac{1}{x} dx = \frac{\ln^2(x)}{2} + C.$$

6.10 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 6.3

Evaluate the following indefinite integrals.

1.
$$\int x^2 e^{x^3} dx,$$
See Solution 8.6.15

2.
$$\int x \sin\left(x^2\right) dx,$$
See Solution 8.6.16

3.
$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx,$$
See Solution 8.6.17

4.
$$\int \sin\left(\frac{1}{2}x + 3\right) dx$$
See Solution 8.6.18

5.
$$\int e^{7x+1} dx,$$
See Solution 8.6.19

6.
$$\int 5^{3x-9} dx,$$
See Solution 8.6.20

7.
$$\int \sin(2 - 3x) dx,$$
See Solution 8.6.21

8.
$$\int \cos\left(1 - \frac{1}{2}x\right) dx,$$
See Solution 8.6.22

9.
$$\int e^{5-x} dx,$$
See Solution 8.6.23

10.
$$\int \tan(x) dx,$$
See Solution 8.6.24

11.
$$\int \frac{e^{2x}}{e^{2x}+4}dx,$$
See Solution 8.6.25

12.
$$\int \frac{\sin(x)}{1+\cos(x)}dx,$$
See Solution 8.6.26

13.
$$\int \frac{4}{7x+5}dx,$$
See Solution 8.6.27

14.
$$\int \frac{x-2}{x^2-4x+1}dx,$$
See Solution 8.6.28

15.
$$\int \frac{x^2+1}{x^3+3x+4}dx,$$
See Solution 8.6.29

16.
$$\int \frac{1}{\tan(x)\cos^2(x)}dx,$$
See Solution 8.6.30

17.
$$\int \frac{\tan(x)}{\cos^2(x)}dx,$$
See Solution 8.6.31

18.
$$\int x\left(x^2+5\right)^{10}dx,$$
See Solution 8.6.32

19.
$$\int \sqrt{(2x+5)^3}dx,$$
See Solution 8.6.33

20.
$$\int \sin(x)\cos(x)\,dx,$$
See Solution 8.6.34

21.
$$\int \frac{\sqrt{\ln(x)}}{x}dx,$$
See Solution 8.6.35

22.
$$\int \frac{1}{x\sqrt{\ln(x)}}dx,$$
See Solution 8.6.36

23.
$$\int \frac{1}{\sqrt[4]{\tan(x)}\cos^2(x)}dx,$$
See Solution 8.6.37

24.
$$\int \frac{\cos(x)}{\sqrt[3]{\sin(x)}}dx,$$

See Solution 8.6.38

25.
$$\int \frac{x}{\sqrt{x^2+1}}dx,$$

See Solution 8.6.39

26.
$$\int \frac{1}{(3x+1)^2}dx,$$

See Solution 8.6.40

27.
$$\int \frac{1}{\sqrt[4]{3x+1}}dx,$$

See Solution 8.6.41

28.
$$\int \frac{x^2}{\sqrt[4]{x^3+1}}dx.$$

See Solution 8.6.42

7 *Definite Integrals of Real Functions*

The **area** "*under the graph of a function*", i.e. the area of the *region* between the x-axis, the vertical lines $x = a$ and $x = b$ and the graph of $y = f(x)$, assuming $[a,b] \subseteq dom(f)$ and $0 \le f(x)$ for $a \le x \le b$, is called the **definite integral** of the function $f(x)$ on the interval $[a,b]$, and is denoted by

$$\int_a^b f(x)\,dx \quad \text{or by} \quad \int_{[a,b]} f(x)\,dx.$$

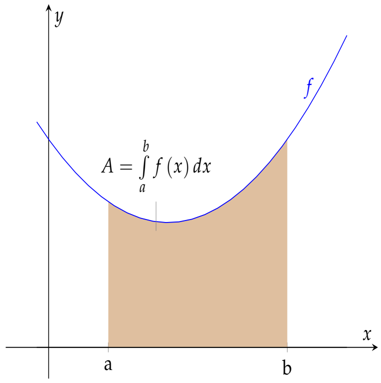
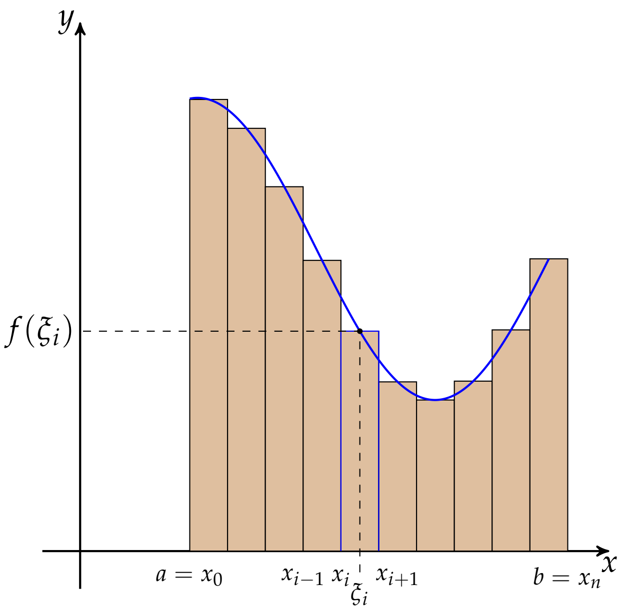


Figure 7.1: Area under the graph of a nonnegative continuous function f.

We do not give here the precise (mathematical) definition, involving

$$\lim_{\max|x_i - x_{i-1}| \to 0} \sum_{i=1}^{n} f(x_i^*) \cdot (x_i - x_{i-1})$$

and the below illustration, since in practice we mainly use the "Rule" of Newton and Leibniz (see Theorem 7.1).



We just mention, that the words "*integral / to integrate*" mean in general (not only in mathematics) "*formed of constituent parts, united, make a complete thing from several others*", and especially in sciences and in engineering "summing a huge umber of extremally (infinitesimally) small quantities ".

7.1 The Fundamental Theorem of Calculus

Although notation for *definite* integral $\int_a^b f(x)\,dx$ look similar to the notation for *indefinite* integral $\int f(x)\,dx$, they are not the same. *Indefinite* integrals are *families of functions*, *definite* integrals are *numbers*. Since finding an antiderivative is usually easier than calculating the value of a definite integral, we look at the relationship between definite and indefinite integrals. The following theorem is often cited as the **Fundamental Theorem of Calculus** or **Newton-Leibniz Rule**[1].

[1] Isaac Newton (1642-1727) English mathematician, physician,
Gottfried Wilhelm Leibniz (1646-1716) German mathematician, physician.

Theorem 7.1: Newton-Leibniz Rule

If $[a,b] \subseteqq dom(f)$ and f has a primitive function F on the interval $[a,b]$, $0 \leq f(x)$ for $a \leq x \leq b$, then the integral

$$\int_a^b f(x)\,dx$$

exists, and

$$\int_a^b f(x)\,dx = F(b) - F(a) \; . \qquad \square$$

Definition 7.1

The difference $F(b) - F(a)$ is often called the **net change** of function F over an interval $[a,b]$ and denoted by $[F(x)]_a^b$ or $[F(x)]_{x=a}^{x=b}$. $\square$

Obviously Theorem 7.1 says *first* compute a primitive function F and *then* substract values $F(b)$ and $F(a)$.

Remark 7.1.1 *We have to highlight, that* $dom(f)$ *and* $dom(F)$ *both must contain the whole* closed *interval* $[a,b]$, *not only because of Theorem 7.1; this problem will be discussed in Section Improper Integrals.*

Let us mention, that the "tail" $+C$ in

$$\int f(x)\,dx = F(x) + C.$$

can be omitted, since

$$\begin{aligned}[F(x) + C]_a^b &= (F(b) + C) - (F(a) + C) = \\ &= F(b) + C - F(a) - C = F(b) - F(a).\end{aligned}$$

Theorem 6.3 says that for any *continuous* function f a primitive function F of f (i.e. $F' = f$) exists. So, for any continuous f also *has* the area under its graph, which can be calculated using Theorem 7.1.

So Newton's theorem suggest us to be able to calculate the definite integral $\int_{[a,b]} f(x)\,dx$ easily (using Theorem 7.1) for *any* continuous function f , *once* we have found a primitive function F for f .
However, Liouville's theorem states, that this is not always the case.

See details here: 6.1 in Chapter *Antiderivatives and Indefinite Integrals of Real Functions*.

WARNING: The most important to our studies is: *always try to solve problems from problem books, but avoid "ad hoc" problems* **!**

7.2 *Step-by-Step Examples*

Now we give step-by-step solution to some basic problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 7.1
Definite Integral

Evaluate the following definite integral.

$$\int_0^1 \left(x^2 + 5x^3 - \sqrt{x}\right) dx.$$

SOLUTION
From Solved Example 6.1, we have

$$\int \left(x^2 + 5x^3 - \sqrt{x}\right) dx = \frac{x^3}{3} + 5 \cdot \frac{x^4}{4} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C.$$

So from Theorem 7.1 with $F(x) = \frac{x^3}{3} + 5 \cdot \frac{x^4}{4} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$, we get

$$\begin{aligned} \int_0^1 \left(x^2 + 5x^3 - \sqrt{x}\right) dx &= [F(x)]_0^1 = \left[\frac{x^3}{3} + 5 \cdot \frac{x^4}{4} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^1 = \\ &= \frac{1^3}{3} + 5 \cdot \frac{1^4}{4} - \frac{1^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 0. \end{aligned}$$

SOLVED EXAMPLE 7.2
Definite Integral

Evaluate the following definite integral.

$$\int_0^{2\pi} \sin(x)\, dx.$$

SOLUTION
Using the 6.2 Table of Standard Indefinite Integrals, we have

$$\int \sin(x)\, dx = -\cos(x) + C.$$

So from Theorem 7.1 with $F(x) = -\cos(x)$, we get

$$\int_0^{2\pi} \sin(x)\, dx = [F(x)]_0^{2\pi} = [-\cos(x)]_0^{2\pi} = -\cos(2\pi) - (-\cos(0)) = -1 + 1 = 0.$$

7.3 *Exercises*

The solutions of the following problems can be found in Chapter 8. *Solutions*.

Exercises 7.1

Evaluate the following definite integrals.

1.
$$\int_0^1 \sqrt{x\sqrt{x}}dx,$$

See Solution 8.7.1

2.
$$\int_0^\pi (x-1)\cos(x)\,dx,$$

See Solution 8.7.2

3.
$$\int_0^1 x\cdot 2^x dx,$$

See Solution 8.7.3

4.
$$\int_1^e \ln(x)\,dx,$$

See Solution 8.7.4

5.
$$\int_{-1}^0 e^{5-x}dx,$$

See Solution 8.7.5

6.
$$\int_0^{\frac{\pi}{4}} \tan(x)\,dx,$$

See Solution 8.7.6

7.
$$\int_0^\pi \sin(x)\cos(x)\,dx.$$

See Solution 8.7.7

7.4 *Applications of Definite Integral*

Most of mathematical, physical, ... , or even everyday quantities can be calculated (or, at least approximated) by measuring all its small components and summing them. As we mentioned in the previous subsection, the process "*integrating*" is precisely the same. So, it must not be surprising, that many these quantities are calculated using integrals.

In what follows, we only list the formulae and present several exercises.

7.4.1 *Areas Under, Above and Between Curves*

Computing the area *under* the graph of a *nonnegative* continuous function f (see Figure 7.1), we can use formula

$$A = \int_a^b f(x)\,dx.$$

Computing the area *above* the graph of a *nonpositive* continuous function f, we can use formula

$$A = \int_a^b (-f(x))\,dx.$$

Computing the area *between* the x-axis and the graph of *any* continuous function f, we can use formula

$$A = \int_a^b |f(x)|dx.$$

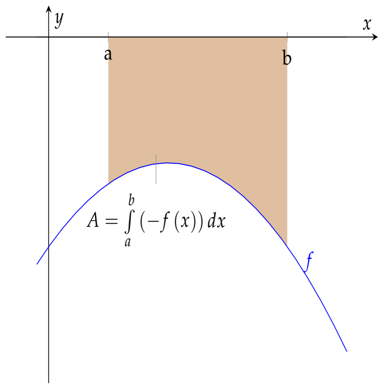


Figure 7.2: Area above the graph of a nonpositive continuous function f.

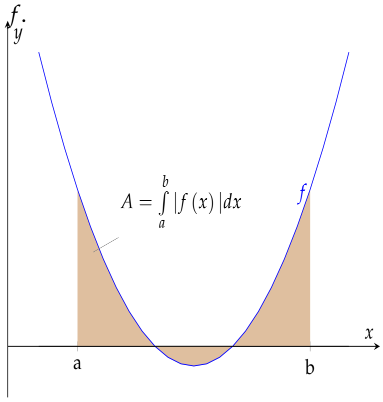


Figure 7.3: Area between the x-axis and the graph of a continuous function f.

Computing the area *between* the graphs of *two* continuous functions f and g, the following formula is an easy but useful variant of Theorem 7.1.

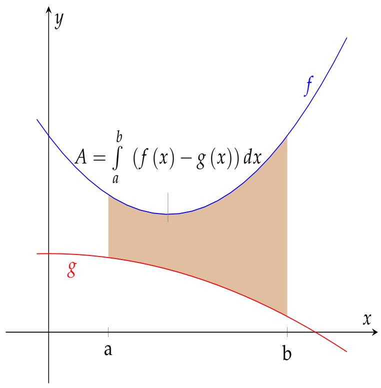


Figure 7.4: Area between graphs of two functions.

Theorem 7.2

Considering *two* continuous functions, f and g on interval $[a, b]$, and assuming

$$g(x) \leq f(x) \qquad \text{for } \forall x \in [a, b] ,$$

the area of the region "between f and g" (i.e. below f and above g), and between the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b (f(x) - g(x))\, dx . \qquad \square$$

7.4.2 *Volume of Revolution*

A solid of revolution is created by rotating (revolving) the graph of a positive function around the x-axis. Many goods in our everyday life are made in this way, with a rotating (revolving) machine called **potter's wheel** or **lathe**[2]

[2] but not by *revolver*.

Figure 7.5: A revolved solid, around the x -*axis**

*source: `\https://de.wikipedia.org/wiki/Drehmaschine`.

Theorem 7.3

For any function f, continuous on the interval $[a,b] \subset dom(f)$, the volume of the solid, created by rotating (revolving) the graph of f around the x-axis is

$$V_x = \pi \int_a^b f^2(x)\,dx.$$

Only *monotone* (increasing or decreasing) functions can be rotated around the y -axis, the volume is

$$\begin{aligned} V_y &= \pi \left| \int_{f(a)}^{f(b)} \left(f^{-1}(y)\right)^2 dy \right| = \\ &= \pi \left| \int_a^b x^2 \cdot f'(x)\,dx \right| = \\ &= \pi \left| \left[x^2 f(x)\right]_a^b - 2\int_a^b x \cdot f(x)\,dx \right| \end{aligned}$$

(please choose any of the above three formulas). □

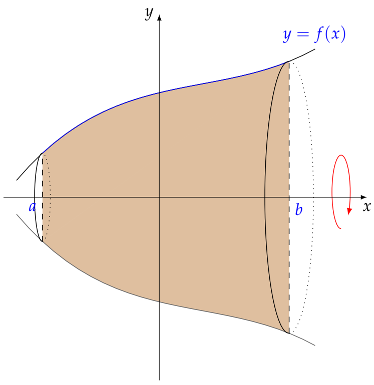


Figure 7.6: Volume of the solid, created by rotating the graph of f around the x-axis.

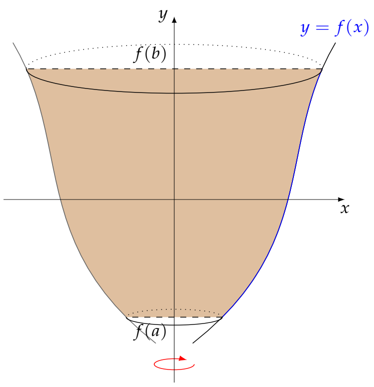


Figure 7.7: Volume of the solid, created by rotating the graph of f around the y-axis.

7.4.3 *Length of a Curve*

Theorem 7.4

The *length* of a function graph between the points $(a, f(a))$ and $(b, f(b))$ is

$$\ell = \int_a^b \sqrt{1 + (f'(x))^2}\, dx\,. \qquad \square$$

We have to warn the Reader, that even for simple functions f the integrand in Theorem 7.4 belongs to Liouville's Theorem 6.1.

7.4.4 *Area of Revolution*

Theorem 7.5

The *area* of the solid's surface when rotating (revolving) the graph of a positive function around the x-axis is

$$A_x = 2\pi \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2}\, dx\,. \qquad \square$$

7.4.5 *Other*

As we mentioned in the introduction, *integration* both verbally and (not only) mathematically means "*forming of constituent parts, summing something from* "infinitely" *many but small parts*". So, it not surprising, that $\int$ is a fundamental tool in physics, chemics, economy, and in all sciences. E.g. route=$\int$velocity, (mechanical) work=$\int$force, (electrical) work=$\int I(t) \cdot U(t) dt$, velocity=$\int$acceleration are only some examples.

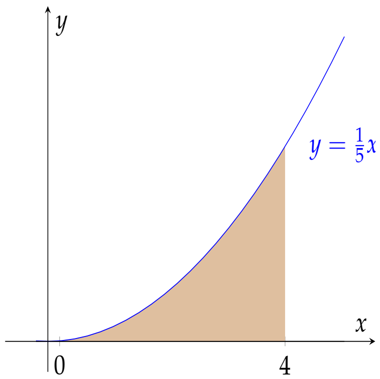
7.5 *Step-by-Step Examples*

Now we give step-by-step solutions to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 7.3
Definite Integral - Area

Find the area under parabola $y = \frac{1}{5}x^2$ over the interval $[0, 4]$.

SOLUTION



Since

$$\int \frac{1}{5}x^2 dx = \frac{1}{5} \cdot \frac{x^3}{3} + C \,,$$

using Theorem 7.1 with $F(x) = \frac{1}{5} \cdot \frac{x^3}{3}$, we have

$$\int_0^4 \frac{1}{5}x^2\, dx = [F(x)]_0^4 = \left[\frac{1}{5} \cdot \frac{x^3}{3}\right]_0^4 = \frac{1}{5} \cdot \left(\frac{4^3}{3} - \frac{0^3}{3}\right) \approx 4.2667 \,.$$

Since $0 \le f(x)$ for each $0 \le x \le 4$, the above number is the real geometrical area of the region between the graph of $f(x)$ and the x axis.

SOLVED EXAMPLE 7.4
Definite Integral - Area

Find the area between $y = 4 - x^2$ and $y = (x + 1)^2$ over the interval $[-1, 0.5]$.

SOLUTION

First we have to decide which function from the above is $f(x)$ and $g(x)$ for satisfying $g(x) \leq f(x)$ in Theorem 7.2. Substituting e.g. $x = 0$ yields

$$(0+1)^2 = 1 < 4 - 0^2 = 4$$

makes us to *think* that $g(x) = (x+1)^2$ and $f(x) = 4 - x^2$.

Now we have to check $g(x) \leq f(x)$ for all x between $a = -1$ and $b = 0.5$. But

$$(x+1)^2 < 4 - x^2$$

is equivalent to

$$0 < h(x) = -2x^2 - 2x + 3$$

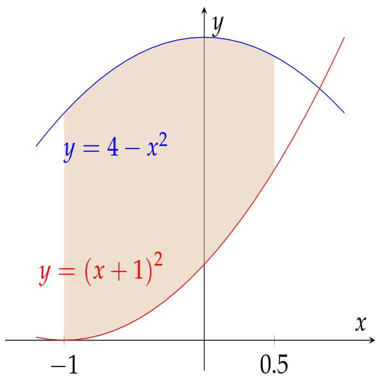
which is valid since $h(x)$ has roots

$$x_1 = \frac{-2 - \sqrt{28}}{4} \approx -1.8229,$$

and

$$x_2 = \frac{-2 + \sqrt{28}}{4} \approx +0.82288,$$

and $h(x)$ is a concave function.



Then

$$\int \left(\left(4 - x^2\right) - (x+1)^2 \right) dx = \int \left(-2x^2 - 2x + 3\right) dx = \frac{-2x^3}{3} - x^2 + 3x + C\,,$$

and from Theorem 7.1 with $F(x) = \dfrac{-2x^3}{3} - x^2 + 3x$, the required area is

$$\int_{-1}^{0.5} \left(4 - x^2\right) - (x+1)^2 \; dx = \left[\frac{-2x^3}{3} - x^2 + 3x\right]_{-1}^{0.5} =$$

$$= \left(\frac{-2 \cdot 0.5^3}{3} - 0.5^2 + 3 \cdot 0.5\right) - \left(\frac{-2 \cdot (-1)^3}{3} - (-1)^2 + 3 \cdot (-1)\right) = 4.5.$$

Note, that the above problem asked the area over the interval $[-1, 0.5]$ only.

However, the whole region, determined by the functions is located in the interval $[x_1, x_2]$ where x_1 and x_2 are the solutions of the equation $f(x) = g(x)$, i.e.

$$4 - x^2 = (x+1)^2 \Longleftrightarrow 0 = 2x^2 + 2x - 3,$$

and the roots of the latest equation are

$$x_1 = \frac{-2-\sqrt{28}}{4} \approx -1.82288,$$

and

$$x_2 = \frac{-2+\sqrt{28}}{4} \approx 0.82288.$$

So the whole area is

$$\int_{x_1}^{x_2} (4-x^2) - (x+1)^2 \; dx = \left[\frac{-2x^3}{3} - x^2 + 3x\right]_{x_1}^{x_2} =$$

$$= \left(\frac{-2\cdot\left(\frac{-2+\sqrt{28}}{4}\right)^3}{3} - \left(\frac{-2+\sqrt{28}}{4}\right)^2 + 3\cdot\left(\frac{-2+\sqrt{28}}{4}\right)\right) -$$

$$- \left(\frac{-2\cdot\left(\frac{-2-\sqrt{28}}{4}\right)^3}{3} - \left(\frac{-2-\sqrt{28}}{4}\right)^2 + 3\cdot\left(\frac{-2-\sqrt{28}}{4}\right)\right) \approx$$

6.17342 . □

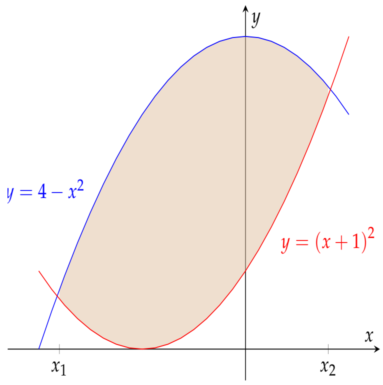


Figure 7.8: The whole area between the graphs of function f and g.

If the interval $[a, b]$ is not given, we have to determine it for ensuring the region between the graphs of $f(x)$ and $g(x)$ to be finite, as in the above example.

SOLVED EXAMPLE 7.5
Definite Integral - Area

Find the area between the function curves $1-\cos(x)$ and $\sin(x)$.

SOLUTION

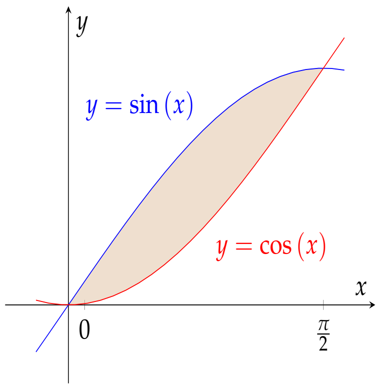
The default interval $[a,b]$ is where a and b are two consecutive intersection points of the curves f and g, i.e. solutions of the equation

$$f(x) = g(x).$$

In our case

$$\begin{aligned}
\sin(x) &= 1-\cos(x)\\
&\Updownarrow\\
\sin(x)+\cos(x) &= 1\\
&\Updownarrow\\
\frac{1}{\sqrt{2}}\sin(x)+\frac{1}{\sqrt{2}}\cos(x) &= \frac{1}{\sqrt{2}}\\
&\Updownarrow\\
\cos\left(\frac{\pi}{4}\right)\sin(x)+\sin\left(\frac{\pi}{4}\right)\cos(x) &= \cos\left(x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\\
&\Updownarrow\\
x-\frac{\pi}{4} &= \pm\frac{\pi}{4}+2k\pi\\
&\Updownarrow\\
x_1=\frac{\pi}{2}+2k\pi, &\qquad x_2=2k\pi, \qquad (k\in\mathbb{Z}).
\end{aligned}$$

Because of periodicity, the interval is $[a,b]=\left[0,\frac{\pi}{2}\right]$.



Since over this interval $\sin(x)$ is concave and $1-\cos(x)$ is convex, we have

$$1-\cos(x) \le \sin(x) .$$

So

$$\begin{aligned}\int (\sin(x) - (1-\cos(x)))\ dx &= \int (\sin(x) + \cos(x) - 1)\ dx = \\ &= -\cos(x) + \sin(x) - x + C.\end{aligned}$$

From Theorem 7.1 with $F(x) = -\cos(x) + \sin(x) - x$, the area is

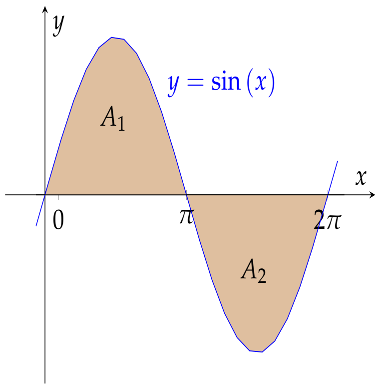
$$\int_0^{\pi/2} (\sin(x) - (1-\cos(x)))\ dx = [-\cos(x) + \sin(x) - x\ ]_0^{\pi/2} =$$

$$= \left(-\cos\left(\tfrac{\pi}{2}\right) + \sin\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2}\right) - (-\cos(0) + \sin(0) - 0) = 2 - \frac{\pi}{2} \approx 0.4292 .$$

SOLVED EXAMPLE 7.6
Definite Integral - Area

Find the area between the x-axis and the graph of $f(x) = \sin(x)$ over the interval $[0, 2\pi]$.

SOLUTION



Since $f(x) = \sin(x)$ is nonnegative over the interval $[0,\pi]$ and nonpositive over $[\pi, 2\pi]$ the area is

$$A = A_1 + A_2 = \int_0^{\pi} \sin(x)\, dx + \int_{\pi}^{2\pi} (-\sin(x))\, dx.$$

As

$$\int \sin(x)\,dx = -\cos(x) + C,$$

using Theorem 7.1 with $F(x) = -\cos(x)$, we have

$$A = \int_0^{\pi} \sin(x)\,dx + \int_{\pi}^{2\pi} (-\sin(x))\,dx = \int_0^{\pi} \sin(x)\,dx - \int_{\pi}^{2\pi} \sin(x)\,dx =$$

$$= [-\cos(x)]_0^{\pi} - [-\cos(x)]_{\pi}^{2\pi} =$$

$$= -\cos(\pi) - (-\cos(0)) - (-\cos(2\pi) - [-\cos(\pi))) = 4.$$

We showed in Solved Example 7.2 that

$$\int_0^{2\pi} \sin(x)\,dx = 0.$$

As region A_1 and A_2 are congruent,

$$\int_0^{2\pi} \sin(x)\,dx = A_1 - A_2 \neq A.$$

WARNING

The assumption $0 \leq f(x)$ is essential for the correct calculation of the (geometrical) area between the graph of $f(x)$ and the x axis! Region A_1 is *above* the x axis while A_2 is *below* it and the area of congruent regions are equal. If we learn, that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

is *negative* for functions $f(x) \leq 0$, we can understand

$$\int_0^{2\pi} \sin(x)\,dx = A_1 - A_2 = 0.$$

The **moral** of the above discussion is that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

summarizes the areas of the function regions *above* and *below* the x axis with signs (+/-), which is not always the geometrical area of these regions.

Definition 7.2: Signed Area

The quantity calculated by

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

is called **signed area** of $f(x)$, which is understood as follows.

i) if $0 \leq f(x)$ for each $a \leq x \leq b$, then $F(b) - F(a)$ is the (geometrical) area of the region between the graph of $f(x)$ and the x axis *(positive)*,

ii) if $f(x) \leq 0$ for each $a \leq x \leq b$, then $F(b) - F(a)$ is (-1) *-times* the (geometrical) area of the region between the graph of $f(x)$ and the x axis *(negative)*,

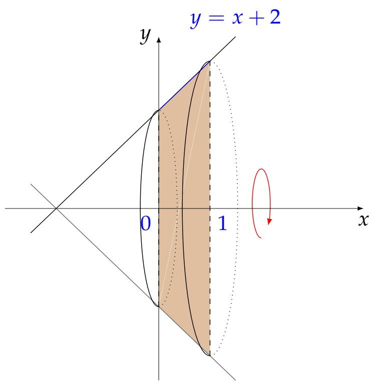
iii) if $f(x)$ is continuous and has both positive and negative values in the interval $[a, b]$, then first find all the roots $x_1, \ldots, x_k$ of $f(x)$ in the interval, put $x_0 := a$ and $x_{k+1} := b$, then add the *signed areas* $\int_{x_i}^{x_{i+1}} f(x)\,dx$ (as defined in i) and ii)), i.e.

$$\int_a^b f(x)\,dx = \sum_{i=0}^{k} \left(\int_{x_i}^{x_{i+1}} f(x)\,dx \right) = F(b) - F(a). \qquad \square$$

SOLVED EXAMPLE 7.7
Definite Integral - Volume

Find the volume of the solid of revolution formed by revolving the graph of function $f(x) = x + 2$, $0 \leq x \leq 1$ around the x -axis (truncated cone).

SOLUTION



From Theorem 7.3 with $f(x) = x + 2$, we get

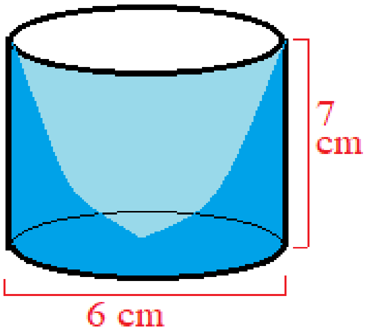
$$V_x = \pi \int_0^1 (x+2)^2 \, dx = \pi \int_0^1 x^2 + 4x + 4 \, dx = \pi \left[\frac{x^3}{3} + 2x^2 + 4x \right]_0^1 =$$

$$= \pi \left[\left(\frac{1^3}{3} + 2 \cdot 1^2 + 4 \cdot 1 \right) - 0 \right] = \frac{19}{3}\pi \approx 19.897 \, .$$

SOLVED EXAMPLE 7.8
Definite Integral - Volume

How much water has to be poured into a cylindric glass of diameter $6cm$ and height $7cm$, which, when stirred with a spoon, reaches exactly from bottom to top?

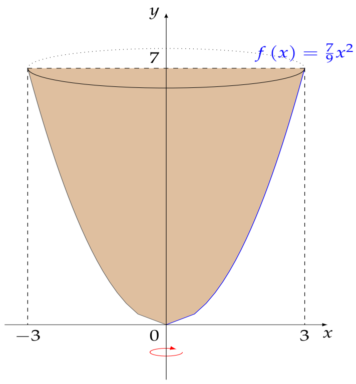
SOLUTION
The *vertical* intersection of the surface of the water (the y -axis is exactly the rotating axe of the glass), is a parabola $y = f(x) = ax^2$ where we must have $7 = f(3) = a \cdot 3^2$ which implies $a = \frac{7}{9}$, i.e. $y = f(x) = \frac{7}{9} \cdot x^2$, $0 \leq x \leq 3$ and $0 \leq y \leq 7$.

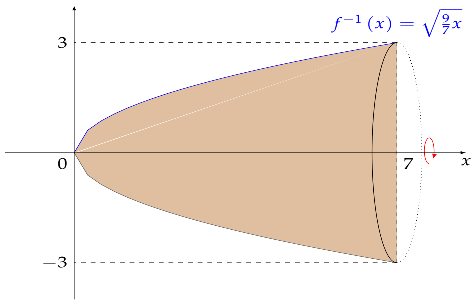


Since formula

$$V_x = \pi \int_a^b f^2(x)\, dx.$$

in Theorem 7.3 for rotating around the x -axis is much easier than any of stated for rotating around the y -axis, we turn the glass down the table, i.e. take the inverse of $f(x)$ and rotate it around the x -axis.





The inverse of

$$y = f(x) = \frac{7}{9}x^2$$

is

$$x = \sqrt{\frac{9}{7}y}$$

i.e.

$$y = g(x) = f^{-1}(x) = \sqrt{\frac{9}{7}x}$$

for $0 \le x \le 7$.

So the volume is

$$\begin{aligned} V_x &= \pi \int_a^b g^2(x)\, dx = \pi \int_0^7 \left(\sqrt{\frac{9}{7}x}\right)^2 dx = \pi \int_0^7 \frac{9}{7}x\, dx = \pi \left[\frac{9}{7}\cdot\frac{x^2}{2}\right]_0^7 = \\ &= \pi \cdot \frac{9}{7}\cdot\frac{7^2}{2} - 0 = \frac{63}{2}\pi \approx 98.9602\ \left(cm^3\right). \end{aligned}$$

However V_x is the volume *under* the graph of $g(x)$, which is the volume of the *air* contained (left) in the glass. This yields, the volume of the water is

$$V_{water} = V_{glass} - V_{air} = 3^2 \cdot \pi \cdot 7 - \pi \frac{63}{2} = \frac{63}{2}\pi \approx 98.9602 \ (cm^3) \ .$$

No wonder: *exactly half* of the glass must be filled water (and half volume left for the air), stir carefully and observe the phenomenon.

Smart Readers may observe, that turning the glass, i.e. taking the inverse of $f(x)$ and rotating it around the x -axis, correspond to the formula

$$V_y = \pi \left| \int_{f(a)}^{f(b)} \left(f^{-1}(y) \right)^2 dy \right|$$

in Theorem 7.3.

7.6 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

7.6.1 *Areas Under, Above and Between Curves*

Exercises 7.2

In each of the following exercises find the area of the indicated region.
It is advisable to sketch the region (and also of a small enclosing rectangle with horizontal and vertical sides) before computing.

1. Between the vertical lines $x = 0$, $x = 1$, the x -axis and the graph of $f(x) = x^3$.

See Solution 8.7.8

2. Above the graph of $f(x) = \sqrt{x}$, below the line $y = 2$ and between $x = 0$ and $x = 4$.

See Solution 8.7.9

3. Above the x -axis and below the graph of $f(x) = x^2 - x^3$.

See Solution 8.7.10

4. Above the x -axis and below the graph of $f(x) = 4x^2 - x^4$.

See Solution 8.7.11

5. Above the x -axis and below the graph of $f(x) = \dfrac{1}{1+x^2}$ between $x = 0$ and $x = 1$.

See Solution 8.7.12

   (In general, the curves $f(x) = \frac{8a^3}{4a^2+x^2}$ for $a \in \mathbb{R}$ are known as *Maria Agnesi*'s[3] "*witch*", which word is a *mistranslation* of the italian word "*sailing sheet*".

[3] Italian mathematician (1718-1799) women.
See also https://en.wikipedia.org/wiki/Witch_of_Agnesi or https://en.wikipedia.org/wiki/Maria_Gaetana_Agnesi.

6. The region between the x -axis and the graph of $f(x) = \dfrac{1}{1+x} + \dfrac{x}{2} - 1$.

See Solution 8.7.13

Exercises 7.3

Find the area of the region bounded by the given curves.

1. $$y = x^2 - x \text{ and } y = 5x - 5.$$

See Solution 8.7.14

2. $$y = x^2 - 6 \text{ and } y = x + 6.$$

See Solution 8.7.15

3. $$y = -(x+1)^2 \text{ and } y = 5x + 11.$$

See Solution 8.7.16

4. $$y = x^2 - 12 \text{ and } y = 2x - x^2.$$

See Solution 8.7.17

5. $$y = (x-1)^2 \text{ and } y = 1 - x^2.$$

See Solution 8.7.18

6. * $$y^2 = 4x \text{ and } y = 2x.$$

See Solution 8.7.19

7. $$y = x(2-x) \text{ and } x = 2y.$$

See Solution 8.7.20

8. $$x^2 = 4y \text{ and } x = 4y - 2.$$

See Solution 8.7.21

9. * $$x = y^2 \text{ and } y = x^2.$$

See Solution 8.7.22

10. * $$y^2 = x \text{ and } x + y = 2.$$

See Solution 8.7.23

11. $$y = \sqrt{x} \text{ and } y = x.$$

See Solution 8.7.24

12.
$$y = x^2 \text{ and } y = 3/(2 + x^2).$$

See Solution 8.7.25

13.
$$y = (1/2) \cdot x^2 + 1 \text{ and } y = x + 1.$$

See Solution 8.7.26

14. *
$$y^2 = x \text{ and } x^2 = 16y.$$

See Solution 8.7.27

15. *
$$y^2 = 4ax \text{ and } y = mx.$$

where $a, m \in \mathbb{R}$ are parameters.

See Solution 8.7.28

Exercises 7.4

Use integration to calculate the following areas.

1. The triangular region bounded by the given lines.
$$y = 2x + 1, \quad y = 3x + 1 \quad \text{and} \quad x = 4.$$

See Solution 8.7.29

2. The triangular region bounded by the given lines.
$$y = x + 3, \quad y = 2x + 1 \quad \text{and} \quad y = 4 - x.$$

See Solution 8.7.30

3. * Find a so that the curves $y = x^2$ and $y = a\cos x$ intersect at the points $(x, y) = (\pi/4, \pi^2/16)$. Then find the area between these curves.

See Solution 8.7.31

4. * The area of the region between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.

   (You might check the results after by elementary geometrical calculations.)

See Solution 8.7.32

7.6.2 *Volume of Revolution*

Exercises 7.5

Calculate the volume of solids over the given interval, when f is revolved around the x -axis.

1.
$$f(x) = x, \qquad 0 \leq x \leq 2,$$
See Solution 8.7.33

2.
$$f(x) = \sqrt{2-x}, \qquad 0 \leq x \leq 2,$$
See Solution 8.7.34

3.
$$f(x) = \left(1+x^2\right)^{-1/2}, \qquad |x| \leq 1,$$
See Solution 8.7.35

4.
$$f(x) = \sin(x), \qquad 0 \leq x \leq \pi,$$
See Solution 8.7.36

5.
$$f(x) = 1 - x^2, \qquad |x| \leq 1,$$
See Solution 8.7.37

6. *
$$f(x) = \cos(x), \qquad 0 \leq x \leq \pi,$$
See Solution 8.7.38

7.
$$f(x) = \frac{1}{\cos(x)}, \qquad 0 \leq x \leq \pi/4,$$
See Solution 8.7.39

8.
$$f(x) = \sqrt{r^2 - x^2}, \qquad 0 \leq x \leq r, \quad \text{(semicircle)}$$
See Solution 8.7.40

9.
$$f(x) = \sqrt{(5x+1) \cdot e^x}, \qquad 0 \leq x \leq 1,$$
See Solution 8.7.41

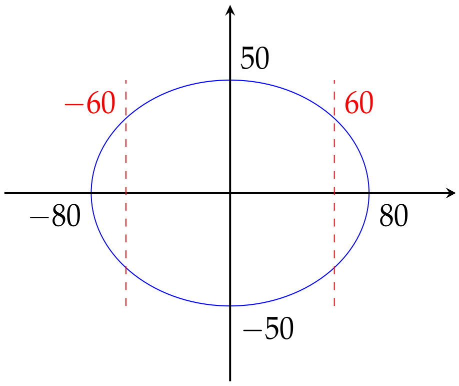
10.
$$f(x) = \sqrt{(x+1) \cdot \ln(x)}, \qquad 1 \leq x \leq e,$$
See Solution 8.7.42

11.
$$f(x) = \sqrt[3]{x}, \qquad 0 \leq x \leq 1,$$
See Solution 8.7.43

12.
$$f(x) = \sqrt{\sin(x)}, \qquad 0 \le x \le \pi/2.$$

See Solution 8.7.44

13. A traditional (wine) barrel has shape (approximately) a part of a rotated ellipse around the x -axis.



See also https://en.wikipedia.org/wiki/Spheroid.

The ellipse has the equality (in centimeters)
$$\left(\frac{x}{80}\right)^2 + \left(\frac{y}{50}\right)^2 = 1$$
from which we need the part $-60 \le x \le 60$.
Calculate the volume of the barrel.

See Solution 8.7.45

7.7 *Improper Integrals*

In many cases, even in practice, we cannot use Newton-Leibniz Rule (Theorem 7.1), not because of Liouville' Theorem.
We may have two difficulties: *either* the interval $[a, b]$ is not a finite one, *or* the integrand function $f(x)$ is not finite in the interval $[a, b]$, i.e. either $f(a)$ or $f(b)$ or $f(c)$ does not exists for some c as $a < c < b$. These cases are called **"improper integrals"**: it is not proper for the first glance, what to do (though the definitions below make them clearer). Note, that "improper" and "indefinite" integrals should not to be confused!

7.7.1 *Integrating over Infinite Intervals*

Integrating over Infinite Intervals is easy to observe (in exams): in the notations $\int\limits_{-\infty}^{b} f(x)\,dx$, $\int\limits_{a}^{\infty} f(x)\,dx$ or $\int\limits_{-\infty}^{\infty} f(x)\,dx$ we must observe the $\pm\infty$ symbol, the other type of problem is much harder to observe. We discuss these problems and their solutions separately. We write simply ∞ instead of $+\infty$.

Definition 7.3

Denote $I := (-\infty, c]$ for any $c \in \mathbb{R}$. Suppose that $I \subseteq dom(f)$ and for *each* $[a, b] \subset I$ there is a primitive function F for f on the subinterval $[a, b]$ **(**i.e. we can apply Newton-Leibniz Rule on $[a, b]$**)**. Then the **improper integral**

$$\int\limits_{-\infty}^{b} f(x)\,dx$$

is defined as

$$\int\limits_{-\infty}^{b} f(x)\,dx := \lim_{\omega\to-\infty}\left(\int\limits_{\omega}^{b} f(x)\,dx\right)$$

assuming that the limit does exist.
In this case the integral is called **convergent**, other-

wise it is **divergent**. □

Definition 7.4

Denote $I := [c, \infty)$ for any $c \in \mathbb{R}$. Suppose that $I \subseteq dom(f)$ and for *each* $[a, b] \subset I$ there is a primitive function F for f on the subinterval $[a, b]$ **(**i.e. we can apply Newton-Leibniz Rule on $[a, b]$**)**. Then the **improper integral**

$$\int_a^\infty f(x)\,dx$$

is defined as

$$\int_a^\infty f(x)\,dx := \lim_{\omega \to \infty} \left(\int_a^\omega f(x)\,dx \right)$$

assuming that the limit does exist.
In this case the integral is called **convergent**, otherwise it is **divergent**. □

Definition 7.5

The integral $\int\limits_{-\infty}^{\infty} f(x)\,dx$ is defined as

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,, \qquad (7.7.1)$$

assuming that *both* improper integrals on the right do exist, where $c \in \mathbb{R}$ is an artbitrary number. □

Let us emphasize that the integral (7.7.1) on the left does exist only when *both* integrals on the right do exist, even for *even* or *odd* functions f. Further explanations can be found in Remark 7.7.1, based on Solved Examples 7.11 and 7.12.

Remark 7.7.1 *Note, that for even functions $\int\limits_{-\infty}^{0} f(x)\,dx$ and $\int\limits_{0}^{\infty} f(x)\,dx$ both have the same value: either both are convergent or both are divergent with the same sign. This implies that $\int\limits_{-\infty}^{\infty} f(x)\,dx$ is convergent just in the case $\int\limits_{0}^{\infty} f(x)\,dx$ is, which implies the equality for even functions*

$$\int_{-\infty}^\infty f(x)\,dx = 2 \cdot \int_0^\infty f(x)\,dx$$

holds in all cases.

However, for odd functions (symmetric to the origin, i.e.

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$), the equality

$$\int_{-\infty}^0 f(x)\,dx = -\int_0^\infty f(x)\,dx$$

*might lead us to the **false conclusion***

$$\int_{-\infty}^\infty f(x)\,dx = \int_0^\infty f(x)\,dx - \int_0^\infty f(x)\,dx = 0.$$

While this is surely true for convergent improper integrals $\int\limits_{-\infty}^{0} f(x)\,dx$ and $\int\limits_{0}^{\infty} f(x)\,dx$, for divergent integrals we must say that $\int\limits_{-\infty}^{\infty} f(x)\,dx$ is divergent. See e.g. the integral $\int\limits_{-\infty}^{\infty} \frac{x}{x^2+1}dx$ in the above example.

7.7.2 *Integrating Discontinuous Functions*

We have to recall that the Newton-Leibniz Rule (Theorem 7.1) is valid only for functions which are continuous on the whole *closed* interval $[a, b]$. So, on *finite* intervals the integrand function (i.e. which we have to *integrate* = summarize) may cause problems. Either the function may have not have any value at one of the endpoints of the interval, i.e. either $f(a)$ or $f(b)$ or both does not exist. Or, what is more hard to observe: some functions have no values at certain *inner* points c in the interval: $a < c < b$.

In one word: we have to check the continuity of the function $f(x)$ at *all* points x in the closed interval $[a, b]$, including the endpoints: for all x such that $a \leq x \leq b$ for calculating $\int_a^b f(x)\,dx$.

Definition 7.6

Suppose that the function $f(x)$ is continuous on interval $(a, b]$.
In other words: only $f(a)$ is the exception, that is $\int\limits_{\omega}^{b} f(x)\,dx$ does exist for all $a < \omega \leq b$.
Then we define

$$\int\limits_a^b f(x)\,dx := \left(\lim_{\omega \to a+} \int\limits_{\omega}^{b} f(x)\,dx\right) . \qquad (7.7.2)$$

Similarly, if $f(x)$ is continuous on interval $[a, b)$ then we have

$$\int\limits_a^b f(x)\,dx := \lim_{\omega \to b-} \left(\int\limits_a^{\omega} f(x)\,dx\right) . \qquad \square \qquad (7.7.3)$$

Remark 7.7.2 *Observe, that the special value of c is* unimportant, *since, if $F(x)$ is a primitive function of $f(x)$, then (7.7.5) calculates*

$$\lim_{\omega \to a+} \int\limits_{\omega}^{c} f(x)\,dx + \lim_{\omega \to b-} \int\limits_{c}^{\omega} f(x)\,dx =$$

$$= \lim_{\omega \to a+} [F(x)]_{\omega}^{c} + \lim_{\omega \to b-} [F(x)]_{c}^{\omega} =$$

$$= \lim_{\omega \to a+} (F(c) - F(\omega)) +$$

$$+ \lim_{\omega \to b-} (F(\omega) - F(c)) =$$

$$= F(c) - \lim_{\omega \to a+} F(\omega) +$$

$$+ \lim_{\omega \to b-} F(\omega) - F(c) =$$

$$= \lim_{\omega \to b-} F(\omega) - \lim_{\omega \to a+} F(\omega) .$$

This final formula could be considered as a generalized Newton-Leibniz Rule. $\square$

Definition 7.7

If $f(x)$ is continuous only on an open interval (a, b) then we have to cut the interval into two parts at an (arbitrary) point c, $a < c < b$.

$$\int\limits_a^b f(x)\,dx := \int\limits_a^c f(x)\,dx + \int\limits_c^b f(x)\,dx , \qquad (7.7.4)$$

detailed by (7.7.2) and (7.7.3)

$$\int_a^b f(x)\,dx := \lim_{\omega\to a+}\int_\omega^c f(x)\,dx + \lim_{\omega\to b-}\int_c^\omega f(x)\,dx \tag{7.7.5}$$

assuming that *both* above limits do exist. (c can be any fixed number such that $a < c < b$.) □

Let us emphasize, that *both limits* in (7.7.5) must be convergent in order that the integral $\int_a^b$ would be convergent. Even in the case "$\int_a^c = -\infty$ and $\int_c^b = +\infty$" we should *not* say "$-\infty + \infty = 0$" **/FALSE!/** , rather "$\int_a^b$" is *divergent* only!

As mentioned, the function may have no value at some *inner* point c in the interval: $a < c < b$.

Definition 7.8

Suppose that the function $f(x)$ is continuous on the (whole) closed interval $[a, b]$ *except* at an intermediate point c , i.e. $a < c < b$. Then we define

$$\int_a^b f(x)\,dx := \lim_{\omega\to c-}\int_a^\omega f(x)\,dx + \lim_{\omega\to c+}\int_\omega^b f(x)\,dx\ . \quad \square \tag{7.7.6}$$

Remark 7.7.3 *Let us emphasize that (7.7.5) and (7.7.6) look very similar but they have very different caclulations and meanings!*

7.8 Step-by-Step Examples

Now we give step-by-step solutions to some problems. At the end of this section there are more exercises for practice.

SOLVED EXAMPLE 7.9
Improper Integral over Infinite Interval

Evaluate the following improper integral.

$$\int_1^\infty \frac{1}{x^2} dx.$$

SOLUTION

Using Definition 7.4 first, we rewrite the improper integral as a limit. That is

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{\omega\to\infty} \int_1^\omega \frac{1}{x^2} dx.$$

Next, we evaluate the indefinite integral, that is

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C.$$

Now, we evaluate the definite integral. From Theorem 7.1 with $F(x) = -\frac{1}{x}$, we get

$$\int_1^\omega \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^\omega = -\frac{1}{\omega} - (-\frac{1}{1}) = -\frac{1}{\omega} + 1.$$

Finally, we evaluate the limit, that is

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{\omega\to\infty} \int_1^\omega \frac{1}{x^2} dx = \lim_{\omega\to\infty} (-\frac{1}{\omega} + 1) = 1.$$

So the improper integral is convergent.

SOLVED EXAMPLE 7.10
Improper Integral over Infinite Interval

Evaluate the following improper integral.

$$\int_{\sqrt{2}}^{\infty} \frac{2x}{x^2+1} dx.$$

SOLUTION
Using Definition 7.4 first, we rewrite the improper integral as a limit. That is

$$\int_{\sqrt{2}}^{\infty} \frac{2x}{x^2+1} dx = \lim_{\omega \to \infty} \int_{\sqrt{2}}^{\omega} \frac{2x}{x^2+1} dx.$$

Next, we evaluate the indefinite integral. Using Theorem 6.7, we have

$$\int \frac{2x}{x^2+1} dx = \ln\left|x^2+1\right| + C\,.$$

Now, we evaluate the definite integral. From Theorem 7.1 with $F(x) = \ln\left|x^2+1\right|$, we get

$$\int_{\sqrt{2}}^{\omega} \frac{2x}{x^2+1} dx = \left[\ln\left|x^2+1\right|\right]_{\sqrt{2}}^{\omega} = \ln\left(\omega^2+1\right) - \ln\left(\left(\sqrt{2}\right)^2+1\right).$$

Finally, we evaluate the limit, that is

$$\int_{\sqrt{2}}^{\infty} \frac{2x}{x^2+1} dx = \lim_{\omega \to \infty} \int_{\sqrt{2}}^{\omega} \frac{2x}{x^2+1} dx = \lim_{\omega \to \infty} (\ln\left(\omega^2+1\right) - \ln\left(\left(\sqrt{2}\right)^2+1\right)) = \infty.$$

This means, that this improper integral is divergent.

SOLVED EXAMPLE 7.11
Improper Integral over Infinite Interval

Evaluate the following improper integral.

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx.$$

SOLUTION

From equation 7.7.1, we have

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = \int_{-\infty}^{c} \frac{x}{x^2+1} dx + \int_{c}^{\infty} \frac{x}{x^2+1} dx$$

for any arbitrary $c \in \mathbb{R}$. Now, we rewrite both of the improper integrals as limits. That is

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = \int_{-\infty}^{c} \frac{x}{x^2+1} dx + \int_{c}^{\infty} \frac{x}{x^2+1} dx = \lim_{\omega \to -\infty} \int_{\omega}^{c} \frac{x}{x^2+1} dx + \lim_{\omega \to \infty} \int_{c}^{\omega} \frac{x}{x^2+1} dx.$$

Next, we evaluate the indefinite integral

$$\int \frac{x}{x^2+1} dx.$$

Theorem 6.7 follows, that

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \ln \left(x^2+1\right) + C,$$

as

$$x^2+1 > 0.$$

Now we evaluate the definite integrals. From Theorem 7.1 with $F(x) = \frac{1}{2} \ln \left(x^2+1\right)$, we get

$$\int_{\omega}^{c} \frac{x}{x^2+1} dx = \left[\frac{1}{2} \ln \left(x^2+1\right)\right]_{\omega}^{c} = \frac{1}{2}(\ln \left(c^2+1\right) - \ln \left(\omega^2+1\right)),$$

and

$$\int_{c}^{\omega} \frac{x}{x^2+1} dx = \left[\frac{1}{2} \ln \left(x^2+1\right)\right]_{c}^{\omega} = \frac{1}{2}(\ln \left(\omega^2+1\right) - \ln \left(c^2+1\right)).$$

Finally, we evaluate the limits. That is

$$\lim_{\omega\to-\infty}\int_{\omega}^{c}\frac{x}{x^2+1}dx = \lim_{\omega\to-\infty}\left[\frac{1}{2}\ln\left(x^2+1\right)\right]_{\omega}^{c} =$$

$$= \lim_{\omega\to-\infty}\frac{1}{2}(\ln\left(c^2+1\right)-\ln\left(\omega^2+1\right)) =$$

$$= \frac{1}{2}\ln\left(c^2+1\right)-\infty.$$

Similarly,

$$\lim_{\omega\to\infty}\int_{c}^{\omega}\frac{x}{x^2+1}dx = \lim_{\omega\to\infty}\left[\frac{1}{2}\ln\left(x^2+1\right)\right]_{c}^{\omega} =$$

$$= \lim_{\omega\to\infty}\frac{1}{2}(\ln\left(\omega^2+1\right)-\ln\left(c^2+1\right)) =$$

$$= \infty-\frac{1}{2}\ln\left(c^2+1\right).$$

This yields,

$$\int_{-\infty}^{\infty}\frac{x}{x^2+1}dx = \lim_{\omega\to-\infty}\int_{\omega}^{c}\frac{x}{x^2+1}dx+\lim_{\omega\to\infty}\int_{c}^{\omega}\frac{x}{x^2+1}dx =$$

$$= \frac{1}{2}\ln\left(c^2+1\right)-\infty+\infty-\frac{1}{2}\ln\left(c^2+1\right).$$

So the improper integral is divergent.

SOLVED EXAMPLE 7.12
Improper Integral over Infinite Interval

Evaluate the following improper integral.

$$\int_{-\infty}^{+\infty} \frac{3}{x^2+4} dx.$$

SOLUTION

Function $f(x) = \dfrac{3}{x^2+4}$ is *even* (symmetric to the y axis, i.e. $f(-x) = f(x)$ for all $x \in \mathbb{R}$), so

$$\int_{-\infty}^{+\infty} \frac{3}{x^2+4} dx = 2 \cdot \int_{0}^{+\infty} \frac{3}{x^2+4} dx.$$

Now, we rewrite the improper integral as a limit. That is

$$\int_{0}^{+\infty} \frac{3}{x^2+4} dx = \lim_{\omega\to\infty} \int_{0}^{\omega} \frac{3}{x^2+4} dx.$$

Next, we evaluate the indefinite integral. Theorem 6.6 follows, that

$$\int \frac{3}{x^2+4} dx = \frac{3}{4} \int \frac{1}{\frac{x^2}{4}+1} dx = \frac{3}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2+1} dx =$$

$$= \frac{3}{4} \cdot \frac{\arctan\left(\frac{x}{2}\right)}{1/2} + C = \frac{6}{4} \cdot \arctan\left(\frac{x}{2}\right) + C.$$

Next, we evaluate the definite integral.

From Theorem 7.1 with $F(x) = \dfrac{6}{4} \cdot \arctan\left(\dfrac{x}{2}\right)$, we get

$$\int_{0}^{\omega} \frac{3}{x^2+4} dx = \frac{6}{4} \left[\arctan\left(\frac{x}{2}\right)\right]_0^{\omega} = \frac{6}{4} \left(\arctan\left(\frac{\omega}{2}\right) - \arctan(0)\right) = \frac{6}{4} \arctan\left(\frac{\omega}{2}\right).$$

Finally, we evaluate the limit. That is

$$\int_{0}^{+\infty} \frac{3}{x^2+4} dx = \lim_{\omega\to\infty} \int_{0}^{\omega} \frac{3}{x^2+4} dx = \lim_{\omega\to\infty} \frac{6}{4} \arctan\left(\frac{\omega}{2}\right) = \frac{6}{4} \cdot \frac{\pi}{2} = \frac{3\pi}{4}.$$

So the result is

$$\int_{-\infty}^{+\infty} \frac{3}{x^2+4} dx = 2 \cdot \int_{0}^{+\infty} \frac{3}{x^2+4} dx = 2 \cdot \frac{3\pi}{4} = \frac{3\pi}{2} \approx 4.712389.$$

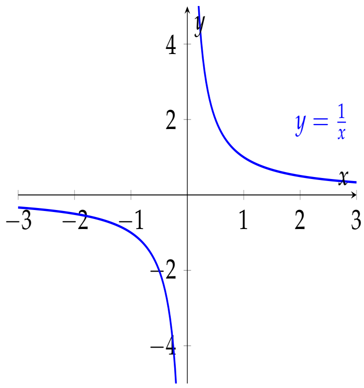
This means, that this improper integral is convergent.

SOLVED EXAMPLE 7.13
Improper Integral - Discontinuous Function

Evaluate the following improper integral.

$$\int_0^2 \frac{1}{x} dx.$$

SOLUTION
Function $\frac{1}{x}$ is discontinous at $x = 0$.



Using Definition 7.6 first, we rewrite the improper integral as a limit. That is

$$\int_0^2 \frac{1}{x} dx = \lim_{\omega \to 0+} \int_\omega^2 \frac{1}{x} dx.$$

Next, we evaluate the indefinite integral, that is

$$\int \frac{1}{x} dx = \ln|x| + C.$$

Now, we evaluate the definite integrals. From Theorem 7.1 with $F(x) = \ln|x|$, we get

$$\int_\omega^2 \frac{1}{x} dx = [\ln|x|]_\omega^2 = \ln|2| - \ln|\omega| = \ln(2) - \ln(\omega),$$

as $\omega > 0$. Finally, we evaluate the limit, that is

$$\int_0^2 \frac{1}{x} dx = \lim_{\omega \to 0+} \int_\omega^2 \frac{1}{x} dx = \lim_{\omega \to 0+} (\ln(2) - \ln(\omega)) = +\infty.$$

So the improper integral is divergent.

SOLVED EXAMPLE 7.14

Improper Integral - Discontinuous Function

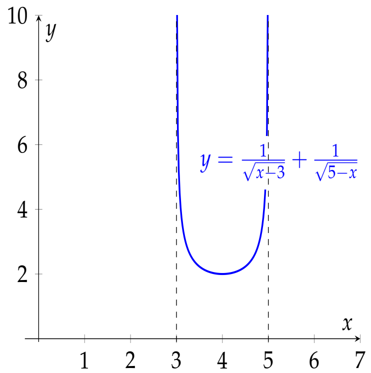
Evaluate the following improper integral.

$$\int_3^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx.$$

SOLUTION

Function $f(x) = \frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}$ is continous on the whole *open* interval $(3,5)$ while can not be calculated at the endpoints $a = 3$ and $b = 5$. So we have to use (7.7.5) (choosing e.g. $c = 4$)

$$\int_3^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx = \int_3^c \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx + \int_c^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx.$$



Using Definition 7.6 first, we rewrite the improper integrals as limits. That is

$$\int_3^c \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx + \int_c^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx =$$

$$= \lim_{\omega \to 3+} \int_\omega^c \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx + \lim_{\omega \to 5-} \int_c^\omega \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx.$$

Next, we evaluate the indefinite integral, that is

$$\int \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}} \right) dx = 2\sqrt{x-3} - 2\sqrt{5-x} + C.$$

Now, we evaluate the definite integrals. From Theorem 7.1 with

$$F(x) = 2\sqrt{x-3} - 2\sqrt{5-x},$$

we get

$$\begin{aligned}\int_{\omega}^{c} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}\right) dx &= \left[2\sqrt{x-3} - 2\sqrt{5-x}\right]_{\omega}^{c} = \\ &= 2\sqrt{c-3} - 2\sqrt{5-c} - \left(2\sqrt{\omega-3} - 2\sqrt{5-\omega}\right),\end{aligned}$$

and

$$\begin{aligned}\int_{c}^{\omega} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}\right) dx &= \left[2\sqrt{x-3} - 2\sqrt{5-x}\right]_{c}^{\omega} = \\ &= 2\sqrt{\omega-3} - 2\sqrt{5-\omega} - \left(2\sqrt{c-3} - 2\sqrt{5-c}\right).\end{aligned}$$

Finally, we evaluate the limits, that is

$$\begin{aligned}\int_{3}^{c} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}\right) dx &= \lim_{\omega\to 3+} \int_{\omega}^{c} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}\right) dx = \\ &= \lim_{\omega\to 3+} \left(2\sqrt{c-3} - 2\sqrt{5-c} - \left(2\sqrt{\omega-3} - 2\sqrt{5-\omega}\right)\right) = \\ &= 2\sqrt{c-3} - 2\sqrt{5-c} - 0 + 2\sqrt{2}.\end{aligned}$$

and

$$\begin{aligned}\int_{c}^{5} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}\right) dx &= \lim_{\omega\to 5-} \int_{c}^{\omega} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}\right) dx = \\ &= \lim_{\omega\to 5-} \left(2\sqrt{\omega-3} - 2\sqrt{5-\omega} - \left(2\sqrt{c-3} - 2\sqrt{5-c}\right)\right) = \\ &= 2\sqrt{2} - 0 - 2\sqrt{c-3} + 2\sqrt{5-c}.\end{aligned}$$

So, the improper integral

$$\begin{aligned}\int_{3}^{5} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{\sqrt{5-x}}\right) dx &= 2\sqrt{c-3} - 2\sqrt{5-c} + 2\sqrt{2} + 2\sqrt{2} - 2\sqrt{c-3} + 2\sqrt{5-c} = \\ &= 4\sqrt{2} \approx 5.65685.\end{aligned}$$

is convergent.

SOLVED EXAMPLE 7.15
Improper Integral - Discontinuous Function

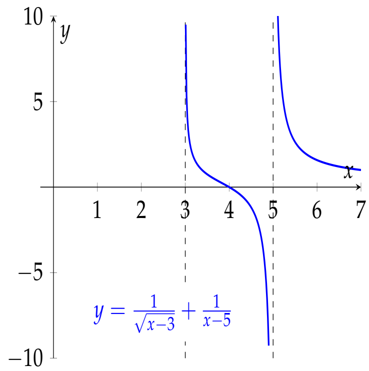
Evaluate the following improper integral.

$$\int_3^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx.$$

SOLUTION

Function $f(x) = \frac{1}{\sqrt{x-3}} + \frac{1}{x-5}$ is continous on the whole *open* interval $(3,5)$ while can not be calculated at the endpoints $a = 3$ and $b = 5$. So we have to use (7.7.5) (choosing e.g. $c = 4$)

$$\int_3^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = \int_3^c \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx + \int_c^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx.$$



Using Definition 7.6 first, we rewrite the improper integrals as limits. That is

$$\int_3^c \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx + \int_c^5 \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx =$$

$$= \lim_{\omega \to 3+} \int_\omega^c \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx + \lim_{\omega \to 5-} \int_c^\omega \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx.$$

Next, we evaluate the indefinite integral, that is

$$\int \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = 2\sqrt{x-3} + \ln|x-5| + C.$$

Now, we evaluate the definite integrals. From Theorem 7.1 with $F(x) = 2\sqrt{x-3} + \ln|x-5|$, we get

$$\int_{\omega}^{c} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = \left[2\sqrt{x-3} + \ln|x-5| \right]_{\omega}^{c} = $$
$$= 2\sqrt{c-3} + \ln|c-5| - \left(2\sqrt{\omega-3} + \ln|\omega-5| \right),$$

and

$$\int_{c}^{\omega} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = \left[2\sqrt{x-3} + \ln|x-5| \right]_{c}^{\omega} = $$
$$= 2\sqrt{\omega-3} + \ln|\omega-5| - \left(2\sqrt{c-3} + \ln|c-5| \right).$$

Finally, we evaluate the limits, that is

$$\int_{3}^{c} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = \lim_{\omega \to 3+} \int_{\omega}^{c} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = $$
$$= \lim_{\omega \to 3+} \left(2\sqrt{c-3} + \ln|c-5| - \left(2\sqrt{\omega-3} + \ln|\omega-5| \right) \right) = $$
$$= 2\sqrt{c-3} + \ln|c-5| - 0 - \ln(2).$$

and

$$\int_{c}^{5} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = \lim_{\omega \to 5-} \int_{c}^{\omega} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = $$
$$= \lim_{\omega \to 5-} \left(2\sqrt{\omega-3} + \ln|\omega-5| - \left(2\sqrt{c-3} + \ln|c-5| \right) \right) = $$
$$= 2\sqrt{2} - \infty - 2\sqrt{c-3} - \ln|c-5| = -\infty.$$

So, the improper integral

$$\int_{3}^{5} \left(\frac{1}{\sqrt{x-3}} + \frac{1}{x-5} \right) dx = 2\sqrt{c-3} + \ln|c-5| - 0 - \ln(2) + (-\infty)$$

is divergent, since one of the above integrals is divergent (namely $\int_c^5 = -\infty$).

SOLVED EXAMPLE 7.16

Improper Integral - Discontinuous Function

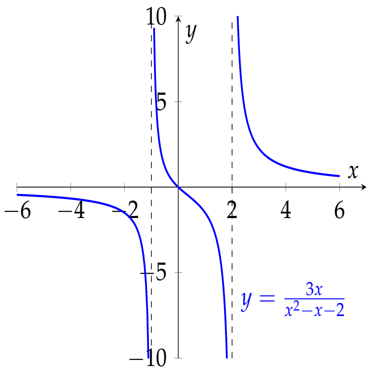
Evaluate the following improper integral.

$$\int_0^3 \frac{3x}{x^2-x-2}\,dx.$$

SOLUTION

Since function $f(x) = \dfrac{3x}{x^2-x-2}$ is continuous for all real numbers $x \in \mathbb{R}$ *except* $x_1 = -1$ and $x_2 = 2$, over interval $[0,3]$, we must use (7.7.6) with $c = x_2 = 2$.

$$\int_0^3 \frac{3x}{x^2-x-2}\,dx = \int_0^2 \frac{3x}{x^2-x-2}\,dx + \int_2^3 \frac{3x}{x^2-x-2}\,dx.$$



Using Definition 7.6 first, we rewrite the improper integrals as limits. That is

$$\int_0^2 \frac{3x}{x^2-x-2}\,dx + \int_2^3 \frac{3x}{x^2-x-2}\,dx = \lim_{\omega\to 2-}\int_0^{\omega} \frac{3x}{x^2-x-2}\,dx + \lim_{\omega\to 2+}\int_{\omega}^3 \frac{3x}{x^2-x-2}\,dx.$$

Next, we evaluate the indedinite integral. Denumerator can be factorized as

$$x^2-x-2 = (x+1)(x-2),$$

and we decompose the function before calculating its primitive function. (See details in `https://math.uni-pannon.hu/~szalkai/ParcTort-pdfw.pdf`.)

$$f(x) = \frac{3x}{x^2-x-2} = \frac{3x}{(x+1)(x-2)} = \frac{1}{x+1} + \frac{2}{x-2},$$

so

$$\int \frac{3x}{x^2-x-2}dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx = \ln|x+1| + 2\ln|x-2| + C =$$

$$= \ln\left|(x+1)\cdot(x-2)^2\right| + C.$$

Now, we evaluate the definite integrals. From Theorem 7.1 with

$$F(x) = \ln\left|(x+1)\cdot(x-2)^2\right| = \ln\left(|x+1|\cdot(x-2)^2\right),$$

we get

$$\begin{aligned}\int_0^{\omega} \frac{3x}{x^2-x-2}\,dx &= \left[\ln|x+1|\cdot(x-2)^2\right]_0^{\omega} = \\ &= \ln|\omega+1|\cdot(\omega-2)^2 - \ln|0+1|\cdot(0-2)^2 = \\ &= \ln|\omega+1|\cdot(\omega-2)^2 - \ln(4),\end{aligned}$$

and

$$\begin{aligned}\int_{\omega}^{3} \frac{3x}{x^2-x-2}\,dx &= \left[\ln|x+1|\cdot(x-2)^2\right]_0^{\omega} = \\ &= \ln|3+1|\cdot(3-2)^2 - \ln|\omega+1|\cdot(\omega-2)^2 = \\ &= \ln(4) - \ln|\omega+1|\cdot(\omega-2)^2.\end{aligned}$$

Finally, we evaluate the limits. That is

$$\begin{aligned}\int_0^{2} \frac{3x}{x^2-x-2}\,dx &= \lim_{\omega\to 2-}\int_0^{\omega} \frac{3x}{x^2-x-2}\,dx = \\ &= \lim_{\omega\to 2-}\left(\ln|\omega+1|\cdot(\omega-2)^2 - \ln(4)\right) = \infty - \ln(4) = \infty.\end{aligned}$$

and

$$\begin{aligned}\int_2^{3} \frac{3x}{x^2-x-2}\,dx &= \lim_{\omega\to 2+}\int_{\omega}^{3} \frac{3x}{x^2-x-2}\,dx = \\ &= \lim_{\omega\to 2+}\left(\ln(4) - \ln|\omega+1|\cdot(\omega-2)^2\right) = (4) - \infty = -\infty.\end{aligned}$$

So, the improper integral

$$\int_0^{3} \frac{3x}{x^2-x-2}\,dx = \infty + (-\infty)$$

is divergent, since both of the above integrals are divergent.

WARNING

We would have a disaster if the discontinuous point $x_2 = 2$ were not observed, since automatic application of Newton-Leibniz Rule would give the following **bad** result.

$$\int_0^3 \frac{3x}{x^2 - x - 2}\, dx = \quad \textbf{/FALSE!/} \quad = \left[\ln|x+1| \cdot (x-2)^2\right]_0^3 =$$

$$= \ln|3+1| \cdot (3-2)^2 - \ln|0+1| \cdot (0-2)^2 = 0\,.$$

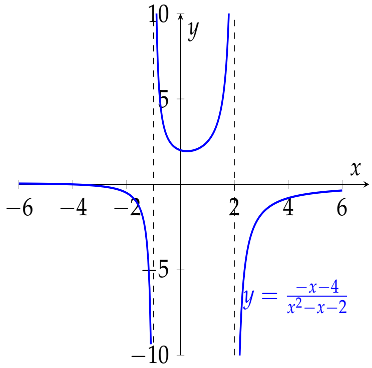
SOLVED EXAMPLE 7.17
Improper Integral - Discontinuous Function

Evaluate the following improper integral.

$$\int_0^3 \frac{-x-4}{x^2-x-2}\, dx.$$

SOLUTION
Since function $f(x) = \dfrac{-x-4}{x^2-x-2}$ is continuous for all real numbers $x \in \mathbb{R}$ *except* $x_1 = -1$ and $x_2 = 2$, over interval $[0,3]$, we must use (7.7.6) with $c = x_2 = 2$.

$$\int_0^3 \frac{-x-4}{x^2-x-2}\, dx = \int_0^2 \frac{-x-4}{x^2-x-2}\, dx + \int_2^3 \frac{-x-4}{x^2-x-2}\, dx.$$



Using Definition 7.6 first, we rewrite the improper integrals as limits. That is

$$\int_0^2 \frac{-x-4}{x^2-x-2}\, dx + \int_2^3 \frac{-x-4}{x^2-x-2}\, dx = \lim_{\omega \to 2-} \int_0^{\omega} \frac{-x-4}{x^2-x-2}\, dx + \lim_{\omega \to 2+} \int_{\omega}^3 \frac{-x-4}{x^2-x-2}\, dx.$$

Next, we evaluate the indedinite integral. Denumerator can be factorized as

$$x^2 - x - 2 = (x+1)(x-2),$$

and we decompose the function before calculating its primitive function. (See details in https://math.uni-pannon.hu/~szalkai/ParcTort-pdfw.pdf.)

$$f(x) = \frac{-x-4}{x^2-x-2} = \frac{-x-4}{(x+1)(x-2)} = \frac{1}{x+1} - \frac{2}{x-2},$$

so

$$\int \frac{-x-4}{x^2-x-2}dx = \int \left(\frac{1}{x+1} - \frac{2}{x-2}\right) dx = \ln|x+1| - 2\ln|x-2| + C =$$

$$= \ln\left|\frac{x+1}{(x-2)^2}\right| + C.$$

Now, we evaluate the definite integrals. From Theorem 7.1 with

$$F(x) = \ln\left|\frac{x+1}{(x-2)^2}\right|,$$

we get

$$\int_0^\omega \frac{-x-4}{x^2-x-2}\,dx = \left[\ln\left|\frac{x+1}{(x-2)^2}\right|\right]_0^\omega = \ln\left|\frac{\omega+1}{(\omega-2)^2}\right| - \ln\left|\frac{0+1}{(0-2)^2}\right| =$$

$$= \ln\left|\frac{\omega+1}{(\omega-2)^2}\right| - \ln\left(\frac{1}{4}\right),$$

and

$$\int_\omega^3 \frac{-x-4}{x^2-x-2}\,dx = \left[\ln\left|\frac{x+1}{(x-2)^2}\right|\right]_\omega^3 = \ln\left|\frac{3+1}{(3-2)^2}\right| - \ln\left|\frac{\omega+1}{(\omega-2)^2}\right| =$$

$$= \ln(4) - \ln\left|\frac{\omega+1}{(\omega-2)^2}\right|.$$

Finally, we evaluate the limits, that is

$$\int_0^2 \frac{-x-4}{x^2-x-2}\,dx = \lim_{\omega\to 2-} \int_0^\omega \frac{-x-4}{x^2-x-2}\,dx =$$

$$= \lim_{\omega\to 2-} \left(\ln\left|\frac{\omega+1}{(\omega-2)^2}\right| - \ln\left(\frac{1}{4}\right)\right) = \infty - \ln\left(\frac{1}{4}\right) = \infty.$$

and

$$\int_2^3 \frac{-x-4}{x^2-x-2}\,dx = \lim_{\omega\to 2+} \int_\omega^3 \frac{-x-4}{x^2-x-2}\,dx =$$

$$= \lim_{\omega\to 2+} \left(\ln(4) - \ln\left|\frac{\omega+1}{(\omega-2)^2}\right|\right) = (4) - \infty = -\infty.$$

So improper integral

$$\int_0^3 \frac{-x-4}{x^2-x-2}\,dx = \infty + (-\infty)$$

is divergent, since both of the above integrals is divergent.

WARNING

We would have a disaster if the discontinuous point $x_2 = 2$ were not observed, since automatic application of Newton-Leibniz Rule would give the following **bad** result.

$$\int_0^3 \frac{-x-4}{x^2-x-2}\,dx = \quad /\textbf{FALSE!}/ \quad = \left[\ln\left|\frac{x+1}{(x-2)^2}\right|\right]_0^3 =$$

$$= \ln\left|\frac{3+1}{(3-2)^2}\right| - \ln\left|\frac{0+1}{(0-2)^2}\right| = 2\ln 4 \approx 2.77259\,.$$

7.9 Exercises

The solutions of the following problems can be found in Chapter 8. *Solutions*.

7.9.1 Integrating over Infinite Intervals

Exercises 7.6

Calculate the following improper integrals.

1.
$$\int_{-\infty}^{-4} \frac{x+1}{x^2+2x-3} dx,$$

See Solution 8.7.46

2. [4]
$$\int_{-\infty}^{-4} \frac{7}{x^2+2x-3} dx,$$

[4] See also Exercise 7.9 1.

See Solution 8.7.47

3.
$$\int_{\sqrt{2}}^{\infty} \frac{2x-3}{x^2+1} dx,$$

See Solution 8.7.48

4.
$$\int_{0}^{\infty} e^{-x}\, dx,$$

See Solution 8.7.49

5.
$$\int_{0}^{\infty} \frac{2}{e^x+e^{-x}} dx,$$

See Solution 8.7.50

6.
$$\int_{1}^{\infty} \frac{1}{x^3} \cdot \exp\left(\frac{-1}{x^2}\right)\, dx,$$

See Solution 8.7.51

7.
$$\int_{1}^{\infty} \frac{dx}{\arctan(x) \cdot (x^2+1)},$$

See Solution 8.7.52

8.

$$\int_{-\infty}^{7} \frac{1}{x^2 + 2x + 10}\, dx,$$

See Solution 8.7.53

9.

$$\int_{0}^{\infty} \frac{1}{x^2 + 4x + 6}\, dx.$$

See Solution 8.7.54

Exercises 7.7

Calculate the following improper integrals.

1.

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 8}\, dx,$$

See Solution 8.7.55

2.

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 5},$$

See Solution 8.7.56

3.

$$\int_{-\infty}^{\infty} \frac{\arctan(x)}{x^2 + 1} dx,$$

See Solution 8.7.57

4. *

$$\int_{-\infty}^{+\infty} \frac{8a^3}{x^2 + 4a^2} dx,$$

where $a \in \mathbb{R}$.

See Solution 8.7.58

7.9.2 *Integrating Discontinuous Functions*

Exercises 7.8

Calculate the following improper integrals. Find all the points in the interval where the function is discontinuous.

1.

$$\int_{0}^{1} \frac{2x}{1 - x^2}\, dx,$$

See Solution 8.7.59

2.
$$\int_0^1 \frac{e^x+1}{e^{2x}-1}dx,$$

See Solution 8.7.60

3.
$$\int_0^1 \frac{\ln x}{2\sqrt{x}}\,dx,$$

See Solution 8.7.61

4.
$$\int_0^1 \frac{\ln x}{x^3}\,dx,$$

See Solution 8.7.62

5.
$$\int_0^1 \frac{\ln x}{\sqrt[3]{x}}\,dx,$$

See Solution 8.7.63

6. *
$$\int_0^1 \frac{1}{x\cdot\ln^2 x}\,dx,$$

See Solution 8.7.64

7.
$$\int_0^1 \frac{1}{\sqrt{x}-1}\,dx,$$

See Solution 8.7.65

8.
$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx,$$

See Solution 8.7.66

9.
$$\int_0^9 \frac{1}{\sqrt{x}\cdot(x-9)}\,dx,$$

See Solution 8.7.67

10.
$$\int_{-2}^0 \frac{-x^2+x-3}{(x^2+5)\,(x+2)}dx,$$

See Solution 8.7.68

11.
$$\int_{-1}^0 \frac{x^2-x+1}{(x^2+2)\,(x+1)}dx,$$

See Solution 8.7.69

12.

$$\int_{0}^{5} \frac{3x}{x^2 - x - 2}\, dx.$$

See Solution 8.7.70

On top of all, many integrals may contain all of the above improper properties discussed so far.

Exercises 7.9

Calculate the following improper integrals. Be aware of the critical inner points, too!
(See also Exercise 7.6. 2.)

1.

$$\int_{-\infty}^{0} \frac{7}{x^2 + 2x - 3} dx,$$

See Solution 8.7.71

2. *

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + x}\, dx.$$

See Solution 8.7.72

8 Solutions

8.1 The Composite Functions

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$f(x) = x^2 - 9, \quad x \in [0,5],$$
$$g(x) = 3x + 1, \quad x \in [0,7].$$

Solution 8.1.1 *From*

$$0 \le 3x + 1 \le 5,$$

$$-1 \le 3x \le 4,$$

we get

$$\frac{-1}{3} \le x \le \frac{4}{3}.$$

As

$$\left[-\frac{1}{3}, \frac{4}{3}\right] \cap [0,7] = \left[0, \frac{4}{3}\right],$$

the composit funcition exists and

$$dom(f(g)) = \left[0, \frac{4}{3}\right],$$

$$f(g(x)) = (3x+1)^2 - 9.$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$f(x) = 3x^2 + 5, \quad x \in [1,125],$$
$$g(x) = 2x + 3, \quad x \in [0,100].$$

Solution 8.1.2 *From*

$$1 \le 2x + 3 \le 125,$$

$$-2 \leq 2x \leq 122,$$

we get

$$-1 \leq x \leq 61.$$

As

$$[-1, 61] \cap [0, 100] = [0, 61],$$

the composit funcition exists and

$$\begin{aligned} dom\,(f\,(g)) &= [0, 61], \\ f\,(g\,(x)) &= 3\,(2x+3)^2 + 5. \end{aligned}$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f\,(x) &= 8 - \sin\,(x), \quad x \in [1, 125], \\ g\,(x) &= 5^x, \quad x \in [-1, 10]. \end{aligned}$$

Solution 8.1.3 *From*

$$1 \leq 5^x \leq 125,$$

$$\log_5\,(1) \leq \log_5\,(5^x) \leq \log_5\,(125),$$

$$\log_5\left(5^0\right) \leq \log_5\,(5^x) \leq \log_5\left(5^3\right),$$

we get

$$0 \leq x \leq 3.$$

As

$$[0, 3] \cap [-1, 10] = [0, 3],$$

the composit funcition exists and

$$\begin{aligned} dom\,(f\,(g)) &= [0, 3], \\ f\,(g\,(x)) &= 8 - \sin\,(5^x). \end{aligned}$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f\,(x) &= \sin^3(x), \quad x \in [1, 25], \\ g\,(x) &= x^2, \quad x \in [0, 3]. \end{aligned}$$

Solution 8.1.4 *From*

$$1 \leq x^2 \leq 25,$$
$$\sqrt{1} \leq \sqrt{x^2} \leq \sqrt{25},$$
$$1 \leq |x| \leq 5,$$

we get

$$-5 \leq x \leq -1 \qquad or \qquad 1 \leq x \leq 5.$$

As

$$([-5, -1] \cup [1, 5]) \cap [0, 3] = [1, 3],$$

the composit funcition exists and

$$\begin{aligned} dom\,(f\,(g)) &= [1, 3], \\ f\,(g\,(x)) &= \sin^3\left(x^2\right). \end{aligned}$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f\,(x) &= \frac{\cos\,(x)}{5} - x, \quad x \in [1, 4], \\ g\,(x) &= \log_2\,(x), \quad x \in [1, 10]. \end{aligned}$$

Solution 8.1.5 *From*

$$1 \leq \log_2\,(x) \leq 4,$$
$$2^1 \leq 2^{\log_2(x)} \leq 2^4,$$

we get

$$2 \leq x \leq 16.$$

As

$$[2, 16] \cap [1, 10] = [2, 10],$$

the composit funcition exists and

$$\begin{aligned} dom\,(f\,(g)) &= [2, 10], \\ f\,(g\,(x)) &= \frac{\cos\,(\log_2\,(x))}{5}. \end{aligned}$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f(x) &= 6 - 9\cos(x), \quad x \in [0, 15], \\ g(x) &= x^2 + 2x, \quad x \in [1, 10]. \end{aligned}$$

Solution 8.1.6 *To solve*

$$0 \le x^2 + 2x \le 15$$

we use that

$$x^2 + 2x = (x+1)^2 - 1.$$

From

$$\begin{gathered} 0 \le (x+1)^2 - 1 \le 15 \\ 1 \le (x+1)^2 \le 16 \\ \sqrt{1} \le \sqrt{(x+1)^2} \le \sqrt{16} \\ 1 \le |x+1| \le 4, \end{gathered}$$

we get

$$\begin{aligned} -4 \le x+1 \le -1 \quad &or \quad 1 \le x+1 \le 4. \\ -5 \le x \le -2 \quad &or \quad 0 \le x \le 3. \end{aligned}$$

As

$$([-5, -2] \cup [0, 3]) \cap [1, 10] = [1, 3],$$

the composit funcition exists and

$$\begin{aligned} dom(f(g)) &= [1, 3], \\ f(g(x)) &= 6 - 9\cos\left(x^2 + 2x\right). \end{aligned}$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f(x) &= x^2 - 6x - 83, \quad x \in [4, 7], \\ g(x) &= \sqrt{x} + 3, \quad x \in [0, 23]. \end{aligned}$$

Solution 8.1.7 *From*

$$\begin{gathered} 4 \le \sqrt{x} + 3 \le 7, \\ 1 \le \sqrt{x} \le 4, \end{gathered}$$

we get

$$1 \leq x \leq 16.$$

As

$$[1,16] \cap [0,23] = [1,16],$$

the composit funcition exists and

$$\begin{aligned} dom\left(f\left(g\right)\right) &= [1,16], \\ f\left(g\left(x\right)\right) &= \left(\sqrt{x}+3\right)^2 - 6\left(\sqrt{x}+3\right) - 83. \end{aligned}$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f\left(x\right) &= \sqrt{4x^2+2}+3x, \quad x \in [4,7], \\ g\left(x\right) &= \ln\left(x\right)+3, \quad x \in [0,23]. \end{aligned}$$

Solution 8.1.8 *From*

$$4 \leq \ln\left(x\right)+3 \leq 7,$$

$$1 \leq \ln\left(x\right) \leq 4,$$

$$e^1 \leq e^{\ln(x)} \leq e^4,$$

we get

$$e \leq x \leq e^4.$$

As

$$\left[e, e^4\right] \cap [0,23] = [e,23],$$

the composit funcition exists and

$$\begin{aligned} dom\left(f\left(g\right)\right) &= [e,23], \\ f\left(g\left(x\right)\right) &= \sqrt{4\left(\ln\left(x\right)+3\right)^2+2}+3\left(\ln\left(x\right)+3\right). \end{aligned}$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f\left(x\right) &= \sqrt{x+3}, \quad x \in [3,4], \\ g\left(x\right) &= \cos\left(x\right)+3, \quad x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]. \end{aligned}$$

Solution 8.1.9 *From*

$$3 \le \cos(x) + 3 \le 4,$$

$$0 \le \cos(x) \le 1,$$

we get

$$-\frac{\pi}{2} + 2k\pi \le x \le \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}.$$

As

$$\left(\bigcup_{k \in \mathbb{Z}} \left[-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi \right] \right) \cap \left[\frac{\pi}{4}, \frac{3\pi}{4} \right] = \left[\frac{\pi}{4}, \frac{\pi}{2} \right],$$

the composit funcition exists and

$$dom(f(g)) = \left[\frac{\pi}{4}, \frac{\pi}{2} \right],$$

$$f(g(x)) = \sqrt{\cos(x) + 3 + 3}.$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$f(x) = \sqrt{1 - x}, \quad x \le 1,$$

$$g(x) = x^2, \quad x \in \mathbb{R}.$$

Solution 8.1.10 *From*

$$x^2 \le 1,$$

$$0 \le x^2 \le 1,$$

$$\sqrt{0} \le \sqrt{x^2} \le \sqrt{1},$$

we get

$$0 \le |x| \le 1,$$

so

$$-1 \le x \le 1.$$

As

$$[-1, 1] \cap \mathbb{R} = [-1, 1],$$

the composit funcition exists and

$$dom(f(g)) = [-1, 1],$$

$$f(g(x)) = \sqrt{1 - x^2}.$$

Step-by-Step Solution

Find the composite function $f \circ g$, if it exists.

$$\begin{aligned} f(x) &= \sqrt{1-x}, \quad x \leq 1, \\ g(x) &= x^2, \quad x \in [2,4]. \end{aligned}$$

Solution 8.1.11 *From*

$$x^2 \leq 1,$$

$$0 \leq x^2 \leq 1,$$

$$\sqrt{0} \leq \sqrt{x^2} \leq \sqrt{1},$$

we get

$$0 \leq |x| \leq 1,$$

so

$$-1 \leq x \leq 1.$$

As

$$[-1,1] \cap [2,4] = \emptyset,$$

the composit funcition does not exist.

8.2 The Inverse Function

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = (x-2)^2, \quad x \in [1,3].$$

Solution 8.2.1 1. *Is f a one-to-one function?*

Let $1 \le x_1, x_2 \le 3$ be any and

$$(x_1 - 2)^2 = (x_2 - 2)^2 \overset{?}{\Longrightarrow} x_1 = x_2.$$

As $f(1) = (1-2)^2 = (-1)^2 = 1$ and $f(3) = (3-2)^2 = 1^2 = 1$ f is not a one-to-one function, so the inverse function does not exist.

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = 5x + 1, \quad x \in [1,5].$$

Solution 8.2.2 1. *Is f a one-to-one function?*

Let $1 \le x_1, x_2 \le 5$ be any and

$$5x_1 + 1 = 5x_2 + 1 \overset{?}{\Longrightarrow} x_1 = x_2.$$

From

$$5x_1 + 1 = 5x_2 + 1,$$
$$5x_1 = 5x_2,$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f exists.

2. *Now we determine the domain of function f^{-1}. As dom $(f^{-1}) = im(f)$, we determine the range of function f. As dom $(f) = [1,5]$, we have*

$$1 \le x \le 5.$$

From this

$$5 \le 5x \le 25,$$
$$6 \le 5x + 1 \le 26.$$

Thus,

$$6 \le (y = f(x) =) \, 5x + 1 \le 25,$$

$$dom\left(f^{-1}\right) = im\,(f) = [6, 26]\,.$$

3. *Finally, we solve* $y = f(x)$ *for "*$x =$*". So consider*

$$y = 5x + 1,$$

where $6 \leq y \leq 26$ *and* $1 \leq x \leq 5$. *As*

$$y - 1 = 5x,$$

we find

$$\frac{1}{5}y - \frac{1}{5} = x.$$

So the inverse of f *exists and*

$$f^{-1}(x) = \frac{1}{5}x - \frac{1}{5}, \quad x \in [6, 26]\,,$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = x^2 + 2x, \quad x \in [-1, 2]\,.$$

Solution 8.2.3 *To determine the inverse of function* f *first we have to rewrite* $f(x)$. *To solve this problem we write*

$$f(x) = (x+1)^2 - 1, \quad x \in [-1, 2]\,.$$

1. *Is* f *a one-to-one function?*

   Let $-1 \leq x_1, x_2 \leq 2$ *be any and*

$$(x_1 + 1)^2 - 1 = (x_2 + 1)^2 - 1 \overset{?}{\Longrightarrow} x_1 = x_2.$$

   From

$$(x_1 + 1)^2 - 1 = (x_2 + 1)^2 - 1,$$
$$(x_1 + 1)^2 = (x_2 + 1)^2,$$
$$|x_1 + 1| = |x_2 + 1|$$

   but $-1 \leq x_1, x_2 \leq 2$, *so from the definition of absolute value we get*

$$x_1 + 1 = x_2 + 1,$$

   this follows

$$x_1 = x_2.$$

   This implies that the inverse of f *exists.*

2. *Now we determine the domain of function* f^{-1}. *As dom* $\left(f^{-1}\right) = im\,(f)$, *we determine the range of function* f. *As dom* $(f) = [-1, 2]$, *we have*

$$-1 \leq x \leq 2.$$

From this

$$0 \leq x + 1 \leq 3,$$
$$0 \leq (x+1)^2 \leq 9,$$
$$-1 \leq (x+1)^2 - 1 \leq 8.$$

Thus,

$$-1 \leq (y = f\,(x) =)\,(x+1)^2 - 1 \leq 8,$$
$$dom\left(f^{-1}\right) = im\,(f) = [-1, 8]\,.$$

3. *Finally, we solve* $y = f\,(x)$ *for "*$x =$*". So consider*

$$y = (x+1)^2 - 1,$$

where $-1 \leq y \leq 8$ *and* $-1 \leq x \leq 2$. *As*

$$y = (x+1)^2 - 1,$$
$$y + 1 = (x+1)^2,$$
$$\sqrt{y+1} = |x+1|\,,$$

but $-1 \leq x_1, x_2 \leq 2$, *so from the definition of absolute value we get*

$$\sqrt{y+1} = x + 1,$$
$$\sqrt{y+1} - 1 = x.$$

So the inverse of f *exists and*

$$f^{-1}\,(x) = \sqrt{x+1} - 1, \quad x \in [-1, 8]\,,$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f\,(x) = 5^x + 1, \quad x \in [-1, 1]\,.$$

Solution 8.2.4 1. *Is* f *a one-to-one function?*

Let $-1 \leq x_1, x_2 \leq 1$ *be any and*

$$5^{x_1} + 1 = 5^{x_2} + 1 \stackrel{?}{\Longrightarrow} x_1 = x_2.$$

From

$$5^{x_1} + 1 = 5^{x_2} + 1,$$

we get

$$5^{x_1} = 5^{x_2},$$

$$\log_5 (5^{x_1}) = \log_5 (5^{x_2}),$$

so we have

$$x_1 = x_2.$$

This implies that the inverse of f exists.

2. *Now we determine the domain of function f^{-1}. As dom $\left(f^{-1}\right) = im\,(f)$, we determine the range of function f. As dom $(f) = [-1, 1]$, we have*

$$-1 \leq x \leq 1.$$

From this

$$5^{-1} \leq 5^x \leq 5^1,$$

$$\frac{1}{5} \leq 5^x \leq 5,$$

$$\frac{6}{5} \leq 5^x + 1 \leq 6.$$

Thus,

$$\frac{6}{5} \leq (y = f(x) =)\, 5^x + 1 \leq 6,$$

$$dom\left(f^{-1}\right) = im\,(f) = \left[\frac{6}{5}, 6\right].$$

3. *Finally, we solve $y = f(x)$ for "$x =$". So consider*

$$y = 5^x + 1,$$

where $\frac{6}{5} \leq y \leq 6$ and $-1 \leq x \leq 1$. As

$$y = 5^x + 1,$$

$$y - 1 = 5^x,$$

$$\log_5 (y - 1) = \log_5 (5^x),$$

$$\log_5 (y - 1) = x,$$

we find

$$\log_5 (x - 1) = y.$$

So the inverse of f exists and

$$f^{-1}(x) = \log_5 (x - 1), \quad x \in \left[\frac{6}{5}, 6\right],$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = 1 - \log_3(x), \quad x \in [1, 27].$$

Solution 8.2.5 1. *Is f a one-to-one function?*

Let $1 \le x_1, x_2 \le 27$ be any and

$$1 - \log_3(x_1) = 1 - \log_3(x_2) \overset{?}{\Longrightarrow} x_1 = x_2.$$

From

$$1 - \log_3(x_1) = 1 - \log_3(x_2),$$
$$-\log_3(x_1) = -\log_3(x_2),$$
$$\log_3(x_1) = \log_3(x_2),$$
$$3^{\log_3(x_1)} = 3^{\log_3(x_2)},$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f exists.

2. *Now we determine the domain of function f^{-1}. As dom $(f^{-1}) =$ im (f), we determine the range of function f. As dom $(f) = [1, 27]$, we have*

$$1 \le x \le 27.$$

From this

$$\log_3(1) \le \log_3(x) \le \log_3(27),$$
$$\log_3\left(3^0\right) \le \log_3(x) \le \log_3\left(3^3\right),$$
$$0 \le \log_3(x) \le 3,$$
$$0 \ge -\log_3(x) \ge -3,$$
$$1 \ge 1 - \log_3(x) \ge -2.$$

Thus,

$$1 \ge (y = f(x) =)\, 1 - \log_3(x) \ge -2,$$
$$dom\left(f^{-1}\right) = im\,(f) = [-2, 1].$$

3. *Finally, we solve $y = f(x)$ for "$x =$". So consider*

$$y = 1 - \log_3(x),$$

where $-2 \le y \le 1$ *and* $1 \le x \le 27$. *As*

$$y = 1 - \log_3(x),$$
$$\log_3(x) = 1 - y,$$
$$3^{\log_3(x)} = 3^{1-y},$$

we find

$$x = 3^{1-y}.$$

So the inverse of f *exists and*

$$f^{-1}(x) = 3^{1-x}, \quad x \in [-2, 1],$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = \sqrt{x-1} + 5, \quad x \in [1, 37].$$

Solution 8.2.6 *1. Is* f *a one-to-one function?*

Let $1 \le x_1, x_2 \le 37$ *be any and*

$$\sqrt{x_1 - 1} + 5 = \sqrt{x_2 - 1} + 5 \overset{?}{\Longrightarrow} x_1 = x_2.$$

From

$$\sqrt{x_1 - 1} + 5 = \sqrt{x_2 - 1} + 5,$$
$$\sqrt{x_1 - 1} = \sqrt{x_2 - 1},$$
$$x_1 - 1 = x_2 - 1,$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f *exists.*

2. *Now we determine the domain of function* f^{-1}. *As* $\mathrm{dom}\left(f^{-1}\right) = \mathrm{im}(f)$, *we determine the range of function* f. *As* $\mathrm{dom}(f) = [1, 37]$, *we have*

$$1 \le x \le 37.$$

From this

$$0 \le x - 1 \le 36,$$
$$\sqrt{0} \le \sqrt{x-1} \le \sqrt{36},$$
$$5 \le \sqrt{x-1} + 5 \le 11.$$

Thus,

$$5 \leq (y = f(x) =) \sqrt{x-1} + 5 \leq 11,$$
$$dom\left(f^{-1}\right) = im(f) = [5, 11].$$

3. *Finally, we solve* $y = f(x)$ *for "*$x =$*". So consider*

$$y = \sqrt{x-1} + 5,$$

where $5 \leq y \leq 11$ *and* $1 \leq x \leq 37$. *As*

$$y = \sqrt{x-1} + 5,$$
$$y - 5 = \sqrt{x-1},$$
$$(y-5)^2 = x - 1,$$

we find

$$(y-5)^2 + 1 = x.$$

So the inverse of f *exists and*

$$f^{-1}(x) = (x-5)^2 + 1, \quad x \in [5, 11],$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = x^2 - 4x + 3, \quad x < 0.$$

Solution 8.2.7 *To determine the inverse of function* f *first we have to rewrite* $f(x)$. *To solve this problem we write*

$$f(x) = x^2 - 4x + 3 = (x-2)^2 - 1.$$

1. *Is* f *a one-to-one function?*

   Let $x_1, x_2 < 0$ *be any and*

$$(x_1 - 2)^2 - 1 = (x_2 - 2)^2 - 1 \overset{?}{\Longrightarrow} x_1 = x_2.$$

   From

$$(x_1 - 2)^2 - 1 = (x_2 - 2)^2 - 1,$$
$$(x_1 - 2)^2 = (x_2 - 2)^2,$$
$$|x_1 - 2| = |x_2 - 2|,$$

   But $x_1, x_2 < 0$, *so from the definition of absolute value we get*

$$-(x_1 - 2) = -(x_2 - 2),$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f exists.

2. *Now we determine the domain of function f^{-1}. As dom $\left(f^{-1}\right) =$ im (f), we determine the range of function f. As dom $(f) = (-\infty, 0)$, we have*

$$x < 0.$$

From this

$$x - 2 < -2,$$
$$(x-2)^2 > 4,$$
$$(x-2)^2 - 1 > 3.$$

Thus,

$$3 < (y = f(x) =)\,(x-2)^2 - 1,$$
$$dom\left(f^{-1}\right) = im\,(f) = (3, \infty).$$

3. *Finally, we solve $y = f(x)$ for "$x =$". So consider*

$$y = (x-2)^2 - 1,$$

where $3 < y$ and $x < 0$. As

$$y = (x-2)^2 - 1,$$
$$y + 1 = (x-2)^2,$$
$$\sqrt{y+1} = |x-2|,$$

But $x_1, x_2 < 0$, so from the definition of absolute value we get

$$\sqrt{y+1} = -(x-2),$$
$$-\sqrt{y+1} = x - 2,$$

so we find

$$2 - \sqrt{y+1} = x.$$

So the inverse of f exists and

$$f^{-1}(x) = 2 - \sqrt{x+1}, \quad 3 < x.$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = \sqrt{x} - 3, \quad x \in [4, 16].$$

Solution 8.2.8 *1. Is f a one-to-one function?*

Let $4 \leq x_1, x_2 \leq 16$ be any and

$$\sqrt{x_1} - 3 = \sqrt{x_2} - 3 \overset{?}{\Longrightarrow} x_1 = x_2.$$

From

$$\sqrt{x_1} - 3 = \sqrt{x_2} - 3,$$
$$\sqrt{x_1} = \sqrt{x_2},$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f exists.

2. Now we determine the domain of function f^{-1}. As dom $\left(f^{-1}\right) =$ im (f), we determine the range of function f. As dom $(f) = [4, 16]$, we have

$$4 \leq x \leq 16.$$

From this

$$2 \leq \sqrt{x} \leq 4,$$
$$-1 \leq \sqrt{x} - 3 \leq 1.$$

Thus,

$$-1 \leq (y = f(x) =) \sqrt{x} - 3 \leq 1,$$
$$\text{dom}\left(f^{-1}\right) = \text{im}(f) = [-1, 1].$$

3. Finally, we solve $y = f(x)$ for "$x =$". So consider

$$y = \sqrt{x} - 3,$$

where $-1 \leq y \leq 1$ and $4 \leq x \leq 16$. As

$$y = \sqrt{x} - 3,$$
$$y + 3 = \sqrt{x},$$

so we find

$$(x + 3)^2 = x.$$

So the inverse of f exists and

$$f^{-1}(x) = (x + 3)^2, \quad x \in [-1, 1].$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = \log_{\frac{1}{2}}(x), \quad x \in \left[\frac{1}{2}, 4\right].$$

Solution 8.2.9 *1. Is f a one-to-one function?*

Let $\frac{1}{2} \leq x_1, x_2 \leq 4$ be any and

$$\log_{\frac{1}{2}}(x_1) = \log_{\frac{1}{2}}(x_2) \stackrel{?}{\Longrightarrow} x_1 = x_2.$$

From

$$\log_{\frac{1}{2}}(x_1) = \log_{\frac{1}{2}}(x_2),$$
$$\left(\frac{1}{2}\right)^{\log_{\frac{1}{2}}(x_1)} = \left(\frac{1}{2}\right)^{\log_{\frac{1}{2}}(x_2)},$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f exists.

2. *Now we determine the domain of function f^{-1}. As dom $\left(f^{-1}\right) = im(f)$, we determine the range of function f. As dom $(f) = \left[\frac{1}{2}, 4\right]$, we have*

$$\frac{1}{2} \leq x \leq 4.$$

As $f(x) = \log_{\frac{1}{2}}(x)$ is monotone decreasing function we get

$$\log_{\frac{1}{2}}\left(\frac{1}{2}\right) \geq \log_{\frac{1}{2}}(x) \geq \log_{\frac{1}{2}}(4),$$
$$1 \geq \log_{\frac{1}{2}}(x) \geq -2.$$

Thus,

$$-2 \leq (y = f(x) =) \log_{\frac{1}{2}}(x) \leq 1,$$
$$dom\left(f^{-1}\right) = im(f) = [-2, 1].$$

3. *Finally, we solve $y = f(x)$ for "$x =$". So consider*

$$y = \log_{\frac{1}{2}}(x),$$

where $-2 \leq y \leq 1$ and $\frac{1}{2} \leq x \leq 4$. As

$$y = \log_{\frac{1}{2}}(x),$$
$$\left(\frac{1}{2}\right)^{y} = \left(\frac{1}{2}\right)^{\log_{\frac{1}{2}}(x)},$$

so we find

$$\left(\frac{1}{2}\right)^{y} = x.$$

So the inverse of f exists and

$$f^{-1}(x) = \left(\frac{1}{2}\right)^{x}, \quad x \in [-2, 1],$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = \ln(x) + 4, \quad x \in \left[1, e^2\right].$$

Solution 8.2.10 1. *Is f a one-to-one function?*

Let $1 \leq x_1, x_2 \leq e^2$ be any and

$$\ln(x_1) + 4 = \ln(x_2) + 4 \overset{?}{\Longrightarrow} x_1 = x_2.$$

From

$$\ln(x_1) + 4 = \ln(x_2) + 4,$$
$$\ln(x_1) = \ln(x_2),$$
$$e^{\ln(x_1)} = e^{\ln(x_2)},$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f exists.

2. *Now we determine the domain of function f^{-1}. As dom $\left(f^{-1}\right) = im(f)$, we determine the range of function f. As dom $(f) = \left[1, e^2\right]$, we have*

$$1 \leq x \leq e^2.$$

From this we get

$$\ln(1) \leq \ln(x) \leq \ln\left(e^2\right),$$
$$0 \leq \ln(x) \leq 2,$$
$$4 \leq \ln(x) + 4 \leq 6.$$

Thus,

$$4 \leq (y = f(x) =) \ln(x) + 4 \leq 6,$$
$$dom\left(f^{-1}\right) = im(f) = [4, 6].$$

3. *Finally, we solve* $y = f(x)$ *for "*$x =$*". So consider*

$$y = \ln(x) + 4,$$

where $4 \le y \le 6$ *and* $1 \le x \le e^2$. *As*

$$y = \ln(x) + 4,$$

$$y - 4 = \ln(x),$$

$$e^{y-4} = e^{\ln(x)},$$

so we find

$$e^{y-4} = x.$$

So the inverse of f *exists and*

$$f^{-1}(x) = e^{x-4}, \quad x \in [4,6],$$

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$f(x) = 3^{x-2}, \quad x \in [1,2].$$

Solution 8.2.11 1. *Is* f *a one-to-one function?*

Let $1 \le x_1, x_2 \le 2$ *be any and*

$$3^{x_1-2} = 3^{x_2-2} \overset{?}{\Longrightarrow} x_1 = x_2.$$

From

$$3^{x_1-2} = 3^{x_2-2},$$

$$\log_3\left(3^{x_1-2}\right) = \log_3\left(3^{x_2-2}\right),$$

$$x_1 - 2 = x_2 - 2,$$

so we get

$$x_1 = x_2.$$

This implies that the inverse of f *exists.*

2. *Now we determine the domain of function* f^{-1}. *As* $\operatorname{dom}\left(f^{-1}\right) = \operatorname{im}(f)$, *we determine the range of function* f. *As* $\operatorname{dom}(f) = [1,2]$, *we have*

$$1 \le x \le 2.$$

From this we get

$$-1 \le x - 2 \le 0.$$

$$3^{-1} \leq 3^{x-2} \leq 3^0.$$

Thus,

$$\frac{1}{3} \leq (y = f(x) =) \, 3^{x-2} \leq 1,$$

$$dom\left(f^{-1}\right) = im\,(f) = \left[\frac{1}{3}, 1\right].$$

3. *Finally, we solve* $y = f(x)$ *for "*$x =$*". So consider*

$$y = 3^{x-2},$$

where $\frac{1}{3} \leq y \leq 1$ *and* $1 \leq x \leq 2$. *As*

$$y = 3^{x-2},$$

$$\log_3(y) = \log_3\left(3^{x-2}\right),$$

$$\log_3(y) = x - 2,$$

so we find

$$\log_3(y) + 2 = x.$$

So the inverse of f *exists and*

$$f^{-1}(x) = \log_3(x) + 2, \quad x \in \left[\frac{1}{3}, 1\right].$$

8.3 Sequences

8.3.1 Sequences with Limit "$\frac{\infty}{\infty}$"

Step-by-Step Solution

Find the inverse function of the given f function, if it exists.

$$a_n = \frac{2016n^{32} - 5n^6 + 1}{n^{32} + n^5 - 76}.$$

Solution 8.3.1

$$a_n = \frac{2016n^{32} - 5n^6 + 1}{n^{32} + n^5 - 76} = \frac{n^{32}\left(2016 - \frac{5n^6}{n^{32}} + \frac{1}{n^{32}}\right)}{n^{32}\left(1 + \frac{n^5}{n^{32}} - \frac{76}{n^{32}}\right)} =$$

$$= \frac{2016 - \frac{5}{n^{26}} + \frac{1}{n^{32}}}{1 + \frac{1}{n^{27}} - \frac{76}{n^{32}}} \to \frac{2016 - 0 + 0}{1 + 0 - 0} = 2016.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{n^{2017} + 8n^5 + 1}{n^{62} + n^7 - 98}.$$

Solution 8.3.2

$$a_n = \frac{n^{2017} + 8n^5 + 1}{n^{62} + n^7 - 98} = \frac{n^{62}\left(\frac{n^{2017}}{n^{62}} - \frac{8n^5}{n^{62}} + \frac{1}{n^{62}}\right)}{n^{62}\left(1 + \frac{n^7}{n^{62}} - \frac{98}{n^{62}}\right)} =$$

$$= \frac{n^{1955} - \frac{8}{n^{57}} + \frac{1}{n^{62}}}{1 + \frac{1}{n^{55}} - \frac{98}{n^{62}}} \to \frac{\infty - 0 + 0}{1 + 0 - 0} = \infty.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{7 - 54\sqrt{n^4+1}}{n^2+\sqrt{n-9}}.$$

Solution 8.3.3

$$a_n = \frac{7 - 54\sqrt{n^4+1}}{n^2+\sqrt{n-9}} = \frac{n^2\left(\frac{7}{n^2} - \frac{54\sqrt{n^4+1}}{n^2}\right)}{n^2\left(1+\frac{\sqrt{n-9}}{n^2}\right)} = \frac{\frac{7}{n^2} - \frac{54\sqrt{n^4+1}}{n^2}}{1+\frac{\sqrt{n-9}}{n^2}} =$$

$$= \frac{\frac{7}{n^2} - 54\sqrt{\frac{n^4+1}{n^4}}}{1+\sqrt{\frac{n-9}{n^4}}} = \frac{\frac{7}{n^2} - 54\sqrt{1+\frac{1}{n^4}}}{1+\sqrt{\frac{1}{n^3}-\frac{9}{n^4}}} \to \frac{0-54\sqrt{1+0}}{1+\sqrt{0-0}} = -54.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{\sqrt[3]{5n+1}-6}{n^2+\sqrt[3]{n}-8}.$$

Solution 8.3.4

$$a_n = \frac{\sqrt[3]{5n+1}-6}{n^2+\sqrt[3]{n}-8} = \frac{n^2\left(\frac{\sqrt[3]{5n+1}}{n^2} - \frac{6}{n^2}\right)}{n^2\left(1+\frac{\sqrt[3]{n}}{n^2}-\frac{8}{n^2}\right)} = \frac{\frac{\sqrt[3]{5n+1}}{n^2} - \frac{6}{n^2}}{1+\frac{\sqrt[3]{n}}{n^2}-\frac{8}{n^2}} =$$

$$= \frac{\sqrt[3]{\frac{5n+1}{n^6}} - \frac{6}{n^2}}{1+\sqrt[3]{\frac{n}{n^6}}-\frac{8}{n^2}} = \frac{\sqrt[3]{\frac{5}{n^5}+\frac{1}{n^6}} - \frac{6}{n^2}}{1+\sqrt[3]{\frac{1}{n^5}}-\frac{8}{n^2}} \to \frac{\sqrt[3]{0+0}-0}{1+\sqrt[3]{0}-0} = 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{87n^{13} + 42}{1 - 2n^{25} + 87n}.$$

Solution 8.3.5

$$a_n = \frac{87n^{13} + 42}{1 - 2n^{25} + 87n} = \frac{n^{25}\left(\frac{87n^{13}}{n^{25}} + \frac{42}{n^{25}}\right)}{n^{25}\left(\frac{1}{n^{25}} - 2 + \frac{87n}{n^{25}}\right)} =$$

$$= \frac{\frac{87n^{13}}{n^{25}} + \frac{42}{n^{25}}}{\frac{1}{n^{25}} - 2 + \frac{87n}{n^{25}}} = \frac{\frac{87}{n^{12}} + \frac{42}{n^{25}}}{\frac{1}{n^{25}} - 2 + \frac{87}{n^{24}}} \to \frac{0+0}{0-2+0} = 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{\sqrt[3]{n+1} + \sqrt{n} - 6}{\sqrt{n^2 - 8} + \sqrt[3]{n}}.$$

Solution 8.3.6

$$a_n = \frac{\sqrt[3]{n+1} + \sqrt{n} - 6}{\sqrt{n^2 - 8} + \sqrt[3]{n}} = \frac{\sqrt{n^2}\left(\frac{\sqrt[3]{n+1}}{\sqrt{n^2}} + \frac{\sqrt{n}}{\sqrt{n^2}} - \frac{6}{\sqrt{n^2}}\right)}{\sqrt{n^2}\left(\frac{\sqrt{n^2-8}}{\sqrt{n^2}} + \frac{\sqrt[3]{n}}{\sqrt{n^2}}\right)} =$$

$$= \frac{\frac{\sqrt[3]{n+1}}{\sqrt{n^2}} + \frac{\sqrt{n}}{\sqrt{n^2}} - \frac{6}{\sqrt{n^2}}}{\frac{\sqrt{n^2-8}}{\sqrt{n^2}} + \frac{\sqrt[3]{n}}{\sqrt{n^2}}} = \frac{\frac{\sqrt[3]{n+1}}{n} + \sqrt{\frac{n}{n^2}} - \frac{6}{n}}{\sqrt{\frac{n^2-8}{n^2}} + \frac{\sqrt[3]{n}}{n}} =$$

$$= \frac{\sqrt[3]{\frac{n+1}{n^3}} + \sqrt{\frac{1}{n}} - \frac{6}{n}}{\sqrt{1 - \frac{8}{n^2}} + \sqrt[3]{\frac{n}{n^3}}} = \frac{\sqrt[3]{\frac{1}{n^2} + \frac{1}{n^3}} + \sqrt{\frac{1}{n}} - \frac{6}{n}}{\sqrt{1 - \frac{8}{n^2}} + \sqrt[3]{\frac{1}{n^2}}} \to \frac{\sqrt[3]{0+0} + \sqrt{0} - 0}{\sqrt{1-0} + 0} = 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{1-n}{\sqrt{n^2+1}+\sqrt{n^2+n}}.$$

Solution 8.3.7

$$\begin{aligned}
a_n &= \frac{1-n}{\sqrt{n^2+1}+\sqrt{n^2+n}} = \frac{\sqrt{n^2}\left(\dfrac{1-n}{\sqrt{n^2}}\right)}{\sqrt{n^2}\left(\dfrac{\sqrt{n^2+1}}{\sqrt{n^2}}+\dfrac{\sqrt{n^2+n}}{\sqrt{n^2}}\right)} = \\
&= \frac{\dfrac{1-n}{n}}{\dfrac{\sqrt{n^2+1}}{\sqrt{n^2}}+\dfrac{\sqrt{n^2+n}}{\sqrt{n^2}}} = \frac{\dfrac{1}{n}-1}{\sqrt{\dfrac{n^2+1}{n^2}}+\sqrt{\dfrac{n^2+n}{n^2}}} = \\
&= \frac{\dfrac{1}{n}-1}{\sqrt{1+\dfrac{1}{n^2}}+\sqrt{1+\dfrac{1}{n}}} \to \frac{0-1}{\sqrt{1+0}+\sqrt{1+0}} = \frac{-1}{2}.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{\sqrt{5n+1}-6}{n^2+\sqrt[3]{n}-7}.$$

Solution 8.3.8

$$\begin{aligned}
a_n &= \frac{\sqrt{5n+1}-6}{n^2+\sqrt[3]{n}-7} = \frac{n^2\left(\dfrac{\sqrt{5n+1}}{n^2}-\dfrac{6}{n^2}\right)}{n^2\left(1+\dfrac{\sqrt[3]{n}}{n^2}-\dfrac{7}{n^2}\right)} = \frac{\dfrac{\sqrt{5n+1}}{n^2}-\dfrac{6}{n^2}}{1+\dfrac{\sqrt[3]{n}}{n^2}-\dfrac{7}{n^2}} \\
&= \frac{\sqrt{\dfrac{5n+1}{n^4}}-\dfrac{6}{n^2}}{1+\sqrt[3]{\dfrac{n}{n^6}}-\dfrac{7}{n^2}} = \frac{\sqrt{\dfrac{5}{n^3}+\dfrac{1}{n^4}}-\dfrac{6}{n^2}}{1+\sqrt[3]{\dfrac{1}{n^5}}-\dfrac{7}{n^2}} \to \frac{\sqrt{0+0}-0}{1+\sqrt[3]{0}-0} = 0.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{65\sqrt[3]{n} - 6}{13\sqrt[3]{n} - 8}.$$

Solution 8.3.9

$$a_n = \frac{65\sqrt[3]{n} - 6}{13\sqrt[3]{n} - 8} = \frac{\sqrt[3]{n}\left(65 - \frac{6}{\sqrt[3]{n}}\right)}{\sqrt[3]{n}\left(13 - \frac{8}{\sqrt[3]{n}}\right)} = \frac{65 - \frac{6}{\sqrt[3]{n}}}{13 - \frac{8}{\sqrt[3]{n}}} \to \frac{65 - 0}{13 - 0} = \frac{65}{13}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{-1}{\sqrt{n} + \sqrt{n+1}}.$$

Solution 8.3.10

$$a_n = \frac{-1}{\sqrt{n} + \sqrt{n+1}} \to \frac{-1}{\infty + \infty} = 0.$$

8.3.2 *Geometric Sequences*

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{(-1)^n + 1}{5^n + 3^n + 2}.$$

Solution 8.3.11

$$a_n = \frac{(-1)^n + 1}{5^n + 3^n + 2} = \frac{5^n\left(\frac{(-1)^n}{5^n} + \frac{1}{5^n}\right)}{5^n\left(1 + \frac{3^n}{5^n} + \frac{2}{5^n}\right)} =$$

$$= \frac{\left(\frac{-1}{5}\right)^n + \left(\frac{1}{5}\right)^n}{1 + \left(\frac{3}{5}\right)^n + \frac{2}{5^n}} \to \frac{0+0}{1+0+0} = 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{3 \cdot 5^n + (-4)^n + 12}{2 \cdot 5^n + 3^n + 2}.$$

Solution 8.3.12

$$a_n = \frac{3 \cdot 5^n + (-4)^n + 12}{2 \cdot 5^n + 3^n + 2} = \frac{5^n\left(3 + \frac{(-4)^n}{5^n} + \frac{12}{5^n}\right)}{5^n\left(2 + \frac{3^n}{5^n} + \frac{2}{5^n}\right)} =$$

$$= \frac{3 + \left(\frac{-4}{5}\right)^n + 12\left(\frac{1}{5}\right)^n}{2 + \left(\frac{3}{5}\right)^n + 2\left(\frac{1}{5}\right)^n} \to \frac{3+0+0}{2+0+0} = \frac{3}{2}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{5^{n+1} + 2 \cdot 3^n}{2 \cdot 5^n + 2^n + 9}.$$

Solution 8.3.13

$$a_n = \frac{5^{n+1} + 2 \cdot 3^n}{2 \cdot 5^n + 2^n + 9} = \frac{5 \cdot 5^n + 2 \cdot 3^n}{2 \cdot 5^n + 2^n + 9} = \frac{5^n \left(5 + \frac{2 \cdot 3^n}{5^n}\right)}{5^n \left(2 + \frac{2^n}{5^n} + \frac{9}{5^n}\right)} =$$

$$= \frac{5 + 2\left(\frac{3}{5}\right)^n}{2 + \left(\frac{2}{5}\right)^n + 9\left(\frac{1}{5}\right)^n} \to \frac{5+0}{2+0+0} = \frac{5}{2}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{3 \cdot 7^n + 12 \cdot 4^n}{9 \cdot 4^n + 3^n + 29}.$$

Solution 8.3.14

$$a_n = \frac{3 \cdot 7^n + 12 \cdot 4^n}{9 \cdot 4^n + 3^n + 29} = \frac{4^n \left(\frac{3 \cdot 7^n}{4^n} + 12\right)}{4^n \left(9 + \frac{3^n}{4^n} + \frac{29}{4^n}\right)} =$$

$$= \frac{3\left(\frac{7}{4}\right)^n + 12}{9 + \left(\frac{3}{4}\right)^n + 29\left(\frac{1}{4}\right)^n} \to \frac{\infty + 12}{9+0+0} = \infty.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{3 \cdot 8^{n-1} + 12 \cdot 4^n - 1}{2 \cdot 7^n + (-3)^n + 2}.$$

Solution 8.3.15

$$a_n = \frac{3 \cdot 8^{n-1} + 12 \cdot 4^n - 1}{2 \cdot 7^n + (-3)^n + 2} = \frac{\frac{3}{8} \cdot 8^n + 12 \cdot 4^n - 1}{2 \cdot 7^n + (-3)^n + 2} = \frac{7^n \left(\frac{3}{8} \cdot \frac{8^n}{7^n} + \frac{12 \cdot 4^n}{7^n} - \frac{1}{7^n} \right)}{7^n \left(2 + \frac{(-3)^n}{7^n} + \frac{2}{7^n} \right)} =$$

$$= \frac{\frac{3}{8} \left(\frac{8}{7} \right)^n + 12 \left(\frac{4}{7} \right)^n - \left(\frac{1}{7} \right)^n}{2 + \left(\frac{-3}{7} \right)^n + 2 \left(\frac{1}{7} \right)^n} \to \frac{\infty + 0 - 0}{2 + 0 + 0} = \infty.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{3 \cdot 5^n + 12 \cdot 4^n}{2 \cdot (-9)^n + 3^n + 2}.$$

Solution 8.3.16

$$a_n = \frac{3 \cdot 5^n + 12 \cdot 4^n}{2 \cdot (-9)^n + 3^n + 2} = \frac{(-9)^n \left(\frac{3 \cdot 5^n}{(-9)^n} + \frac{12 \cdot 4^n}{(-9)^n} \right)}{(-9)^n \left(2 + \frac{3^n}{(-9)^n} + \frac{2}{(-9)^n} \right)} =$$

$$= \frac{3 \left(\frac{5}{-9} \right)^n + 12 \left(\frac{4}{-9} \right)^n}{2 + \left(\frac{3}{-9} \right)^n + 2 \left(\frac{1}{-9} \right)^n} \to \frac{0 + 0}{2 + 0 + 0} = 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{7 \cdot (-5)^n + 92 \cdot 7^n}{2 \cdot 5^n + (-8)^n + 2}.$$

Solution 8.3.17

$$\begin{aligned} a_n &= \frac{7 \cdot (-5)^n + 92 \cdot 7^n}{2 \cdot 5^n + (-8)^n + 2} = \frac{(-8)^n \left(\frac{7 \cdot (-5)^n}{(-8)^n} + \frac{92 \cdot 7^n}{(-8)^n} \right)}{(-8)^n \left(\frac{2 \cdot 5^n}{(-8)^n} + 1 + \frac{2}{(-8)^n} \right)} = \\ &= \frac{7 \left(\frac{5}{8} \right)^n + 92 \left(\frac{7}{-8} \right)^n}{\left(\frac{5}{-8} \right)^n + 1 + 2 \left(\frac{1}{-8} \right)^n} \to \frac{0+0}{0+1+0} = 0. \end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{(-1)^n + 9 \cdot 4^n}{3^n + 2}.$$

Solution 8.3.18

$$\begin{aligned} a_n &= \frac{(-1)^n + 9 \cdot 4^n}{3^n + 2} = \frac{3^n \left(\frac{(-1)^n}{3^n} + \frac{9 \cdot 4^n}{3^n} \right)}{3^n \left(1 + \frac{2}{3^n} \right)} = \\ &= \frac{\left(\frac{-1}{3} \right)^n + 9 \left(\frac{4}{3} \right)^n}{1 + 2 \left(\frac{1}{3} \right)^n} \to \frac{0 + \infty}{1 + 0} = \infty. \end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{(-1)^n + 9 \cdot 4^n}{3^n + 2 \cdot 4^n}.$$

Solution 8.3.19

$$\begin{aligned} a_n &= \frac{(-1)^n + 9 \cdot 4^n}{3^n + 2 \cdot 4^n} = \frac{4^n \left(\frac{(-1)^n}{4^n} + 9 \right)}{4^n \left(\frac{3^n}{4^n} + 2 \right)} = \\ &= \frac{\left(\frac{-1}{4} \right)^n + 9}{\left(\frac{3}{4} \right)^n + 2} \to \frac{0+9}{0+2} = \frac{9}{2}. \end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{(-1)^{n+1} + 5 \cdot 3^n}{3^{n-2} + 9}.$$

Solution 8.3.20

$$\begin{aligned} a_n &= \frac{(-1)^{n+1} + 5 \cdot 3^n}{3^{n-2} + 9} = \frac{-(-1)^n + 5 \cdot 3^n}{\frac{1}{9} \cdot 3^n + 9} = \frac{3^n \left(\frac{-(-1)^n}{3^n} + 5 \right)}{3^n \left(\frac{1}{9} + \frac{9}{3^n} \right)} = \\ &= \frac{-\left(\frac{-1}{3} \right)^n + 5}{\frac{1}{9} + 9 \left(\frac{1}{3} \right)^n} \to \frac{0+5}{\frac{1}{9}+0} = 45. \end{aligned}$$

8.3.3 *Sequences with Limit "$\infty - \infty$"*

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = 12 \cdot 4^n - 3 \cdot 7^n + 3.$$

Solution 8.3.21

$$a_n = 12 \cdot 4^n - 3 \cdot 7^n + 3 = 7^n \left(12 \left(\frac{4}{7} \right)^n - 3 + 3 \left(\frac{1}{7} \right)^n \right) \to \infty \cdot (0 - 3 + 0) = -\infty.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = 26n^{312} - 59n^6 + 31.$$

Solution 8.3.22

$$a_n = 26n^{312} - 59n^6 + 31 = n^{312} \left(26 - \frac{59n^6}{n^{312}} + \frac{31}{n^{312}} \right) \to \infty \cdot (26 - 0 + 0) = \infty.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = n^{32} - 9n^5 - 796.$$

Solution 8.3.23

$$a_n = n^{32} - 9n^5 - 796 = n^{32} \left(1 - \frac{9n^5}{n^{32}} - \frac{796}{n^{32}} \right) \to \infty \cdot (1 - 0 - 0) = \infty.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = 7 \cdot (-5)^n + 92 \cdot 7^n.$$

Solution 8.3.24 *Although $(-5)^n$ is a divergent sequence, we can find the limit of this sequence with the previous technique.*

$$a_n = 7 \cdot (-5)^n + 92 \cdot 7^n = 7^n \left(7 \left(\frac{-5}{7} \right)^n + 92 \right) \to \infty \cdot (0 + 92) = \infty.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt{n^3+n} - \sqrt{n^3+2}.$$

Solution 8.3.25

$$\begin{aligned}
a_n &= \sqrt{n^3+n} - \sqrt{n^3+2} = \left(\sqrt{n^3+n} - \sqrt{n^3+2}\right) \cdot \frac{\sqrt{n^3+n} + \sqrt{n^3+2}}{\sqrt{n^3+n} + \sqrt{n^3+2}} = \\
&= \frac{\left(\sqrt{n^3+n}\right)^2 - \left(\sqrt{n^3+2}\right)^2}{\sqrt{n^3+n} + \sqrt{n^3+2}} = \frac{n^3+n-(n^3+2)}{\sqrt{n^3+n} + \sqrt{n^3+2}} = \\
&= \frac{n-2}{\sqrt{n^3+n} + \sqrt{n^3+2}} = \frac{\sqrt{n^3} \cdot \dfrac{n-2}{\sqrt{n^3}}}{\sqrt{n^3}\left(\dfrac{\sqrt{n^3+n}}{\sqrt{n^3}} + \dfrac{\sqrt{n^3+2}}{\sqrt{n^3}}\right)} = \\
&= \frac{\dfrac{n}{\sqrt{n^3}} - \dfrac{2}{\sqrt{n^3}}}{\sqrt{\dfrac{n^3+n}{n^3}} + \sqrt{\dfrac{n^3+2}{n^3}}} = \frac{\dfrac{1}{\sqrt{n}} - \dfrac{2}{\sqrt{n^3}}}{\sqrt{1+\dfrac{1}{n^2}} + \sqrt{1+\dfrac{2}{n^3}}} \to \frac{0-0}{1+1} = 0.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt{n^2+1} - n.$$

Solution 8.3.26

$$\begin{aligned}
a_n &= \sqrt{n^2+1} - n = \left(\sqrt{n^2+1} - n\right) \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \\
&= \frac{\left(\sqrt{n^2+1}\right)^2 - n^2}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} \to 0.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt{n^2 + n - 1} - n.$$

Solution 8.3.27

$$a_n = \sqrt{n^2+n-1} - n = \left(\sqrt{n^2+n-1} - n\right) \cdot \frac{\sqrt{n^2+n-1}+n}{\sqrt{n^2+n-1}+n} =$$

$$= \frac{\left(\sqrt{n^2+n-1}\right)^2 - n^2}{\sqrt{n^2+n-1}+n} = \frac{n^2+n-1-n^2}{\sqrt{n^2+n-1}+n} =$$

$$= \frac{n-1}{\sqrt{n^2+n-1}+n} = \frac{n \cdot \dfrac{n-1}{n}}{n\left(\dfrac{\sqrt{n^2+n-1}}{n}+1\right)} =$$

$$= \frac{\dfrac{n-1}{n}}{\dfrac{\sqrt{n^2+n-1}}{n}+1} = \frac{1-\dfrac{1}{n}}{\sqrt{\dfrac{n^2+n-1}{n^2}}+1} = \frac{1-\dfrac{1}{n}}{\sqrt{1+\dfrac{1}{n}-\dfrac{1}{n^2}}+1} \to \frac{1-0}{1+1} = 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt{n^4+1} - n^2.$$

Solution 8.3.28

$$a_n = \sqrt{n^4+1} - n^2 = \left(\sqrt{n^4+1} - n^2\right) \cdot \frac{\sqrt{n^4+1}+n^2}{\sqrt{n^4+1}+n^2} =$$

$$= \frac{\left(\sqrt{n^4+1}\right)^2 - n^4}{\sqrt{n^4+1}+n^2} = \frac{n^4+1-n^4}{\sqrt{n^4+1}+n^2} = \frac{1}{\sqrt{n^4+1}+n^2} \to 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt{n^4+n-1} - n^2.$$

Solution 8.3.29

$$\begin{aligned}
a_n &= \sqrt{n^4+n-1} - n^2 = \left(\sqrt{n^4+n-1} - n^2\right) \cdot \frac{\sqrt{n^4+n-1}+n^2}{\sqrt{n^4+n-1}+n^2} = \\
&= \frac{\left(\sqrt{n^4+n-1}\right)^2 - n^4}{\sqrt{n^4+n-1}+n^2} = \frac{n^4+n-1-n^4}{\sqrt{n^4+n-1}+n^2} = \frac{n-1}{\sqrt{n^4+n-1}+n^2} = \\
&= \frac{n^2 \cdot \dfrac{n-1}{n^2}}{n^2\left(\dfrac{\sqrt{n^4+n-1}}{n^2}+1\right)} = \frac{\dfrac{n-1}{n^2}}{\dfrac{\sqrt{n^2+n-1}}{n^2}+1} = \frac{\dfrac{1}{n}-\dfrac{1}{n^2}}{\sqrt{\dfrac{n^2+n-1}{n^4}}+1} = \\
&= \frac{\dfrac{1}{n}-\dfrac{1}{n^2}}{\sqrt{\dfrac{1}{n^2}+\dfrac{1}{n^3}-\dfrac{1}{n^4}}+1} \to \frac{0-0}{0+1} = 0.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt{4^n+1} - 2^n.$$

Solution 8.3.30

$$\begin{aligned}
a_n &= \sqrt{4^n+1} - 2^n = \left(\sqrt{4^n+1} - 2^n\right) \cdot \frac{\sqrt{4^n+1}+2^n}{\sqrt{4^n+1}+2^n} = \\
&= \frac{\left(\sqrt{4^n+1}\right)^2 - (2^n)^2}{\sqrt{4^n+1}+2^n} = \frac{4^n+1-4^n}{\sqrt{4^n+1}+2^n} = \frac{1}{\sqrt{4^n+1}+2^n} \to \frac{1}{\infty+\infty} = 0.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt{2^n + 1} - \sqrt{2^n + 3}.$$

Solution 8.3.31

$$\begin{aligned}
a_n &= \sqrt{2^n+1} - \sqrt{2^n+3} = \left(\sqrt{2^n+1} - \sqrt{2^n+3}\right) \cdot \frac{\sqrt{2^n+1} + \sqrt{2^n+3}}{\sqrt{2^n+1} + \sqrt{2^n+3}} = \\
&= \frac{\left(\sqrt{2^n+1}\right)^2 - \left(\sqrt{2^n+3}\right)^2}{\sqrt{2^n+1} + \sqrt{2^n+3}} = \frac{2^n + 1 - (2^n + 3)}{\sqrt{2^n+1} + \sqrt{2^n+3}} = \\
&= \frac{-2}{\sqrt{2^n+1} + \sqrt{2^n+3}} \to \frac{-2}{\infty + \infty} = 0.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{1}{\sqrt{9^n + 1} - \sqrt{9^n - 2}}.$$

Solution 8.3.32

$$\begin{aligned}
a_n &= \frac{1}{\sqrt{9^n+1} - \sqrt{9^n-2}} = \frac{1}{\sqrt{9^n+1} - \sqrt{9^n-2}} \cdot \frac{\sqrt{9^n+1} + \sqrt{9^n-2}}{\sqrt{9^n+1} + \sqrt{9^n-2}} = \\
&= \frac{\sqrt{9^n+1} + \sqrt{9^n-2}}{\left(\sqrt{9^n+1}\right)^2 - \left(\sqrt{9^n-2}\right)^2} = \frac{\sqrt{9^n+1} + \sqrt{9^n-2}}{9^n + 1 - (9^n - 2)} = \\
&= \frac{\sqrt{9^n+1} + \sqrt{9^n-2}}{3} \to \frac{\infty + \infty}{3} = \infty.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{1}{\sqrt{n^4+1} - \sqrt{n^4-2}}.$$

Solution 8.3.33

$$\begin{aligned}
a_n &= \frac{1}{\sqrt{n^4+1} - \sqrt{n^4-2}} = \frac{1}{\sqrt{n^4+1} - \sqrt{n^4-2}} \cdot \frac{\sqrt{n^4+1} + \sqrt{n^4-2}}{\sqrt{n^4+1} + \sqrt{n^4-2}} = \\
&= \frac{\sqrt{n^4+1} + \sqrt{n^4-2}}{\left(\sqrt{n^4+1}\right)^2 - \left(\sqrt{n^4-2}\right)^2} = \frac{\sqrt{n^4+1} + \sqrt{n^4-2}}{n^4+1-(n^4-2)} = \\
&= \frac{\sqrt{n^4+1} + \sqrt{n^4-2}}{3} \to \frac{\infty + \infty}{3} = \infty.
\end{aligned}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \frac{1}{\sqrt{n^4+n} - \sqrt{n^4-2}}.$$

Solution 8.3.34

$$\begin{aligned}
a_n &= \frac{1}{\sqrt{n^4+n} - \sqrt{n^4-2}} = \frac{1}{\sqrt{n^4+n} - \sqrt{n^4-2}} \cdot \frac{\sqrt{n^4+n} + \sqrt{n^4-2}}{\sqrt{n^4+n} + \sqrt{n^4-2}} = \\
&= \frac{\sqrt{n^4+n} + \sqrt{n^4-2}}{\left(\sqrt{n^4+n}\right)^2 - \left(\sqrt{n^4-2}\right)^2} = \frac{\sqrt{n^4+n} + \sqrt{n^4-2}}{n^4+n-(n^4-2)} = \frac{\sqrt{n^4+n} + \sqrt{n^4-2}}{n+2} = \\
&= \frac{n\left(\frac{\sqrt{n^4+n}}{n} + \frac{\sqrt{n^4-2}}{n}\right)}{n\left(1+\frac{2}{n}\right)} = \frac{\sqrt{\frac{n^4+n}{n^2}} + \sqrt{\frac{n^4-2}{n^2}}}{1+\frac{2}{n}} \to \frac{\infty+\infty}{1+0} = \infty.
\end{aligned}$$

8.3.4 *Application of The Squeeze Theorem*

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{n^2+n+2}.$$

Solution 8.3.35 *As*

$$n^2+n+2 = n^2\left(1+\frac{1}{n}+\frac{2}{n^2}\right)$$

and

$$1+\frac{1}{n}+\frac{2}{n^2} \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2} \leq 1+\frac{1}{n}+\frac{2}{n^2} \leq \frac{3}{2}$$

for every $n \geq n_0$. So, if $n \geq n_0$, then

$$n^2 \cdot \frac{1}{2} \leq n^2+n+2 \leq n^2 \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{n^2 \cdot \frac{1}{2}} \leq \sqrt[n]{n^2+n+2} \leq \sqrt[n]{n^2 \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. This follows that for $n \geq n_0$, we have

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^2 = \sqrt[n]{n^2 \cdot \frac{1}{2}} \leq \sqrt[n]{n^2+n+2} \leq \sqrt[n]{n^2 \cdot \frac{3}{2}} = \left(\sqrt[n]{n}\right)^2 \cdot \sqrt[n]{\frac{3}{2}}.$$

As

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^2 \to 1$$

and

$$\left(\sqrt[n]{n}\right)^2 \cdot \sqrt[n]{\frac{3}{2}} \to 1,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{n^2+n+2} \to 1.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{n^4-n^3-n-2}.$$

Solution 8.3.36 *As*

$$n^4 - n^3 - n - 2 = n^4\left(1 - \frac{1}{n} - \frac{1}{n^3} - \frac{2}{n^4}\right)$$

and

$$1 - \frac{1}{n} - \frac{1}{n^3} - \frac{2}{n^4} \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2} \leq 1 - \frac{1}{n} - \frac{1}{n^3} - \frac{2}{n^4} \leq \frac{3}{2}$$

for every $n \geq n_0$. So, if $n \geq n_0$, then

$$n^4 \cdot \frac{1}{2} \leq n^4 - n^3 - n - 2 \leq n^4 \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{n^4 \cdot \frac{1}{2}} \leq \sqrt[n]{n^4 - n^3 - n - 2} \leq \sqrt[n]{n^4 \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. This follows that for $n \geq n_0$, we have

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^4 = \sqrt[n]{n^4 \cdot \frac{1}{2}} \leq \sqrt[n]{n^4 - n^3 - n - 2} \leq \sqrt[n]{n^4 \cdot \frac{3}{2}} = \left(\sqrt[n]{n}\right)^4 \cdot \sqrt[n]{\frac{3}{2}}.$$

As

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^4 \to 1$$

and

$$\left(\sqrt[n]{n}\right)^4 \cdot \sqrt[n]{\frac{3}{2}} \to 1,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{n^4 - n^3 - n - 2} \to 1.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{n^5 + n^4 - 2n^3 + 32}.$$

Solution 8.3.37 *As*

$$n^5 + n^4 - 2n^3 + 32 = n^5\left(1 + \frac{1}{n} - \frac{2}{n^2} + \frac{32}{n^5}\right)$$

and

$$1 + \frac{1}{n} - \frac{2}{n^2} + \frac{32}{n^5} \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2} \leq 1 + \frac{1}{n} - \frac{2}{n^2} + \frac{32}{n^5} \leq \frac{3}{2}$$

for every $n \geq n_0$. *So, if* $n \geq n_0$, *then*

$$n^5 \cdot \frac{1}{2} \leq n^5 + n^4 - 2n^3 + 32 \leq n^5 \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{n^5 \cdot \frac{1}{2}} \leq \sqrt[n]{n^5 + n^4 - 2n^3 + 32} \leq \sqrt[n]{n^5 \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. *This follows that for* $n \geq n_0$, *we have*

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^5 = \sqrt[n]{n^5 \cdot \frac{1}{2}} \leq \sqrt[n]{n^5 + n^4 - 2n^3 + 32} \leq \sqrt[n]{n^5 \cdot \frac{3}{2}} = \left(\sqrt[n]{n}\right)^5 \cdot \sqrt[n]{\frac{3}{2}}.$$

As

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^5 \to 1$$

and

$$\left(\sqrt[n]{n}\right)^5 \cdot \sqrt[n]{\frac{3}{2}} \to 1,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{n^5 + n^4 - 2n^3 + 32} \to 1.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{n^2 + 5\sqrt{n} + 9}.$$

Solution 8.3.38 *As*

$$n^2 + 5\sqrt{n} + 9 = n^2 \left(1 + \frac{5}{\sqrt{n}} + \frac{9}{n^2}\right)$$

and

$$1 + \frac{5}{\sqrt{n}} + \frac{9}{n^2} \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ *such that,*

$$\frac{1}{2} \leq 1 + \frac{5}{\sqrt{n}} + \frac{9}{n^2} \leq \frac{3}{2}$$

for every $n \geq n_0$. *So, if* $n \geq n_0$, *then*

$$n^2 \cdot \frac{1}{2} \leq n^2 + 5\sqrt{n} + 9 \leq n^2 \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{n^2 \cdot \frac{1}{2}} \leq \sqrt[n]{n^2 + 5\sqrt{n} + 9} \leq \sqrt[n]{n^2 \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. *This follows that for* $n \geq n_0$, *we have*

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^2 = \sqrt[n]{n^2 \cdot \frac{1}{2}} \leq \sqrt[n]{n^2 + 5\sqrt{n} + 9} \leq \sqrt[n]{n^2 \cdot \frac{3}{2}} = \left(\sqrt[n]{n}\right)^2 \cdot \sqrt[n]{\frac{3}{2}}.$$

As

$$\sqrt[n]{\frac{1}{2}} \cdot \left(\sqrt[n]{n}\right)^2 \to 1$$

and

$$\left(\sqrt[n]{n}\right)^2 \cdot \sqrt[n]{\frac{3}{2}} \to 1,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{n^2 + 5\sqrt{n} + 9} \to 1.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{7 \cdot 8^n + 3 \cdot 5^n + 9}.$$

Solution 8.3.39 *As*

$$7 \cdot 8^n + 3 \cdot 5^n + 9 = 7 \cdot 8^n \left(1 + \frac{3 \cdot 5^n}{7 \cdot 8^n} + \frac{9}{7 \cdot 8^n}\right) = 7 \cdot 8^n \left(1 + \frac{3}{7} \cdot \frac{5^n}{8^n} + \frac{9}{7} \cdot \frac{1}{8^n}\right)$$

and

$$1 + \frac{3}{7} \cdot \frac{5^n}{8^n} + \frac{9}{7} \cdot \frac{1}{8^n} = 1 + \frac{3}{7} \cdot \left(\frac{5}{8}\right)^n + \frac{9}{7} \cdot \left(\frac{1}{8}\right)^n \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2} \le 1 + \frac{3}{7} \cdot \left(\frac{5}{8}\right)^n + \frac{9}{7} \cdot \left(\frac{1}{8}\right)^n \le \frac{3}{2}$$

for every $n \ge n_0$. So, if $n \ge n_0$, then

$$7 \cdot 8^n \cdot \frac{1}{2} \le 7 \cdot 8^n + 3 \cdot 5^n + 9 \le 7 \cdot 8^n \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{7 \cdot 8^n \cdot \frac{1}{2}} \le \sqrt[n]{7 \cdot 8^n + 3 \cdot 5^n + 9} \le \sqrt[n]{7 \cdot 8^n \cdot \frac{3}{2}}$$

whenever $n \ge n_0$. This follows that for $n \ge n_0$, we have

$$\sqrt[n]{\frac{7}{2}} \cdot \sqrt[n]{8^n} = \sqrt[n]{7 \cdot 8^n \cdot \frac{1}{2}} \le \sqrt[n]{7 \cdot 8^n + 3 \cdot 5^n + 9} \le \sqrt[n]{7 \cdot 8^n \cdot \frac{3}{2}} = \sqrt[n]{8^n} \cdot \sqrt[n]{\frac{21}{2}}.$$

As

$$\sqrt[n]{\frac{7}{2}} \cdot \sqrt[n]{8^n} = \sqrt[n]{\frac{7}{2}} \cdot 8 \to 8$$

and

$$\sqrt[n]{8^n} \cdot \sqrt[n]{\frac{21}{2}} = 8 \cdot \sqrt[n]{\frac{21}{2}} \to 8,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{7 \cdot 8^n + 3 \cdot 5^n + 9} \to 8.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{2 \cdot 9^n + 3 \cdot 6^n + 9 \cdot 2^n}.$$

Solution 8.3.40 *As*

$$2 \cdot 9^n + 3 \cdot 6^n + 9 \cdot 2^n = 2 \cdot 9^n \left(1 + \frac{3 \cdot 6^n}{2 \cdot 9^n} + \frac{9 \cdot 2^n}{2 \cdot 9^n}\right) = 2 \cdot 9^n \left(1 + \frac{3}{2} \cdot \frac{6^n}{9^n} + \frac{9}{2} \cdot \frac{2^n}{9^n}\right)$$

and

$$1 + \frac{3}{2} \cdot \frac{6^n}{9^n} + \frac{9}{2} \cdot \frac{2^n}{9^n} = 1 + \frac{3}{2} \cdot \left(\frac{6}{9}\right)^n + \frac{9}{2} \cdot \left(\frac{2}{9}\right)^n \to 1,$$

from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2} \leq 1 + \frac{3}{2} \cdot \frac{6^n}{9^n} + \frac{9}{2} \cdot \frac{2^n}{9^n} \leq \frac{3}{2}$$

for every $n \geq n_0$. So, if $n \geq n_0$, then

$$2 \cdot 9^n \cdot \frac{1}{2} \leq 2 \cdot 9^n + 3 \cdot 6^n + 9 \cdot 2^n \leq 2 \cdot 9^n \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{2 \cdot 9^n \cdot \frac{1}{2}} \leq \sqrt[n]{2 \cdot 9^n + 3 \cdot 6^n + 9 \cdot 2^n} \leq \sqrt[n]{2 \cdot 9^n \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. This follows that for $n \geq n_0$, we have

$$\sqrt[n]{\frac{2}{2}} \cdot \sqrt[n]{9^n} = \sqrt[n]{2 \cdot 9^n \cdot \frac{1}{2}} \leq \sqrt[n]{2 \cdot 9^n + 3 \cdot 6^n + 9 \cdot 2^n} \leq \sqrt[n]{2 \cdot 9^n \cdot \frac{3}{2}} = \sqrt[n]{9^n} \cdot \sqrt[n]{\frac{6}{2}}.$$

As

$$\sqrt[n]{1} \cdot \sqrt[n]{9^n} = \sqrt[n]{1} \cdot 9 \to 9$$

and

$$\sqrt[n]{9^n} \cdot \sqrt[n]{\frac{6}{2}} = 9 \cdot \sqrt[n]{\frac{6}{2}} \to 9,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{2 \cdot 9^n + 3 \cdot 6^n + 9 \cdot 2^n} \to 9.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{7 \cdot 5^n - 2 \cdot 3^n - 89}.$$

Solution 8.3.41 *As*

$$7 \cdot 5^n - 2 \cdot 3^n - 89 = 7 \cdot 5^n \left(1 - \frac{2 \cdot 3^n}{7 \cdot 5^n} - \frac{89}{7 \cdot 5^n}\right) = 7 \cdot 5^n \left(1 - \frac{2}{7} \cdot \frac{3^n}{5^n} - \frac{89}{7} \cdot \frac{1}{5^n}\right)$$

and

$$1-\frac{2}{7}\cdot\frac{3^n}{5^n}-\frac{89}{7}\cdot\frac{1}{5^n}=1-\frac{2}{7}\cdot\left(\frac{3}{5}\right)^n+\frac{89}{7}\cdot\left(\frac{1}{5}\right)^n\to 1,$$

from Lemma 3.1. we find, that there exists $n_0\in\mathbb{N}$ *such that,*

$$\frac{1}{2}\le 1-\frac{2}{7}\cdot\frac{3^n}{5^n}-\frac{89}{7}\cdot\frac{1}{5^n}\le\frac{3}{2}$$

for every $n\ge n_0$*. So, if* $n\ge n_0$*, then*

$$7\cdot 5^n\cdot\frac{1}{2}\le 7\cdot 5^n-2\cdot 3^n-89\le 7\cdot 5^n\cdot\frac{3}{2}.$$

Thus

$$\sqrt[n]{7\cdot 5^n\cdot\frac{1}{2}}\le\sqrt[n]{7\cdot 5^n-2\cdot 3^n-89}\le\sqrt[n]{7\cdot 5^n\cdot\frac{3}{2}}$$

whenever $n\ge n_0$*. This follows that for* $n\ge n_0$*, we have*

$$\sqrt[n]{\frac{7}{2}}\cdot\sqrt[n]{5^n}=\sqrt[n]{7\cdot 5^n\cdot\frac{1}{2}}\le\sqrt[n]{7\cdot 5^n-2\cdot 3^n-89}\le\sqrt[n]{7\cdot 5^n\cdot\frac{3}{2}}=\sqrt[n]{5^n}\cdot\sqrt[n]{\frac{21}{2}}.$$

As

$$\sqrt[n]{\frac{7}{2}}\cdot\sqrt[n]{5^n}=\sqrt[n]{\frac{7}{2}}\cdot 5\to 5$$

and

$$\sqrt[n]{5^n}\cdot\sqrt[n]{\frac{21}{2}}=5\cdot\sqrt[n]{\frac{21}{2}}\to 5,$$

from the Squeeze Theorem, we get

$$a_n=\sqrt[n]{7\cdot 5^n-2\cdot 3^n-89}\to 5.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n=\sqrt[n]{7\cdot 10^n-31\cdot 6^n+72}.$$

Solution 8.3.42 *As*

$$7\cdot 10^n-31\cdot 6^n+72=7\cdot 10^n\left(1-\frac{31\cdot 6^n}{7\cdot 10^n}+\frac{72}{7\cdot 10^n}\right)=7\cdot 10^n\left(1-\frac{31}{7}\cdot\frac{6^n}{10^n}+\frac{72}{7}\cdot\frac{1}{10^n}\right)$$

and

$$1-\frac{31}{7}\cdot\frac{6^n}{10^n}+\frac{72}{7}\cdot\frac{1}{10^n}=1-\frac{31}{7}\cdot\left(\frac{6}{10}\right)^n+\frac{72}{7}\cdot\left(\frac{1}{10}\right)^n\to 1,$$

from Lemma 3.1. we find, that there exists $n_0\in\mathbb{N}$ *such that,*

$$\frac{1}{2}\le 1-\frac{31}{7}\cdot\frac{6^n}{10^n}+\frac{72}{7}\cdot\frac{1}{10^n}\le\frac{3}{2}$$

for every $n\ge n_0$*. So, if* $n\ge n_0$*, then*

$$7\cdot 10^n\cdot\frac{1}{2}\le 7\cdot 10^n-31\cdot 6^n+72\le 7\cdot 10^n\cdot\frac{3}{2}.$$

Thus

$$\sqrt[n]{7\cdot 10^n\cdot\frac{1}{2}}\leq\sqrt[n]{7\cdot 10^n-31\cdot 6^n+72}\leq\sqrt[n]{7\cdot 10^n\cdot\frac{3}{2}}$$

whenever $n\geq n_0$. *This follows that for* $n\geq n_0$, *we have*

$$\sqrt[n]{\frac{7}{2}}\cdot\sqrt[n]{10^n}=\sqrt[n]{7\cdot 10^n\cdot\frac{1}{2}}\leq\sqrt[n]{7\cdot 10^n-31\cdot 6^n+72}\leq\sqrt[n]{7\cdot 10^n\cdot\frac{3}{2}}=\sqrt[n]{10^n}\cdot\sqrt[n]{\frac{21}{2}}.$$

As

$$\sqrt[n]{\frac{7}{2}}\cdot\sqrt[n]{10^n}=\sqrt[n]{\frac{7}{2}}\cdot 10\to 10$$

and

$$\sqrt[n]{10^n}\cdot\sqrt[n]{\frac{21}{2}}=10\cdot\sqrt[n]{\frac{21}{2}}\to 10,$$

from the Squeeze Theorem, we get

$$a_n=\sqrt[n]{7\cdot 10^n-31\cdot 6^n+72}\to 10.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n=\sqrt[n]{7n+5\sin(n)}.$$

Solution 8.3.43 *First we prove that*

$$\frac{\sin(n)}{n}\to 0.$$

As

$$-1\leq\sin(n)\leq 1$$

we get

$$\frac{-1}{n}\leq\frac{\sin(n)}{n}\leq\frac{1}{n},$$

so from the Squeeze Theorem, we get

$$\frac{\sin(n)}{n}\to 0.$$

As

$$7n+5\sin(n)=7n\left(1+\frac{5\sin(n)}{7n}\right)$$

and

$$1+\frac{5\sin(n)}{7n}=1+\frac{5}{7}\cdot\frac{\sin(n)}{n}\to 1,$$

from Lemma 3.1. we find, that there exists $n_0\in\mathbb{N}$ *such that,*

$$\frac{1}{2}\leq 1+\frac{5\sin(n)}{7n}\leq\frac{3}{2}$$

for every $n \geq n_0$. *So, if* $n \geq n_0$, *then*

$$7n \cdot \frac{1}{2} \leq 7n + 5\sin(n) \leq 7n \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{7n \cdot \frac{1}{2}} \leq \sqrt[n]{7n + 5\sin(n)} \leq \sqrt[n]{7n \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. *This follows that for* $n \geq n_0$, *we have*

$$\sqrt[n]{\frac{7}{2}} \cdot \sqrt[n]{n} = \sqrt[n]{7n \cdot \frac{1}{2}} \leq \sqrt[n]{7n + 5\sin(n)} \leq \sqrt[n]{7n \cdot \frac{3}{2}} = \sqrt[n]{n} \cdot \sqrt[n]{\frac{21}{2}}.$$

As

$$\sqrt[n]{\frac{7}{2}} \cdot \sqrt[n]{n} \to 1$$

and

$$\sqrt[n]{n} \cdot \sqrt[n]{\frac{21}{2}} \to 1,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{7n + 5\sin(n)} \to 1.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \sqrt[n]{7n^2 + 2\sin(n)}.$$

Solution 8.3.44 *As*

$$\frac{\sin(n)}{n} \to 0$$

and

$$7n^2 + 2\sin(n) = 7n^2 \left(1 + \frac{2\sin(n)}{7n^2}\right),$$

we have

$$1 + \frac{2\sin(n)}{7n^2} = 1 + \frac{2}{7n} \cdot \frac{\sin(n)}{n} \to 1.$$

So from Lemma 3.1. we find, that there exists $n_0 \in \mathbb{N}$ *such that,*

$$\frac{1}{2} \leq 1 + \frac{2\sin(n)}{7n^2} \leq \frac{3}{2}$$

for every $n \geq n_0$. *So, if* $n \geq n_0$, *then*

$$7n^2 \cdot \frac{1}{2} \leq 7n^2 + 2\sin(n) \leq 7n^2 \cdot \frac{3}{2}.$$

Thus

$$\sqrt[n]{7n^2 \cdot \frac{1}{2}} \leq \sqrt[n]{7n^2 + 2\sin(n)} \leq \sqrt[n]{7n^2 \cdot \frac{3}{2}}$$

whenever $n \geq n_0$. *This follows that for* $n \geq n_0$, *we have*

$$\sqrt[n]{\frac{7}{2}} \cdot \left(\sqrt[n]{n}\right)^2 = \sqrt[n]{7n^2 \cdot \frac{1}{2}} \leq \sqrt[n]{7n^2 + 2\sin(n)} \leq \sqrt[n]{7n^2 \cdot \frac{3}{2}} = \left(\sqrt[n]{n}\right)^2 \cdot \sqrt[n]{\frac{21}{2}}.$$

As

$$\sqrt[n]{\frac{7}{2}} \cdot \left(\sqrt[n]{n}\right)^2 \to 1$$

and

$$\left(\sqrt[n]{n}\right)^2 \cdot \sqrt[n]{\frac{21}{2}} \to 1,$$

from the Squeeze Theorem, we get

$$a_n = \sqrt[n]{7n^2 + 2\sin(n)} \to 1.$$

8.3.5 *The Sequence* $\left(1+\frac{1}{n}\right)^n$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(\frac{n+9}{n-4}\right)^n.$$

Solution 8.3.45

$$\left(\frac{n+9}{n-4}\right)^n = \left(\frac{n\left(1+\frac{9}{n}\right)}{n\left(1-\frac{4}{n}\right)}\right)^n = \frac{\left(1+\frac{9}{n}\right)^n}{\left(1-\frac{4}{n}\right)^n} = \frac{\left(1+\frac{9}{n}\right)^n}{\left(1+\frac{-4}{n}\right)^n} \to \frac{e^9}{e^{-4}}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(\frac{9n+5}{9n-3}\right)^n.$$

Solution 8.3.46

$$\left(\frac{9n+5}{9n-3}\right)^n = \left(\frac{9n\left(1+\frac{5}{9n}\right)}{9n\left(1-\frac{3}{9n}\right)}\right)^n = \frac{\left(1+\frac{5}{9n}\right)^n}{\left(1-\frac{3}{9n}\right)^n} = \frac{\left(1+\frac{\frac{5}{9}}{n}\right)^n}{\left(1+\frac{-\frac{3}{9}}{n}\right)^n} \to \frac{e^{\frac{5}{9}}}{e^{\frac{-3}{9}}}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(1+\frac{1}{9n}\right)^{3n-1}.$$

Solution 8.3.47

$$\left(1+\frac{1}{9n}\right)^{3n-1} = \left(\left(1+\frac{1}{9n}\right)^{\frac{9n}{9n}}\right)^{3n-1} = \left(\left(1+\frac{1}{9n}\right)^{9n}\right)^{\frac{3n-1}{9n}}.$$

As $9n \to \infty$, using Theorem 3.14, we find

$$\left(1+\frac{1}{9n}\right)^{9n} \to e,$$

whenever $n \to \infty$. *Furthermore,*

$$\frac{3n-1}{9n} \to \frac{3}{9},$$

whenever $n \to \infty$. *So*

$$\left(1+\frac{1}{9n}\right)^{3n-1} \to e^{\frac{3}{9}}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(\frac{n^3+9}{n^3+14}\right)^{n^3}.$$

Solution 8.3.48 *Let* $m = n^3$. *So*

$$\left(\frac{n^3+9}{n^3+14}\right)^{n^3} = \left(\frac{m+9}{m+14}\right)^{m}.$$

As $n \to \infty$, *we have* $m \to \infty$. *After some basic manipulations, we have*

$$\left(\frac{m+9}{m+14}\right)^{m} = \left(\frac{m\left(1+\frac{9}{m}\right)}{m\left(1+\frac{14}{m}\right)}\right)^{m} = \frac{\left(1+\frac{9}{m}\right)^{m}}{\left(1+\frac{14}{m}\right)^{m}} \to \frac{e^9}{e^{14}}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(1+\frac{1}{n^2}\right)^{3n^2-1}.$$

Solution 8.3.49 *Let* $m = n^2$. *So*

$$\left(1+\frac{1}{n^2}\right)^{3n^2-1} = \left(1+\frac{1}{m}\right)^{3m-1}.$$

As $n \to \infty$, *we have* $m \to \infty$. *After some basic manipulations, we have*

$$\left(1+\frac{1}{m}\right)^{3m-1} = \left(\left(1+\frac{1}{m}\right)^{m}\right)^{3} \cdot \left(1+\frac{1}{m}\right)^{-1} \to e^3 \cdot 1^{-1} = e^3.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(\frac{3n^2+9}{3n^2+14}\right)^{n^2}.$$

Solution 8.3.50 *Let* $m = n^2$. *So*

$$\left(\frac{3n^2+9}{3n^2+14}\right)^{n^2} = \left(\frac{3m+9}{3m+14}\right)^{m}.$$

As $n \to \infty$, *we have* $m \to \infty$. *After some basic manipulations, we have*

$$\left(\frac{3m+9}{3m+14}\right)^{m} = \left(\frac{3m\left(1+\frac{9}{3m}\right)}{3m\left(1+\frac{14}{3m}\right)}\right)^{m} = \frac{\left(1+\frac{9}{3m}\right)^m}{\left(1+\frac{14}{3m}\right)^m} = \frac{\left(1+\frac{\frac{9}{3}}{m}\right)^m}{\left(1+\frac{\frac{14}{3}}{m}\right)^m} \to \frac{e^{\frac{9}{3}}}{e^{\frac{14}{3}}} = e^{-\frac{5}{3}}$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(1-\frac{21}{6n-3}\right)^{n}.$$

Solution 8.3.51

$$\left(1-\frac{21}{6n-3}\right)^{n} = \left(\left(1+\frac{-21}{6n-3}\right)^{\frac{6n-3}{6n-3}}\right)^{n} = \left(\left(1+\frac{-21}{6n-3}\right)^{6n-3}\right)^{\frac{n}{6n-3}}.$$

As $6n-3 \to \infty$, *using Theorem 3.14, we find*

$$\left(1+\frac{-21}{6n-3}\right)^{6n-3} \to e^{-21},$$

whenever $n \to \infty$. *Furthermore,*

$$\frac{n}{6n-3} \to \frac{1}{6},$$

whenever $n \to \infty$. *So*

$$\left(1-\frac{21}{6n-3}\right)^{n} \to \left(e^{-21}\right)^{\frac{1}{6}} = e^{\frac{-21}{6}}.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(1 + \frac{1}{5n+1}\right)^{1-3n^2},$$

Solution 8.3.52

$$\left(1+\frac{1}{5n+1}\right)^{1-3n^2} = \left(\left(1+\frac{1}{5n+1}\right)^{\frac{5n+1}{5n+1}}\right)^{1-3n^2} = \left(\left(1+\frac{1}{5n+1}\right)^{5n+1}\right)^{\frac{1-3n^2}{5n+1}}.$$

As $5n+1 \to \infty$, using Theorem 3.14, we find

$$\left(1+\frac{1}{5n+1}\right)^{5n+1} \to e,$$

whenever $n \to \infty$. Furthermore,

$$\frac{1-3n^2}{5n+1} = \frac{n\left(\frac{1}{n}-3n\right)}{n\left(5+\frac{1}{n}\right)} \to -\infty,$$

whenever $n \to \infty$. So

$$\left(1+\frac{1}{5n+1}\right)^{1-3n^2} \to e^{-\infty} = 0.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(1 + \frac{1}{5n^2+1}\right)^{1-3n}.$$

Solution 8.3.53

$$\left(1+\frac{1}{5n^2+1}\right)^{1-3n} = \left(\left(1+\frac{1}{5n^2+1}\right)^{\frac{5n^2+1}{5n^2+1}}\right)^{1-3n} = \left(\left(1+\frac{1}{5n+1}\right)^{5n+1}\right)^{\frac{1-3n^2}{5n+1}}.$$

As $5n^2+1 \to \infty$, using Theorem 3.14, we find

$$\left(1+\frac{1}{5n^2+1}\right)^{5n^2+1} \to e,$$

whenever $n \to \infty$. Furthermore,

$$\frac{1-3n}{5n^2+1} = \frac{n^2\left(\frac{1}{n^2}-\frac{3}{n}\right)}{n^2\left(5+\frac{1}{n^2}\right)} \to 0,$$

whenever $n \to \infty$. *So*

$$\left(1+\frac{1}{5n^2+1}\right)^{1-3n} \to e^0 = 1.$$

Step-by-Step Solution

Find the limit of the following sequence.

$$a_n = \left(\frac{4n-5}{5n+2}\right)^n.$$

Solution 8.3.54 *As*

$$\frac{4n+5}{5n+2} \to \frac{4}{5} \neq 1,$$

this problem cannot be solved as the previous ones. We will use the Squeeze Theorem. As

$$\frac{4n+5}{5n+2} \leq \frac{4n}{5n+2} \leq \frac{4n}{5n} = \frac{4}{5}$$

we get

$$0 \leq \frac{4n+5}{5n+2} \leq \frac{4}{5}$$

and from this

$$0 \leq \left(\frac{4n+5}{5n+2}\right)^n \leq \left(\frac{4}{5}\right)^n$$

As

$$\left(\frac{4}{5}\right)^n \to 0,$$

from the Squeeze Theorem, we get

$$a_n = \left(\frac{4n-5}{5n+2}\right)^n \to 0.$$

8.3.6 *Finding the Limit of a Sequence with Definition*

Step-by-Step Solution

Use *Definition* 3.1 to prove that

$$a_n = \frac{5n-4}{2n+3} \to \frac{5}{2}.$$

Solution 8.3.55 *We want to show that*

$$\lim_{n\to\infty} \frac{5n-4}{2n+3} = \frac{5}{2}.$$

For any given $\varepsilon > 0$, *we have to find* n_0 $(n_0 = n_0(\varepsilon))$, *such that if* $n \geq n_0$, *then*

$$\left|\frac{5n-4}{2n+3} - \frac{5}{2}\right| < \varepsilon.$$

holds. As

$$\left|\frac{5n-4}{2n+3} - \frac{5}{2}\right| < \varepsilon$$

after some equivalent (!) manipulations, we have

$$\begin{aligned}
\left|\frac{5n-4}{2n+3} - \frac{5}{2}\right| &< \varepsilon \\
&\Updownarrow \\
\left|\frac{-23}{4n+6}\right| &< \varepsilon \\
&\Updownarrow \qquad \textit{as } n > 0 \\
\frac{23}{4n+6} &< \varepsilon \\
&\Updownarrow \qquad \textit{as } \varepsilon > 0 \\
\frac{23}{\varepsilon} &< 4n+6 \\
&\Updownarrow \\
\frac{\frac{23}{\varepsilon} - 6}{4} &< n.
\end{aligned}$$

If $\varepsilon \geq \frac{23}{6}$, *then the left side of the last inequality is nonpositive, thus it holds for any* $n \in \mathbb{N}$. *Thus we have*

$$n_0 = \begin{cases} 0, & \text{if } \varepsilon \geq \frac{23}{6}, \\ \left[\frac{\frac{23}{\varepsilon} - 6}{4}\right] + 1, & \text{if } 0 < \varepsilon < \frac{23}{6} \end{cases}.$$

Hence $n \geq n_0$ *implies*

$$\left|\frac{5n-4}{2n+3} - \frac{5}{2}\right| < \varepsilon,$$

Definition 3.1, this proves that
$$\lim_{n\to\infty}\frac{5n-4}{2n+3}=\frac{5}{2}.$$

Step-by-Step Solution

Use *Definition 3.1* to prove that
$$a_n=\frac{6n+1}{9n-2}\to\frac{2}{3}.$$

Solution 8.3.56 *We want to show that*
$$\lim_{n\to\infty}\frac{6n+1}{9n-2}=\frac{2}{3}.$$
For any given $\varepsilon>0$, we have to find n_0 $(n_0=n_0(\varepsilon))$, such that if $n\geq n_0$, then
$$\left|\frac{6n+1}{9n-2}-\frac{2}{3}\right|<\varepsilon.$$
holds. As
$$\left|\frac{6n+1}{9n-2}-\frac{2}{3}\right|<\varepsilon$$
after some equivalent (!) manipulations, we have
$$\begin{array}{rcll}\left|\frac{6n+1}{9n-2}-\frac{2}{3}\right| & < & \varepsilon & \\ & \Updownarrow & & \\ \left|\frac{7}{27n-6}\right| & < & \varepsilon & \\ & \Updownarrow & & \text{as } n>0 \\ \frac{7}{27n-6} & < & \varepsilon & \\ & \Updownarrow & & \text{as } \varepsilon>0 \\ \frac{7}{\varepsilon} & < & 27n-6 & \\ & \Updownarrow & & \\ \frac{\frac{7}{\varepsilon}+6}{27} & < & n. & \end{array}$$
As $\frac{7}{\varepsilon}+6>0$ for any given $\varepsilon>0$, we have
$$n_0=\left[\frac{\frac{7}{\varepsilon}+6}{27}\right]+1.$$
Hence $n\geq n_0$ implies
$$\left|\frac{6n+1}{9n-2}-\frac{2}{3}\right|<\varepsilon,$$

by Definition 3.1, this proves that

$$\lim_{n\to\infty}\frac{6n+1}{9n-2}=\frac{2}{3}.$$

Step-by-Step Solution

Use *Definition 3.1* to prove that

$$a_n=\left(\frac{1}{2}\right)^n\to 0.$$

Solution 8.3.57 *We want to show that*

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n=0.$$

For any given $\varepsilon>0$, *we have to find* n_0 $(n_0=n_0(\varepsilon))$, *such that if* $n\geq n_0$, *then*

$$\left|\left(\frac{1}{2}\right)^n-0\right|<\varepsilon.$$

holds. As

$$\left|\left(\frac{1}{2}\right)^n-0\right|<\varepsilon$$

after some equivalent (!) manipulations, we have

$$\begin{array}{rcl}\left|\left(\frac{1}{2}\right)^n-0\right| & < & \varepsilon\\ & \Updownarrow & \\ \left|\left(\frac{1}{2}\right)^n\right| & < & \varepsilon\\ & \Updownarrow & \\ \left(\frac{1}{2}\right)^n & < & \varepsilon\\ & \Updownarrow & \quad \text{as } \varepsilon>0\\ n & > & \log_{\frac{1}{2}}(\varepsilon)\end{array}$$

If $\varepsilon\geq 1$, *then the right side of the last inequality is nonpositive, thus it holds for any* $n\in\mathbb{N}$. *Thus we have*

$$n_0=\begin{cases}0, & \text{if } \varepsilon\geq 1,\\ \left[\log_{\frac{1}{2}}(\varepsilon)\right]+1, & \text{if } 0<\varepsilon<1\end{cases}.$$

Hence $n\geq n_0$ *implies*

$$\left|\left(\frac{1}{2}\right)^n-0\right|<\varepsilon,$$

by Definition 3.1, this proves that

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n=0.$$

Step-by-Step Solution

Use *Definition 3.1* to prove that

$$a_n = \left(-\frac{1}{5}\right)^n \to 0.$$

Solution 8.3.58 *We want to show that*

$$\lim_{n\to\infty} \left(-\frac{1}{5}\right)^n = 0.$$

For any given $\varepsilon > 0$, *we have to find* n_0 $(n_0 = n_0(\varepsilon))$, *such that if* $n \geq n_0$, *then*

$$\left|\left(-\frac{1}{5}\right)^n - 0\right| < \varepsilon.$$

holds. As

$$\left|\left(-\frac{1}{5}\right)^n - 0\right| < \varepsilon$$

after some equivalent (!) manipulations, we have

$$\begin{array}{rcl} \left|\left(-\frac{1}{5}\right)^n\right| & < & \varepsilon \\ & \Updownarrow & \\ \left|(-1)^n \left(\frac{1}{5}\right)^n\right| & < & \varepsilon \\ & \Updownarrow & \\ \left|(-1)^n\right| \cdot \left|\left(\frac{1}{5}\right)^n\right| & < & \varepsilon \\ & \Updownarrow & \\ \left(\frac{1}{5}\right)^n & < & \varepsilon \\ & \Updownarrow & \quad \textit{as } \varepsilon > 0 \\ n & > & \log_{\frac{1}{5}}(\varepsilon) \end{array}$$

If $\varepsilon \geq 1$, *then the right side of the last inequality is nonpositive, thus it holds for any* $n \in \mathbb{N}$. *Thus we have*

$$n_0 = \begin{cases} 0, & \textit{if } \varepsilon \geq 1, \\ \left[\log_{\frac{1}{5}}(\varepsilon)\right] + 1, & \textit{if } 0 < \varepsilon < 1 \end{cases}.$$

Hence $n \geq n_0$ *implies*

$$\left|\left(-\frac{1}{5}\right)^n - 0\right| < \varepsilon,$$

by Definition 3.1, this proves that

$$\lim_{n\to\infty} \left(\frac{1}{5}\right)^n = 0.$$

Step-by-Step Solution

Use *Definition 3.1* to prove that

$$a_n = \frac{1}{n^2+1} \to 0.$$

Solution 8.3.59 *We want to show that*

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$$

For any given $\varepsilon > 0$*, we have to find* n_0 $(n_0 = n_0(\varepsilon))$ *, such that if* $n \geq n_0$*, then*

$$\left| \frac{1}{n^2+1} - 0 \right| < \varepsilon.$$

holds. As

$$\left| \frac{1}{n^2+1} - 0 \right| < \varepsilon$$

after some equivalent (!) manipulations, we have

$$\begin{aligned}
\left| \frac{1}{n^2+1} - 0 \right| &< \varepsilon \\
&\Updownarrow \\
\left| \frac{1}{n^2+1} \right| &< \varepsilon \\
&\Updownarrow \qquad \text{as } n^2+1>0 \\
\frac{1}{n^2+1} &< \varepsilon \\
&\Updownarrow \qquad \text{as } \varepsilon > 0 \\
\frac{1}{\varepsilon} &< n^2+1 \\
&\Updownarrow \\
\frac{1}{\varepsilon} + 1 &< n^2 \\
&\Updownarrow \qquad \text{as } n, \varepsilon > 0 \\
\sqrt{\frac{1}{\varepsilon}+1} &< n.
\end{aligned}$$

As $\sqrt{\frac{1}{\varepsilon}+1}$ *for any given* $\varepsilon > 0$*, we have*

$$n_0 = \left[\sqrt{\frac{1}{\varepsilon}+1} \right] + 1.$$

Hence $n \geq n_0$ *implies*

$$\left| \frac{1}{n^2+1} - 0 \right| < \varepsilon,$$

by Definition 3.1, this proves that

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$$

Step-by-Step Solution

Use *Definition* 3.5 to prove that

$$a_n = 2^n \to \infty.$$

Solution 8.3.60 *By Definition 3.5, we need to show that, given any $c \in \mathbb{R}$ there exists an n_0 $(n_0 = n_0(c))$, such that if $n \geq n_0$, then*

$$2^n > c$$

holds. If $c \leq 0$, then the right side is nonpositive, so for any $n \in \mathbb{N}$ the inequality holds. Suppose that $c > 0$. Then

$$n > \log_2(c).$$

Thus

$$n_0 = \begin{cases} 0, & \text{if } c \leq 0, \\ \left[\sqrt{c+1}\right] + 1, & \text{if } c > 0 \end{cases},$$

and $n \geq n_0$ implies

$$2^n > c,$$

and by definition, this proves that

$$2^n \to \infty.$$

Step-by-Step Solution

Use *Definition* 3.5 to prove that

$$a_n = \ln(n) \to \infty.$$

Solution 8.3.61 *We need to show that, given any $c \in \mathbb{R}$ there exists an n_0 $(n_0 = n_0(c))$, such that if $n \geq n_0$, then*

$$\ln(n) > c$$

holds. Then

$$n > e^c.$$

Thus

$$n_0 = [e^c] + 1,$$

and $n \geq n_0$ *implies*

$$2^n > c,$$

and by Definition 3.5, this proves that

$$2^n \to \infty.$$

Step-by-Step Solution

Use *Definition 3.5* to prove that

$$a_n = n^2 + 2n + 1 \to \infty.$$

Solution 8.3.62 *Let*

$$a_n = n^2 + 2n + 1 = (n+1)^2.$$

We need to show that, given any $c \in \mathbb{R}$ *there exists an* n_0 $(n_0 = n_0(c))$, *such that if* $n \geq n_0$, *then*

$$(n+1)^2 > c$$

holds. If $c < 0$, *then the right side is negative, so for any* $n \in \mathbb{N}$ *the inequality holds. Suppose that* $c \geq 0$. *Then*

$$(n+1)^2 > c$$

follows

$$n + 1 > \sqrt{c},$$

and

$$n > \sqrt{c} - 1.$$

Thus

$$n_0 = \begin{cases} 0, & \text{if } c < 0, \\ [\sqrt{c} - 1] + 1, & \text{if } c \geq 0 \end{cases},$$

and $n \geq n_0$ *implies*

$$n^2 - 1 > c,$$

and by Definition 3.5, this proves that

$$n^2 - 1 \to \infty.$$

Step-by-Step Solution

Use Definition 3.6 to prove that

$$a_n = 1 - \lg(n) \to -\infty.$$

Solution 8.3.63 *By Definition 3.5, we need to show that, given any* $c \in \mathbb{R}$ *there exists an* n_0 $(n_0 = n_0(c))$, *such that if* $n \geq n_0$, *then*

$$1 - \lg(n) < c$$

holds. As

$$1 - \lg(n) < c,$$

we find

$$1 - c < \lg(n),$$

and

$$e^{1-c} < n.$$

Thus

$$n_0 = \left[e^{1-c}\right] + 1,$$

and $n \geq n_0$ *implies*

$$1 - \lg(n) < c$$

and by definition, this proves that

$$1 - \lg(n) \to -\infty.$$

Step-by-Step Solution

Use *Definition 3.6* to prove that

$$a_n = 1 - 3^n \to -\infty.$$

Solution 8.3.64 *By Definition 3.6, we need to show that, given any* $c \in \mathbb{R}$ *there exists an* n_0 $(n_0 = n_0(c))$, *such that if* $n \geq n_0$, *then*

$$1 - 3^n < c$$

holds. As

$$1 - 3^n < c,$$

we find

$$1 - c < 3^n,$$

If $c \geq 1$, *then the left side is negative, so for any* $n \in \mathbb{N}$ *the inequality holds. Suppose that* $c < 1$. *Then*

$$\log_3(1 - c) < n.$$

Thus

$$n_0 = \begin{cases} 0, & \text{if } c > 1, \\ [\sqrt{1-c}] + 1, & \text{if } c \leq 1 \end{cases}.$$

and $n \geq n_0$ *implies*

$$1 - 3^n < c$$

and by definition, this proves that

$$1 - 3^n \to -\infty.$$

8.3.7 *Divergent Sequences*

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = \frac{(-1)^n n^2}{n^2+7}.$$

Solution 8.3.65 *First let $n = 2k$, where $k \in \mathbb{N}$, so we have*

$$a_{2k} = \frac{(-1)^{2k} (2k)^2}{(2k)^2+1}.$$

As $(-1)^{2k} = 1$, we have

$$a_{2k} = \frac{(2k)^2}{(2k)^2+1} = \frac{4k^2}{4k^2+1} \to \frac{4}{4},$$

whenever $k \to \infty$.
Now let $n = 2k+1$, where $k \in \mathbb{N}$, so we have

$$a_{2k+1} = \frac{(-1)^{2k+1} (2k+1)^2}{(2k+1)^2+1}.$$

As $(-1)^{2k+1} = -1$, we find

$$a_{2k+1} = \frac{(-1)^{2k+1} (2k+1)^2}{(2k+1)^2+1} = \frac{-(2k+1)^2}{(2k+1)^2+1} = -\frac{4k^2+4k+1}{4k^2+4k+2} \to -\frac{4}{4}.$$

whenever $k \to \infty$. This follows

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1}.$$

As a result sequence

$$a_n = \frac{(-1)^n n^2}{n^2+7}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

Solution 8.3.66

$$a_n = \frac{1+(-1)^n n^2}{n+1}.$$

First let $n = 2k$, $k \in \mathbb{N}$. Then

$$a_{2k} = \frac{1+(-1)^{2k} (2k)^2}{2k+1}.$$

As $(-1)^{2k} = 1$, *we have*

$$a_{2k} = \frac{1 + (2k)^2}{2k+1} = \frac{1+4k^2}{2k+1} \to \infty,$$

whenever $k \to \infty$.

Now let $n = 2k+1$, *where* $k \in \mathbb{N}$, *so we have*

$$a_{2k+1} = \frac{1 + (-1)^{2k+1} (2k+1)^2}{2k+1+1}.$$

As $(-1)^{2k+1} = -1$, *we obtain*

$$a_{2k+1} = \frac{1 + (-1)^{2k+1} (2k+1)^2}{2k+1+1} = \frac{1-(2k+1)^2}{2k+2} = \frac{1-\left(4k^2+4k+1\right)}{2k+2} \to -\infty.$$

whenever $k \to \infty$. *This follows*

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = \frac{1 + (-1)^n n^2}{n+1}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = (-1)^n \frac{5n+3}{6n+1}.$$

Solution 8.3.67 *No matter* n *is odd or even, we have*

$$\frac{5n+3}{6n+1} \to \frac{5}{6}$$

holds. From this we get for $n = 2k$ *that* $(-1)^{2k} = 1 \to 1$, *so*

$$a_{2k} \to 1 \cdot \frac{5}{6} = \frac{5}{6}$$

whenever $k \to \infty$.

Now let $n = 2k+1$ *then* $(-1)^{2k+1} = -1 \to -1$, *so*

$$a_{2k+1} \to -1 \cdot \frac{5}{6} = -\frac{5}{6}$$

whenever $k \to \infty$. *This follows*

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = (-1)^n \frac{5n+3}{6n+1}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = \frac{n}{n^{(-1)^n} + 2}.$$

Solution 8.3.68 *First let* $n = 2k$, $k \in \mathbb{N}$. *Then*

$$a_{2k} = \frac{2k}{(2k)^{(-1)^{2k}} + 2}.$$

As $(-1)^{2k} = 1$, *we have*

$$a_{2k} = \frac{2k}{(2k)^{(-1)^{2k}} + 2} = \frac{2k}{2k+2} \to 1,$$

whenever $k \to \infty$.
Now let $n = 2k+1$, *where* $k \in \mathbb{N}$, *so we have*

$$a_{2k+1} = \frac{2\,(k+1)}{(2k+1)^{(-1)^{2k+1}} + 2}.$$

As $(-1)^{2k+1} = -1$, *we find*

$$a_{2k+1} = \frac{2\,(k+1)}{(2k+1)^{(-1)^{2k+1}} + 2} = \frac{2\,(k+1)}{(2k+1)^{-1} + 2}.$$

As

$$\frac{2\,(k+1)}{(2k+1)^{-1} + 2} = \frac{2\,(k+1)}{\frac{1}{2k+1} + 2} = \frac{4k^2 + 6k + 2}{4k+3},$$

so

$$a_{2k+1} = \frac{4k^2 + 6k + 2}{4k+3} \to \infty$$

whenever $k \to \infty$. *We can conclude that*

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = \frac{n}{n^{(-1)^n} + 2}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = \frac{5^n\left(1 + (-1)^n\right) + 3^n}{5^n + 2^n}.$$

Solution 8.3.69 *Preliminary manipulations are necessary before applying the properties of limits.*

$$a_n = \frac{5^n\left(1+(-1)^n\right)+3^n}{5^n+2^n} = \frac{5^n\left(1+(-1)^n+\left(\frac{3}{5}\right)^n\right)}{5^n\left(1+\left(\frac{2}{5}\right)^n\right)} = \frac{1+(-1)^n+\left(\frac{3}{5}\right)^n}{1+\left(\frac{2}{5}\right)^n}.$$

No matter n is odd or even, we have

$$\left(\frac{3}{5}\right)^n \to 0$$

and

$$\left(\frac{2}{5}\right)^n \to 0$$

holds. From this we get for $n = 2k$ that $(-1)^{2k} = 1 \to 1$, so

$$a_{2k} = \frac{1+(-1)^{2k}+\left(\frac{3}{5}\right)^{2k}}{1+\left(\frac{2}{5}\right)^{2k}} = \frac{1+1+\left(\frac{3}{5}\right)^{2k}}{1+\left(\frac{2}{5}\right)^{2k}} \to 2$$

whenever $k \to \infty$.
Now let $n = 2k+1$ then $(-1)^{2k+1} = -1 \to -1$, so

$$a_{2k+1} = \frac{1+(-1)+\left(\frac{3}{5}\right)^{2k}}{1+\left(\frac{2}{5}\right)^{2k}} \to \frac{0}{1} = 0$$

whenever $k \to \infty$. This follows

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = \frac{5^n\left(1+(-1)^n\right)+3^n}{5^n+2^n}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = (n+1)^{(-1)^n}.$$

Solution 8.3.70 *First let $n = 2k$, where $k \in \mathbb{N}$, so we have*

$$a_{2k} = (2k+1)^{(-1)^{2k}}.$$

As $(-1)^{2k} = 1$, we have

$$a_{2k} = (2k+1)^{(-1)^{2k}} = 2k+1 \to \infty,$$

whenever $k \to \infty$.

Now let $n = 2k+1$, *where* $k \in \mathbb{N}$, *so we have*

$$a_{2k+1} = (2k+1+1)^{(-1)^{2k+1}}.$$

As $(-1)^{2k+1} = -1$, *we find*

$$a_{2k+1} = (2k+1+1)^{(-1)^{2k+1}} = (2k+2)^{-1} = \frac{1}{2k+2} \to 0.$$

whenever $k \to \infty$. *This follows*

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = (n+1)^{(-1)^n}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

Solution 8.3.71

$$a_n = (-1)^n \frac{6^n+1}{2\cdot 6^n - 1}.$$

No matter n is odd or even, we have

$$\frac{6^n+1}{2\cdot 6^n-1} \to \frac{1}{2}$$

holds. From this we get for $n = 2k$ *that* $(-1)^{2k} = 1 \to 1$, *so*

$$a_{2k} \to 1 \cdot \frac{1}{2} = \frac{1}{2}$$

whenever $k \to \infty$.

Now let $n = 2k+1$ *then* $(-1)^{2k+1} = -1 \to -1$, *so*

$$a_{2k} \to -1 \cdot \frac{1}{2} = -\frac{1}{2}$$

whenever $k \to \infty$. *We obtain*

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = (-1)^n \frac{6^n+1}{2\cdot 6^n-1}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = (-1)^n \sqrt{n+1} - \sqrt{n}.$$

Solution 8.3.72 *First let $n = 2k$, where $k \in \mathbb{N}$, so we have*

$$a_{2k} = (-1)^{2k} \sqrt{2k+1} - \sqrt{2k}.$$

As $(-1)^{2k} = 1$, we have

$$a_{2k} = \sqrt{2k+1} - \sqrt{2k} \to \infty - \infty,$$

whenever $k \to \infty$. As

$$\sqrt{2k+1} - \sqrt{2k} = \sqrt{n} - \sqrt{n+1} = \left(\sqrt{2k+1} - \sqrt{2k}\right) \frac{\sqrt{2k+1} + \sqrt{2k}}{\sqrt{2k+1} + \sqrt{2k}} =$$

$$= \frac{2k+1-2k}{\sqrt{2k+1} + \sqrt{2k}} = \frac{1}{\sqrt{2k+1} + \sqrt{2k}},$$

we have

$$a_{2k} = \sqrt{2k+1} - \sqrt{2k} \to 0.$$

Now let $n = 2k+1$, where $k \in \mathbb{N}$, so we have

$$a_{2k+1} = (-1)^{2k+1} \sqrt{2k+1+1} - \sqrt{2k+1}.$$

As $(-1)^{2k+1} = -1$, we find

$$a_{2k+1} = -\sqrt{2k+2} - \sqrt{2k+1} \to -\infty - \infty = -\infty.$$

whenever $k \to \infty$. This follows

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = (-1)^n \sqrt{n+1} - \sqrt{n}$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = (-1)^n \sqrt{n^2+1} - n.$$

Solution 8.3.73 *First let $n = 2k$, where $k \in \mathbb{N}$, so we have*

$$a_{2k} = (-1)^{2k} \sqrt{(2k)^2+1} - 2k.$$

As $(-1)^{2k} = 1$, *we have*

$$a_{2k} = \sqrt{4k^2+1} - 2k \to \infty - \infty,$$

whenever $k \to \infty$. *As*

$$\begin{aligned} \sqrt{4k^2+1} - 2k &= \left(\sqrt{4k^2+1} - 2k\right) \frac{\sqrt{4k^2+1}+2k}{\sqrt{4k^2+1}+2k} = \\ &= \frac{4k^2+1-(2k)^2}{\sqrt{4k^2+1}+2k} = \frac{1}{\sqrt{4k^2+1}+2k}, \end{aligned}$$

we have

$$a_{2k} = \sqrt{4k^2+1} - 2k \to 0.$$

Now let $n = 2k+1$, *where* $k \in \mathbb{N}$, *so we have*

$$a_{2k+1} = (-1)^{2k+1} \sqrt{(2k+1)^2+1} - (2k+1).$$

As $(-1)^{2k+1} = -1$, *we find*

$$a_{2k+1} = -\sqrt{(2k+1)^2+1} - (2k+1) \to -\infty - \infty = -\infty.$$

whenever $k \to \infty$. *This follows*

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = (-1)^n \sqrt{n^2+1} - n$$

is divergent.

Step-by-Step Solution

Prove that the sequence is divergent.

$$a_n = (-1)^n \left(\sqrt{n^2+n} - n\right).$$

Solution 8.3.74 *Preliminary manipulations are necessary before applying the properties of limits. No matter n is odd or even, we have*

$$\begin{aligned} \sqrt{n^2+n} - n &= \left(\sqrt{n^2+n} - n\right) \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \\ &= \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2}. \end{aligned}$$

From this we get for $n = 2k$ *that* $(-1)^{2k} = 1 \to 1$, *so*

$$a_{2k} = (-1)^{2k}\left(\sqrt{(2k)^2 + 2k} - 2k\right) \to 1 \cdot \frac{1}{2} = \frac{1}{2}$$

whenever $k \to \infty$.
Now let $n = 2k+1$ *then* $(-1)^{2k+1} = -1 \to -1$, *so*

$$a_{2k} = (-1)^{2k+1}\left(\sqrt{(2k+1)^2 + 2k + 1} - (2k+1)\right) \to -1 \cdot \frac{1}{2} = -\frac{1}{2}$$

whenever $k \to \infty$. *This follows*

$$\lim_{k\to\infty} a_{2k} \neq \lim_{k\to\infty} a_{2k+1},$$

so sequence

$$a_n = (-1)^n\left(\sqrt{n^2 + n} - n\right)$$

is divergent.

8.4 Limit and Continuity of One Variable Real Functions

8.4.1 Limits at Infinity

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to-\infty}\frac{x^3-4x^2+1}{x^3-x^2-1}.$$

Solution 8.4.1

$$\lim_{x\to-\infty}\frac{x^3-4x^2+1}{x^3-x^2-1}=\lim_{x\to-\infty}\frac{x^3\left(1-\frac{4}{x}+\frac{1}{x^3}\right)}{x^3\left(1-\frac{1}{x}-\frac{1}{x^3}\right)}=\lim_{x\to-\infty}\frac{1-\frac{4}{x}+\frac{1}{x^3}}{1-\frac{1}{x}-\frac{1}{x^3}}=\frac{1}{1}=1.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty}\frac{x^3+3x}{x^2-x+1}.$$

Solution 8.4.2

$$\lim_{x\to\infty}\frac{x^3+3x}{x^2-x+1}=\lim_{x\to\infty}\frac{x^2\left(x+\frac{3}{x}\right)}{x^2\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}=\frac{\infty}{1}=\infty.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\longrightarrow\infty}\frac{x^5+x^4}{3x^6+x^2+1}.$$

Solution 8.4.3

$$\lim_{x\longrightarrow\infty}\frac{x^5+x^4}{3x^6+x^2+1}=\lim_{x\longrightarrow\infty}\frac{x^6\left(\frac{1}{x}+\frac{1}{x^6}\right)}{x^6\left(3+\frac{1}{x^4}+\frac{1}{x^6}\right)}=\lim_{x\longrightarrow\infty}\frac{\frac{1}{x}+\frac{1}{x^6}}{3+\frac{1}{x^4}+\frac{1}{x^6}}=\frac{0}{3}=0.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x}+1}{x^2-1}.$$

Solution 8.4.4

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x}+1}{x^2-1} = \lim_{x\to\infty} \frac{x^2\left(\frac{\sqrt{x^2+2x}}{x^2}+\frac{1}{x^2}\right)}{x^2\left(1-\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+2x}{x^4}}+\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{0}{1} = 0.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt[5]{x^2-1}}.$$

Solution 8.4.5

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt[5]{x^2-1}} = \lim_{x\to\infty} \frac{x^{2/5}\left(\frac{2x}{x^{2/5}}+\frac{1}{x^{2/5}}\right)}{x^{2/5}\left(\sqrt[5]{1-\frac{1}{x^2}}\right)} = \lim_{x\to\infty} \frac{2\cdot x^{1-2/5}+\frac{1}{x^{2/5}}}{\sqrt[5]{1-\frac{1}{x^2}}} = \frac{\infty}{1} = \infty.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty} \frac{\sqrt{x}+1}{\sqrt{x}-1}.$$

Solution 8.4.6

$$\lim_{x\to\infty} \frac{\sqrt{x}+1}{\sqrt{x}-1} = \lim_{x\to\infty} \frac{\sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)}{\sqrt{x}\left(1-\frac{1}{\sqrt{x}}\right)} = \lim_{x\to\infty} \frac{1+\frac{1}{\sqrt{x}}}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1} = 1.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty}\left(\sqrt{x+4}-\sqrt{x+2}\right).$$

Solution 8.4.7

$$\lim_{x\to\infty}\left(\sqrt{x+4}-\sqrt{x+2}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x+4}-\sqrt{x+2}\right)\left(\sqrt{x+4}+\sqrt{x+2}\right)}{\sqrt{x+4}+\sqrt{x+2}}=$$

$$=\lim_{x\to\infty}\frac{(x+4)-(x+2)}{\sqrt{x+4}+\sqrt{x+2}}=\lim_{x\to\infty}\frac{2}{\sqrt{x+4}+\sqrt{x+2}}=\frac{2}{\infty}=0.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty}x\left(\sqrt{x+1}-\sqrt{x}\right).$$

Solution 8.4.8

$$\lim_{x\to\infty}x\left(\sqrt{x+1}-\sqrt{x}\right)=\lim_{x\to\infty}\frac{x\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\sqrt{x+1}+\sqrt{x}}=$$

$$=\lim_{x\to\infty}\frac{x\left[(x+1)-x\right]}{\sqrt{x+1}+\sqrt{x}}=\lim_{x\to\infty}\frac{x}{\sqrt{x+1}+\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{x}\cdot\sqrt{x}}{\sqrt{x}\cdot\left(\sqrt{1+\frac{1}{\sqrt{x}}}+1\right)}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{1+\frac{1}{\sqrt{x}}}+1}=\frac{\infty}{2}=\infty.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right).$$

Solution 8.4.9

$$\lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4}-x\right)\left(\sqrt{x^2+4}+x\right)}{\sqrt{x^2+4}+x}=$$

$$\lim_{x\to\infty}\frac{(x^2+4)-x^2}{\sqrt{x^2+4}+x}=\lim_{x\to\infty}\frac{4}{\sqrt{x^2+4}+x}=\frac{4}{\infty}=0.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty}\left(\sqrt{x^2+4}-(x+2)\right).$$

Solution 8.4.10

$$\lim_{x\to\infty}\left(\sqrt{x^2+4}-(x+2)\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4}-(x+2)\right)\left(\sqrt{x^2+4}+(x+2)\right)}{\sqrt{x^2+4}+(x+2)}=$$

$$=\lim_{x\to\infty}\frac{(x^2+4)-(x+2)^2}{\sqrt{x^2+4}+x+2}=\lim_{x\to\infty}\frac{-4x}{\sqrt{x^2+4}+x+2}$$

observing $\sqrt{x^2}=x$

$$=\lim_{x\to\infty}\frac{x\cdot(-4)}{\sqrt{x^2}\cdot\left(\sqrt{1+\frac{4}{x^2}}+1+\frac{2}{x}\right)}=\lim_{x\to\infty}\frac{-4}{\sqrt{1+\frac{4}{x^2}}+1+\frac{2}{x}}=\frac{-4}{2}=-2.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to\infty}\frac{1}{\sqrt{x^2-1}-x}.$$

Solution 8.4.11

$$\lim_{x\to\infty}\frac{1}{\sqrt{x^2-1}-x}=\lim_{x\to\infty}\frac{\sqrt{x^2-1}+x}{\left(\sqrt{x^2-1}-x\right)\left(\sqrt{x^2-1}+x\right)}=$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2-1}+x}{(x^2-1)-x^2}=\lim_{x\to\infty}\frac{\sqrt{x^2-1}+x}{-1}=\frac{\infty}{-1}=-\infty.$$

8.4.2 *Limits at Finite Point*

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^2 - 1}.$$

Solution 8.4.12

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)(x-1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{(x-1)}{(x+1)} = \frac{0}{2} = 0.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 5x + 6}.$$

Solution 8.4.13

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{(x-3)(x+2)}{(x-3)(x-2)} = \lim_{x \to 3} \frac{(x+2)}{(x-2)} = \frac{5}{1} = 5.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x \to -2} \frac{2 - x - x^2}{x^2 + 3x + 2}.$$

Solution 8.4.14

$$\lim_{x \to -2} \frac{2 - x - x^2}{x^2 + 3x + 2} = \lim_{x \to -2} \frac{(-1) \cdot (x-1)(x+2)}{(x+1)(x+2)} = \lim_{x \to -2} \frac{-(x-1)}{(x+1)} = \frac{3}{-1} = -3.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 4}\frac{x^2-5x+4}{x^2-6x+1}.$$

Solution 8.4.15 *Not of type "$\frac{0}{0}$" :*

$$\lim_{x\to 4}\frac{x^2-5x+4}{x^2-6x+1}=\frac{0}{-7}=0.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 1}\frac{x^3-x}{x^2+2x-3}.$$

Solution 8.4.16

$$\lim_{x\to 1}\frac{x^3-x}{x^2+2x-3} = \lim_{x\to 1}\frac{x\left(x^2-1\right)}{(x-1)(x+3)}=\lim_{x\to 1}\frac{x(x-1)(x+1)}{(x-1)(x+3)}=$$

$$=\lim_{x\to 1}\frac{x(x+1)}{(x+3)}=\frac{1\cdot 2}{4}=\frac{1}{2}.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{1-x}}{x}.$$

Solution 8.4.17

$$\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{1-x}}{x}\cdot\frac{\sqrt{x+1}+\sqrt{1-x}}{\sqrt{x+1}+\sqrt{1-x}}=$$

$$=\lim_{x\to 0}\frac{(x+1)-(1-x)}{x\cdot\left(\sqrt{x+1}+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{2x}{x\cdot\left(\sqrt{x+1}+\sqrt{1-x}\right)}=$$

$$=\lim_{x\to 0}\frac{2}{\sqrt{x+1}+\sqrt{1-x}}=\frac{2}{1+1}=1.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 0}\frac{\sqrt{x^2+x+1}-1}{x}.$$

Solution 8.4.18

$$\lim_{x\to 0}\frac{\sqrt{x^2+x+1}-1}{x}=\lim_{x\to 0}\frac{\sqrt{x^2+x+1}-1}{x}\cdot\frac{\sqrt{x^2+x+1}+1}{\sqrt{x^2+x+1}+1}=$$
$$=\lim_{x\to 0}\frac{\left(x^2+x+1\right)-1^2}{x\cdot\left(\sqrt{x^2+x+1}+1\right)}=\lim_{x\to 0}\frac{x^2+x}{x\cdot\left(\sqrt{x^2+x+1}+1\right)}=$$
$$=\lim_{x\to 0}\frac{x+1}{\sqrt{x^2+x+1}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 1}\frac{x^2-x}{\sqrt{x}-1}.$$

Solution 8.4.19

$$\lim_{x\to 1}\frac{x^2-x}{\sqrt{x}-1}=\lim_{x\to 1}\frac{x^2-x}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{x\left(x-1\right)\left(\sqrt{x}+1\right)}{x-1}=$$
$$\lim_{x\to 1}\frac{x\left(\sqrt{x}+1\right)}{1}=\frac{1\cdot 2}{1}=2.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x^2-1}.$$

Solution 8.4.20 *Not of type "$\frac{0}{0}$" :*

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x^2-1}=\frac{0}{-1}=0.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 0} \frac{x}{\sqrt{9+x}-3}.$$

Solution 8.4.21

$$\lim_{x\to 0} \frac{x}{\sqrt{9+x}-3} = \lim_{x\to 0} \frac{x}{\sqrt{9+x}-3} \cdot \frac{\left(\sqrt{9+x}+3\right)}{\left(\sqrt{9+x}+3\right)}.$$

$$= \lim_{x\to 0} \frac{x \cdot \left(\sqrt{9+x}+3\right)}{(9+x)-3^2} = \lim_{x\to 0} \frac{x \cdot \left(\sqrt{9+x}+3\right)}{x} =$$

$$= \lim_{x\to 0} \left(\sqrt{9+x}+3\right) = \sqrt{9}+3 = 6.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 1} \frac{1-x^3}{1-x}.$$

Solution 8.4.22 *Use the identity*

$$1-a^3 = (1-a)\left(1+a+a^2\right)$$

for $a = x$ to get

$$\lim_{x\to 1} \frac{1-x^3}{1-x} = \lim_{x\to 1} \frac{(1-x)\left(1+x+x^2\right)}{1-x} = \lim_{x\to 1} \left(1+x+x^2\right) = 3.$$

Step-by-Step Solution

Calculate the following limit. Please check before that the problem is of type "$\frac{0}{0}$" or not.

$$\lim_{x\to 0} \frac{\sqrt[3]{x+1}-1}{x}.$$

Solution 8.4.23 *Using identity (8.4.22) for* $a = \sqrt[3]{x+1}$, *we can multiply both the numerator and the denominator with a suitable expression to cancel the root as*

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt[3]{x+1}-1}{x} &= \lim_{x\to 0} \frac{\left(\sqrt[3]{x+1}-1\right)\cdot\left[\left(\sqrt[3]{x+1}\right)^2+\left(\sqrt[3]{x+1}\right)+1\right]}{x\cdot\left[\left(\sqrt[3]{x+1}\right)^2+\left(\sqrt[3]{x+1}\right)+1\right]} = \\
&= \lim_{x\to 0} \frac{\left(\sqrt[3]{x+1}\right)^3-1}{x\cdot\left[\left(\sqrt[3]{x+1}\right)^2+\left(\sqrt[3]{x+1}\right)+1\right]} = \\
&= \lim_{x\to 0} \frac{1}{\left(\sqrt[3]{x+1}\right)^2+\left(\sqrt[3]{x+1}\right)+1} = \frac{1}{3}.
\end{aligned}$$

8.4.3 *Famous Limits I.*

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to 0}\frac{\sin(2x)}{x}.$$

Solution 8.4.24

$$\lim_{x\to 0}\frac{\sin(2x)}{x} = \lim_{x\to 0}\frac{\sin(2x)}{2x}\cdot 2 = \lim_{y\to 0}\frac{\sin(y)}{y}\cdot 2 = 1\cdot 2 = 2.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to 0}\frac{\sin(x^2)}{x^2}.$$

Solution 8.4.25

$$\lim_{x\to 0}\frac{\sin(x^2)}{x^2} = \lim_{y\to 0}\frac{\sin(y)}{y} = 1.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to 0}\frac{\sin^2(2x)}{x}.$$

Solution 8.4.26

$$\lim_{x\to 0}\frac{\sin^2(2x)}{x} = \lim_{x\to 0}\left(\frac{\sin(2x)}{2x}\right)^2\cdot\frac{4x}{1} = 1^2\cdot 0 = 0.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to 0+}\frac{\sin(\sqrt{x})}{\sqrt{x}}.$$

Solution 8.4.27

$$\lim_{x\to 0+}\frac{\sin(\sqrt{x})}{\sqrt{x}} = \lim_{y\to 0+}\frac{\sin(y)}{y} = 1.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to -2}\frac{\sin(x+2)}{x+2}.$$

Solution 8.4.28

$$\lim_{x\to -2}\frac{\sin(x+2)}{x+2}=\lim_{y\to 0}\frac{\sin(y)}{y}=1$$

(check $y=x+2\to 0 \iff x\to -2$).

Step-by-Step Solution

Calculate the following limit.

Solution 8.4.29

$$\lim_{x\to -2}\frac{\sin(x^2-4)}{x+2}=\lim_{x\to -2}\frac{\sin(x^2-4)}{x^2-4}\cdot\frac{x^2-4}{x+2}=$$

$$=\lim_{y\to 0}\frac{\sin(y)}{y}\cdot\lim_{x\to -2}\frac{(x+2)(x-2)}{x+2}=1\cdot\lim_{x\to -2}\frac{x-2}{1}=1\cdot(-4)=-4$$

(check $y=x^2-4\to 0 \iff x\to -2$).

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to 0}\frac{\sin(2x)}{\sin(7x)}.$$

Solution 8.4.30

$$\lim_{x\to 0}\frac{\sin(2x)}{\sin(7x)}=\lim_{x\to 0}\frac{\sin(2x)}{2x}\cdot\frac{7x}{\sin(7x)}\cdot\frac{2}{7}=$$

$$=\lim_{y_1\to 0}\frac{\sin(y_1)}{y_1}\cdot\lim_{y_2\to 0}\frac{y_2}{\sin(y_2)}\cdot\frac{2}{7}=1\cdot 1\cdot\frac{2}{7}=\frac{2}{7}.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to -1} \frac{\sin(4x+4)}{x^2+x}.$$

Solution 8.4.31

$$\lim_{x\to -1} \frac{\sin(4x+4)}{x^2+x} = \lim_{x\to -1} \frac{\sin(\ 4(x+1)\ )}{x\cdot 4\cdot (x+1)}\cdot 4 = \frac{1}{-1}\cdot 4 = -4.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to 0} \frac{\sin\left(x^2+x\right)}{x}.$$

Solution 8.4.32

$$\lim_{x\to 0} \frac{\sin\left(x^2+x\right)}{x} = \lim_{x\to 0} \frac{\sin\left(x^2+x\right)}{x^2+x}\cdot \frac{x^2+x}{x} = \lim_{y\to 0} \frac{\sin(y)}{y}\cdot \lim_{x\to 0}(x+1) = 1\cdot 1 = 1.$$

Step-by-Step Solution

Calculate the following limit.

$$\lim_{x\to 0} \frac{\sin(2x)}{\tan(5x)}.$$

Solution 8.4.33

$$\lim_{x\to 0} \frac{\sin(2x)}{\tan(5x)} = \lim_{x\to 0} \frac{\sin(2x)}{\frac{\sin(5x)}{\cos(5x)}} = \lim_{x\to 0} \frac{\sin(2x)}{1}\cdot \frac{\cos(5x)}{\sin(5x)} =$$

$$= \lim_{x\to 0} \frac{\sin(2x)}{2x}\cdot \frac{5x}{\sin(5x)}\cdot \frac{2}{5}\cdot \frac{\cos(5x)}{1} =$$

$$= \lim_{y_1\to 0} \frac{\sin(y_1)}{y_1}\cdot \lim_{y_2\to 0} \frac{y_2}{\sin(y_2)}\cdot \frac{2}{5}\cdot \lim_{x\to 0} \frac{\cos(5x)}{1} =$$

$$= 1\cdot 1\cdot \frac{2}{5}\cdot 1 = \frac{2}{5}.$$

8.4.4 *Famous Limits II.*

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{x}.$$

Solution 8.4.34

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{x} = \lim_{x\to 0} \frac{\sin(2x)}{x} - \lim_{x\to 0} \frac{2x}{x} = \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot 2 - 2 = 1 \cdot 2 - 2 = 0.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0} \frac{\sin(3x)}{\ln(1+5x)}.$$

Solution 8.4.35

$$\lim_{x\to 0} \frac{\sin(3x)}{\ln(1+5x)} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{5x}{\ln(1+5x)} \cdot \frac{3}{5} = 1 \cdot 1 \cdot \frac{3}{5} = \frac{3}{5}.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0} \frac{\ln(1+3x)}{2\sin(x)}.$$

Solution 8.4.36

$$\lim_{x\to 0} \frac{\ln(1+3x)}{2\sin(x)} = \lim_{x\to 0} \frac{1}{2} \cdot \frac{\ln(1+3x)}{3x} \cdot \frac{x}{\sin(x)} \cdot \frac{3}{1} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{3}{1} = \frac{3}{2}.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0} \frac{e^{2x} - 1}{\sin(5x)}.$$

Solution 8.4.37

$$\lim_{x\to 0} \frac{e^{2x} - 1}{\sin(5x)} = \lim_{x\to 0} \frac{e^{2x} - 1}{2x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{2}{5} = 1 \cdot 1 \cdot \frac{2}{5} = \frac{2}{5}.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0}\frac{\tan(x)}{e^x-1}.$$

Solution 8.4.38

$$\lim_{x\to 0}\frac{\tan(x)}{e^x-1}=\lim_{x\to 0}\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{e^x-1}=\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)}\cdot\frac{x}{e^x-1}=1.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0}\frac{1-\cos(x)}{x\cdot\sin(x)}.$$

Solution 8.4.39

$$\lim_{x\to 0}\frac{1-\cos(x)}{x\cdot\sin(x)}=\lim_{x\to 0}\frac{1-\cos(x)}{x^2}\cdot\frac{x}{\sin(x)}=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0}\frac{e^{x^2}-\cos(x)}{x^2}.$$

Solution 8.4.40

$$\begin{aligned}\lim_{x\to 0}\frac{e^{x^2}-\cos(x)}{x^2} &= \lim_{x\to 0}\frac{e^{x^2}-1+1-\cos(x)}{x^2}=\\ &= \lim_{x\to 0}\left(\frac{e^{x^2}-1}{x^2}+\frac{1-\cos(x)}{x^2}\right)=1+\frac{1}{2}=\frac{3}{2}.\end{aligned}$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0} \cot(x) \cdot \ln(1+2x).$$

Solution 8.4.41

$$\lim_{x\to 0} \cot(x) \cdot \ln(1+2x) = \lim_{x\to 0} \frac{\cos(x)}{\sin(x)} \cdot \ln(1+2x) =$$

$$= \lim_{x\to 0} \cos(x) \cdot \frac{x}{\sin(x)} \cdot \frac{\ln(1+2x)}{2x} \cdot 2 = 1^3 \cdot 2 = 2.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to 0} \frac{2\cos(3x) - 2 + 9x^2}{2x}.$$

Solution 8.4.42

$$\lim_{x\to 0} \frac{2\cos(3x) - 2 + 9x^2}{2x} = \lim_{x\to 0} \frac{2\cos(3x) - 2}{2x} + \frac{9x^2}{2x} =$$

$$= \lim_{x\to 0} \frac{\cos(3x) - 1}{(3x)^2} \cdot 9x + \frac{9x}{2} = -1 \cdot 0 + 0 = 0.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x \to 0} \frac{1 - \cos(x)}{\ln(x+1) - (x+1)^2 + 1}.$$

Solution 8.4.43

$$\lim_{x \to 0} \frac{1 - \cos(x)}{\ln(x+1) - (x+1)^2 + 1} = \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} \cdot \frac{x^2}{\ln(x+1) - (x+1)^2 + 1} =$$

$$= \frac{1}{2} \cdot \lim_{x \to 0} \frac{x^2}{\ln(x+1) - (x+1)^2 + 1}.$$

We calculate the reciprocial of the second term, that is

$$\lim_{x \to 0} \frac{\ln(x+1) - (x+1)^2 + 1}{x^2} = \lim_{x \to 0} \frac{\ln(x+1) - x^2 - 2x}{x^2} =$$

$$= \lim_{x \to 0} \frac{\ln(x+1)}{x^2} - \frac{x^2 + 2x}{x^2} =$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \left(\frac{\ln(x+1)}{x} - x - 2 \right) =$$

$$= \pm\infty \cdot (1 - 0 - 2) = \pm\infty \cdot (-1) = \mp\infty.$$

So the final answer is

$$\lim_{x \to 0} \frac{1 - \cos(x)}{\ln(x+1) - (x+1)^2 + 1} \frac{1}{2} \cdot \frac{1}{\mp\infty} = \frac{1}{2} \cdot 0 = 0.$$

8.4.5 Function Limits $"\frac{c}{0}"$ $(c \neq 0)$

Step-by-Step Solution

Calculate the following limit

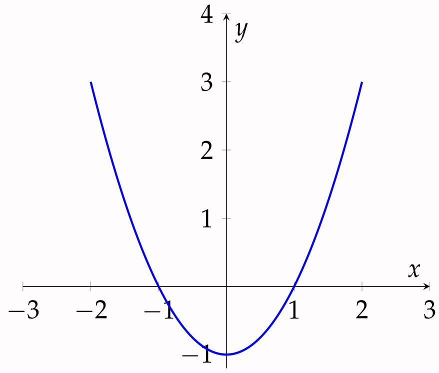
$$\lim_{x\to -1} \frac{x^2-2x+1}{x^2-1}.$$

Solution 8.4.44 *The numerator is*

$$\lim_{x\to -1} \left(x^2-2x+1\right) = 4.$$

The denominator is

$$\lim_{x\to -1} \left(x^2-1\right) = 0.$$



In more detail

$$\lim_{x\to -1-} \left(x^2-1\right) = 0+,$$

and

$$\lim_{x\to -1+} \left(x^2-1\right) = 0-.$$

So

$$\lim_{x\to -1-} \frac{x^2-2x+1}{x^2-1} = "\frac{4}{0+}" = +\infty \quad \textit{and} \quad \lim_{x\to -1+} \frac{x^2-2x+1}{x^2-1} = "\frac{4}{0-}" = -\infty\,.$$

This implies that

$$\lim_{x\to -1} \frac{x^2-2x+1}{x^2-1}$$

does not exist.

Step-by-Step Solution

Calculate the following limit

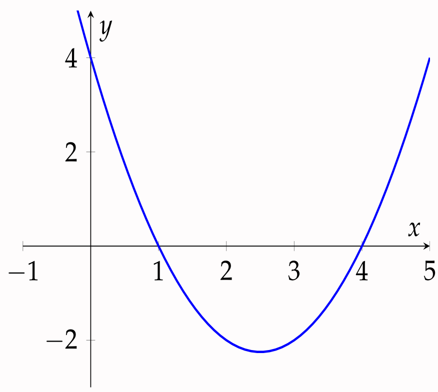
$$\lim_{x\to 4} \frac{x^2-6x+1}{x^2-5x+4}.$$

Solution 8.4.45 *The numerator is*

$$\lim_{x\to 4} \left(x^2-6x+1\right) = -7.$$

The denominator is

$$\lim_{x\to 4} \left(x^2-5x+4\right) = 0.$$



In more detail

$$\lim_{x\to 4-} \left(x^2-5x+4\right) = 0-,$$

and

$$\lim_{x\to 4+} \left(x^2-5x+4\right) = 0+.$$

So

$$\lim_{x\to 4-} \frac{x^2-6x+1}{x^2-5x+4} = \text{"}\frac{-7}{0-}\text{"} = +\infty \quad \textit{and} \quad \lim_{x\to 4+} \frac{x^2-6x+1}{x^2-5x+4} = \text{"}\frac{-7}{0+}\text{"} = -\infty.$$

This implies that

$$\lim_{x\to 4} \frac{x^2-6x+1}{x^2-5x+4}$$

does not exist.

Step-by-Step Solution

Calculate the following limit

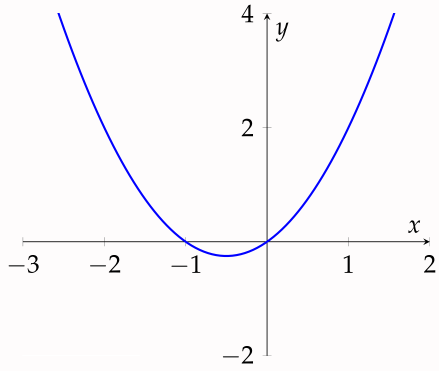
$$\lim_{x\to -1-} \frac{\sin(4x+2)}{x^2+x}.$$

Solution 8.4.46 *The numerator is*

$$\lim_{x\to -1} \sin(4x+2) = \sin\left(-2^{rad}\right) \approx -0.9093.$$

The denominator is

$$\lim_{x\to -1} \left(x^2+x\right) = 0.$$



In more detail

$$\lim_{x\to -1-} \left(x^2+x\right) = 0+,$$

and

$$\lim_{x\to -1+} \left(x^2+x\right) = 0-.$$

So

$$\lim_{x\to -1-} \frac{\sin(4x+2)}{x^2+x} = \frac{"\ -0.9093\ "}{0+} = -\infty,$$

and

$$\lim_{x\to -1+} \frac{\sin(4x+2)}{x^2+x} = \frac{"\ -0.9093\ "}{0-} = +\infty.$$

This implies that

$$\lim_{x\to -1} \frac{\sin(4x+2)}{x^2+x}$$

does not exist.

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to\frac{\pi}{2}+}\frac{\ln\left(1-\sin\left(x\right)\right)}{\frac{\pi}{2}-x}.$$

Solution 8.4.47 *In the numerator we have*

$$\lim_{x\to\frac{\pi}{2}\pm}\left(1-\sin\left(x\right)\right)=0+,$$

so

$$\lim_{x\to\frac{\pi}{2}\pm}\ln\left(1-\sin\left(x\right)\right)=-\infty.$$

The denominator is

$$\lim_{x\to\frac{\pi}{2}-}\left(\frac{\pi}{2}-x\right)=0+,$$

and

$$\lim_{x\to\frac{\pi}{2}+}\left(\frac{\pi}{2}-x\right)=0-.$$

So

$$\lim_{x\to\frac{\pi}{2}-}\frac{\ln\left(1-\sin\left(x\right)\right)}{\frac{\pi}{2}-x}=\frac{"-\infty"}{0+}=-\infty,$$

and

$$\lim_{x\to\frac{\pi}{2}+}\frac{\ln\left(1-\sin\left(x\right)\right)}{\frac{\pi}{2}-x}=\frac{"-\infty"}{0-}=+\infty.$$

This implies that

$$\lim_{x\to\frac{\pi}{2}+}\frac{\ln\left(1-\sin\left(x\right)\right)}{\frac{\pi}{2}-x}$$

does not exist.

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to a}e^{\frac{x}{1-x}}=\lim_{x\to a}\exp\left(\frac{x}{1-x}\right),\quad\text{where } a=0,1,\pm\infty.$$

Solution 8.4.48 *Since function* exp *is continuous everywhere, we have to calculate the exponent first.*

For $a=0$, *we have*

$$\lim_{x\to 0}\left(\frac{x}{1-x}\right)=0,$$

so

$$\lim_{x\to 0}e^{\frac{x}{1-x}}=e^{0}=1.$$

For $a = 1$*, we have*

$$\lim_{x\to 1-}\left(\frac{x}{1-x}\right) = \frac{''\,1\,''}{0+} = +\infty,$$

and

$$\lim_{x\to 1+}\left(\frac{x}{1-x}\right) = \frac{''\,1\,''}{0-} = -\infty,$$

so

$$\lim_{x\to 1-} e^{\frac{x}{1-x}} = ''e^{+\infty}'' = +\infty,$$

and

$$\lim_{x\to 1+} e^{\frac{x}{1-x}} = ''e^{-\infty}'' = 0.$$

This implies that

$$\lim_{x\to 1+} e^{\frac{x}{1-x}}$$

does not exist. For $a = \pm\infty$*, we have*

$$\lim_{x\to\pm\infty}\left(\frac{x}{1-x}\right) = \lim_{x\to\pm\infty}\frac{x\cdot 1}{x\cdot\left(\frac{1}{x}-1\right)} = \lim_{x\to\pm\infty}\frac{1}{\frac{1}{x}-1} = -1,$$

so

$$\lim_{x\to\pm\infty} e^{\frac{x}{1-x}} = e^{-1} = \frac{1}{e} \approx 0.36788.$$

Step-by-Step Solution

Calculate the following limit

$$\lim_{x\to a}\frac{x}{1-e^{1/x}}, \quad \text{where } a = 0, 1, \pm\infty.$$

Solution 8.4.49 *For* $a = 0$*, we have*

$$\lim_{x\to 0+}\left(1-e^{1/x}\right) = ''\left(1-''e^{+\infty}''\right)'' = ''\left(1-''+\infty''\right)'' = -\infty,$$

so

$$\lim_{x\to 0+}\frac{x}{1-e^{1/x}} = \frac{''\,0\,''}{-\infty} = 0,$$

and

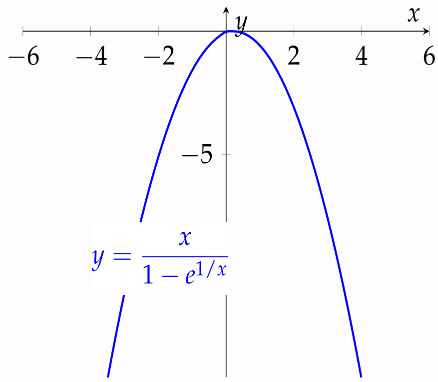
$$\lim_{x\to 0-}\left(1-e^{1/x}\right) = ''\left(1-''e^{-\infty}''\right)'' = (1-0) = 1,$$

which implies

$$\lim_{x\to 0-}\frac{x}{1-e^{1/x}} = \frac{0}{1} = 0.$$

The result is

$$\lim_{x\to 0}\frac{x}{1-e^{1/x}} = \frac{0}{1} = 0.$$



For $a = 1$, we have

$$\lim_{x \to 1} \frac{x}{1-e^{1/x}} = \frac{1}{1-e^1} \approx -0.58198.$$

For $a = +\infty$, we have

$$\lim_{x \to +\infty} \frac{x}{1-e^{1/x}} = \lim_{x \to +\infty} \frac{'' +\infty ''}{'' 1-e^{0+} ''} = \frac{'' +\infty ''}{'' 0- ''} = -\infty.$$

For $a = -\infty$, we have

$$\lim_{x \to -\infty} \frac{x}{1-e^{1/x}} = \lim_{x \to +\infty} \frac{'' -\infty ''}{'' 1-e^{0-} ''} = \frac{'' -\infty ''}{'' 0+ ''} = -\infty.$$

8.4.6 *Continuity of Functions*

Step-by-Step Solution

Decide the continuity of function

$$f(x) = \begin{cases} \dfrac{\sin(8x)}{\sin(4x)} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}.$$

Solution 8.4.50 *As*

$$\lim_{x\to 0} \frac{\sin(8x)}{\sin(4x)} = \lim_{x\to 0} \frac{\sin(8x)}{8x} \cdot \frac{8}{4} \cdot \frac{4x}{\sin(4x)} = \frac{8}{4} = 2 = f(0),$$

function f is *continuous at* $a = 0$.

Step-by-Step Solution

Decide the continuity of function

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x^2 - 2x - 3} & \text{if } x \neq 3 \\ \dfrac{5}{4} & \text{if } x = 3 \end{cases}.$$

Solution 8.4.51 *As*

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 2x - 3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x+1)} = \lim_{x\to 3} \frac{x+2}{x+1} = \frac{5}{4} = f(3),$$

function f is *continuous at* $a = 3$.

Step-by-Step Solution

Decide the continuity of function

$$f(x) = \begin{cases} 2^{x-1} & \text{if } x \leq 0 \\ \dfrac{\sqrt{x+1} - 1}{x} & \text{if } x > 0 \end{cases}.$$

Solution 8.4.52 *As*

$$\lim_{x\to 0-} 2^{x-1} = 2^{0-1} = \frac{1}{2} = f(0),$$

and

$$\lim_{x\to 0+}\frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0+}\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} =$$

$$= \lim_{x\to 0}\frac{(x+1)-1}{x\cdot\left(\sqrt{x+1}+1\right)} = \lim_{x\to 0}\frac{x}{x\cdot\left(\sqrt{x+1}+1\right)} =$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2},$$

function f is *continuous at* $a = 0$.

Step-by-Step Solution

Decide the continuity of function

$$f(x) = \begin{cases} \dfrac{x^2-x}{2-2x} & \text{if } x < 1 \\ \log_{\frac{1}{2}}(2^x+1) & \text{if } x \geq 1 \end{cases}.$$

Solution 8.4.53 *As*

$$\lim_{x\to 1+}\log_{\frac{1}{2}}(2^x+1) = \log_{\frac{1}{2}}\left(2^1+1\right) = \log_{\frac{1}{2}}(3) = f(1) \approx -1.5850,$$

and

$$\lim_{x\to 1-}\frac{x^2-x}{2-2x} = \lim_{x\to 1}\frac{x(x-1)}{2(1-x)} = \lim_{x\to 1}\frac{-x}{2} = \frac{-1}{2},$$

function f *is* not *continuous at* $a = 1$.

8.5 *Derivatives of Real Functions*

Step-by-Step Solution

Derivate the following function.

$$F(x) = \sqrt{x + \sqrt{x}}$$

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Solution 8.5.1 *First, we use Theorem 5.5 with the usual notation*

$$F(x) = f(g(x)) \ ,$$

where

$$f(x) = \sqrt{x}\,,$$

and

$$g(x) = x + \sqrt{x}.$$

Using Theorem 5.1 and some basic mathematics, all the derivations can be evaluated using 5.3 Table of Derivatives, that is

$$F'(x) = f'(g(x)) \cdot g'(x)\,.$$

$$f'(x) = \frac{1}{2\sqrt{x}}\,,$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{x} + \sqrt{\mathbf{x}}) = \frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{x} + \sqrt{\mathbf{x}}}}\,,$$

and

$$g'(x) = 1 + \frac{1}{2}x^{\left(\frac{-1}{2}\right)}.$$

This follows

$$F'(x) = \frac{1}{2}\left(x + \sqrt{x}\right)^{\left(\frac{-1}{2}\right)}\left(1 + \frac{1}{2}x^{\left(\frac{-1}{2}\right)}\right) = \frac{2 + \frac{1}{\sqrt{x}}}{4\sqrt{x + \sqrt{x}}}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = \frac{\ln\left(2x - 4x^3\right)}{\sqrt[3]{4x + 1}}$$

Solution 8.5.2 *Let $N(x)$ be the numerator.*

$$N(x) = ln\left(2x - 4x^3\right)$$

First, we calculate the derivative of the numerator using Theorem 5.5 with the usual notation

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \ln(x), \\ &\quad \textit{and} \\ g(x) &= 2x - 4x^3. \end{aligned}$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all the derivations can be evaluated.

$$N'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = \frac{1}{x},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'\left(\mathbf{2x - 4x^3}\right) = \left(\frac{\mathbf{1}}{\mathbf{2x - 4x^3}}\right).$$

As

$$g'(x) = 2 - 12x^2,$$

we get

$$N'(x) = \frac{1}{2x - 4x^3}\left(2 - 12x^2\right) = \frac{2 - 12x^2}{2x - 4x^3}.$$

Second, we calculate the derivative of the denominator. Let $M(x)$ be the denominator.

$$M(x) = \sqrt[3]{4x + 1}$$

We use Theorem 5.5 again with the usual notation.

$$M(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \sqrt[3]{x}, \\ &\quad \textit{and} \\ g(x) &= 4x + 1. \end{aligned}$$

With the same method as above, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \frac{1}{3}\sqrt[3]{x^2},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{4x}+\mathbf{1}) = \frac{1}{3}\sqrt[3]{(4x+1)^2}.$$

As

$$g'(x) = 4,$$

we get

$$M'(x) = \frac{4}{3\sqrt[3]{(4x+1)^2}}.$$

From Theorem 5.4, we have

$$F'(x) = \left(\frac{\ln(2x-4x^3)}{\sqrt[3]{4x+1}}\right)' = \frac{\left(\ln(2x-4x^3)\right)'\sqrt[3]{4x+1} - \ln(2x-4x^3)\left(\sqrt[3]{4x+1}\right)'}{\left(\sqrt[3]{4x+1}\right)^2}$$

so the result is

$$F'(x) = \frac{\frac{2-12x^2}{2x-4x^3}\sqrt[3]{4x+1} - \ln(2x-4x^3)\frac{4}{3\sqrt[3]{(4x+1)^2}}}{\left(\sqrt[3]{4x+1}\right)^2}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = \frac{\ln(x^2+2x)}{\sin(e^x)}$$

Solution 8.5.3 *Let $N(x)$ be the numerator.*

$$N(x) = ln\left(x^2+2x\right)$$

First, we calculate the derivative of the numerator using Theorem 5.5 with the usual notation

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \ln(x) \\ &\text{and} \\ g(x) &= x^2+2x. \end{aligned}$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all the derivations can be evaluated.

$$N'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = \frac{1}{x},$$

so

$$f'(g(x)) = f'(x^2 + 2x) = \left(\frac{1}{x^2 + 2x}\right).$$

As

$$g'(x) = 2x + 2,$$

we get

$$N'(x) = \frac{1}{x^2 + 2x}(2x + 2) = \frac{2x + 2}{x^2 + 2x}.$$

Second, we calculate the derivative of the denominator. Let $M(x)$ *be the denominator.*

$$M(x) = \sin(e^x)$$

We use Theorem 5.5 again with the usual notation.

$$M(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \sin(x) \\ &\text{and} \\ g(x) &= e^x. \end{aligned}$$

With the same method as above, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \cos(x),$$

so

$$f'(g(x)) = f'(e^x) = \cos(e^x).$$

As

$$g'(x) = e^x,$$

we get

$$M'(x) = \cos(e^x) \cdot e^x.$$

From Theorem 5.4, we have

$$F'(x) = \left(\frac{\ln(x^2 + 2x)}{\sin(e^x)}\right)' = \frac{\left(\ln(x^2 + 2x)\right)' \sin(e^x) - \ln(x^2 + 2x)\left(\sin(e^x)\right)'}{\left(\sin(e^x)\right)^2},$$

so the result is

$$F'(x) = \frac{\frac{2x + 2}{x^2 + 2x}\sin(e^x) - \ln(x^2 + 2x)\cos(e^x)\, e^x}{\sin^2(e^x)}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = \sin^2(x)\tan(x^3 - 5x)$$

Solution 8.5.4 *Let $M(x)$ be the first member of product.*

$$M(x) = \sin^2(x)$$

First, we calculate the derivative of M using Theorem 5.5 with the usual notation

$$M(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= x^2 \\ &\quad \textit{and} \\ g(x) &= \sin(x). \end{aligned}$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = 2x,$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\sin(\mathbf{x})) = (\mathbf{2}\sin(\mathbf{x})).$$

As

$$g'(x) = \cos(x),$$

we get

$$M'(x) = 2\sin(x)\cos(x).$$

Second, we calculate the derivative of the second member. Let

$$N(x) = \tan(x^3 - 5x).$$

We use Theorem 5.5 again with the usual notation.

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \tan(x) \\ &\quad \textit{and} \\ g(x) &= x^3 - 5x. \end{aligned}$$

With the same method as above, all the derivations can be evaluated.

$$N'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \frac{1}{\cos^2(x)},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{x^3 - 5x}) = \frac{1}{\cos^2(x^3 - 5x)}.$$

As

$$g'(x) = 3x^2 - 5,$$

we get

$$N'(x) = \frac{1}{\cos^2(x^3 - 5x)} \cdot \left(3x^2 - 5\right) = \frac{3x^2 - 5}{\cos^2(x^3 - 5x)}.$$

From Theorem 5.3, we have

$$F'(x) = \left(\sin^2(x)\tan(x^3 - 5x)\right)' = \left(\sin^2(x)\right)' \tan(x^3 - 5x) - \sin^2(x)\left(\tan(x^3 - 5x)\right)',$$

so the result is

$$F'(x) = 2\sin(x)\cos(x)\tan(x^3 - 5x) + \sin^2(x)\frac{3x^2 - 5}{\cos^2(x^3 - 5x)}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = \frac{e^{2-3x^4}}{\sqrt[4]{\cot(x)}}$$

Solution 8.5.5 *Let $N(x)$ be the numerator.*

$$N(x) = e^{2-3x^4}$$

First, we calculate the derivative of the numerator using Theorem 5.5 with the usual notation

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= e^x \\ &\quad \textit{and} \\ g(x) &= 2 - 3x^4. \end{aligned}$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all *the derivations can be evaluated.*

$$N'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = e^{x},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'\left(\mathbf{2 - 3x^4}\right) = \left(\mathbf{e^{2-3x^4}}\right).$$

As

$$g'(x) = -12x^3,$$

we get

$$N'(x) = e^{2-3x^4}\left(-12x^3\right).$$

Second, we calculate the derivative of the denominator. Let $M(x)$ *be the denominator.*

$$M(x) = \sqrt[4]{\cot(x)},$$

We use Theorem 5.5 again with the usual notation.

$$M(x) = f(g(x)),$$

where

$$f(x) = \sqrt[4]{x}$$

and

$$g(x) = \cot(x).$$

With the same method as above, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \frac{1}{\left(4\sqrt[4]{x^3}\right)},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\cot(\mathbf{x})) = \frac{1}{\left(4\sqrt[4]{\cot^3(x)}\right)}.$$

As

$$g'(x) = \frac{-1}{\sin^2(x)},$$

we get

$$M'(x) = \frac{1}{\left(4\sqrt[4]{\cot^3(x)}\right)} \cdot \frac{-1}{\sin^2(x)}.$$

From Theorem 5.4, we have

$$F'(x) = \left(\frac{e^{2-3x^4}}{\sqrt[4]{\cot(x)}}\right)' = \frac{\left(e^{2-3x^4}\right)' \sqrt[4]{\cot(x)} - e^{2-3x^4}\left(\sqrt[4]{\cot(x)}\right)'}{\left(\sqrt[4]{\cot(x)}\right)^2},$$

so the result is

$$F'(x) = \frac{e^{2-3x^4}\left(-12x^3\right)\sqrt[4]{\cot(x)} - e^{2-3x^4}\frac{1}{\left(4\sqrt[4]{\cot^3(x)}\right)} \cdot \frac{-1}{\sin^2(x)}}{\left(\sqrt[4]{\cot(x)}\right)^2}.$$

$$F'(x) = \frac{e^{2-3x^4}\left(-12x^3\right)\sqrt[4]{\cot(x)} + \frac{e^{2-3x^4}}{\left(4\sqrt[4]{\cot^3(x)}\right)\sin^2(x)}}{\left(\sqrt[4]{\cot(x)}\right)^2}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = \frac{\tan(e^x)}{\ln(x^2)}$$

Solution 8.5.6 *Let $N(x)$ be the numerator.*

$$N(x) = \tan(e^x)$$

First, we calculate the derivative of the numerator using Theorem 5.5 with the usual notation

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \tan(x) \\ &\textit{and} \\ g(x) &= e^x. \end{aligned}$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all the derivations can be evaluated.

$$N'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = \frac{1}{\cos^2(x)},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{e^x}) = \left(\frac{\mathbf{1}}{\cos^2(\mathbf{e^x})}\right).$$

As

$$g'(x) = e^x,$$

we get

$$N'(x) = \frac{1}{\cos^2(e^x)} e^x = \frac{e^x}{\cos^2(e^x)}.$$

Second, we calculate the derivative of the denominator. Let $M(x)$ be the denominator.

$$M(x) = \ln(x^2),$$

We use Theorem 5.5 again with the usual notation.

$$M(x) = f(g(x)),$$

where

$$f(x) = \ln(x)$$
and
$$g(x) = x^2.$$

With the same method as above, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \frac{1}{x},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{x^2}) = \frac{1}{x^2}.$$

As

$$g'(x) = 2x,$$

we get

$$M'(x) = \frac{1}{x^2} 2x = \frac{2x}{x^2} = \frac{2}{x}.$$

From Theorem 5.4, we have

$$F'(x) = \left(\frac{\tan(e^x))}{\ln(x^2)}\right)' = \frac{(\tan(e^x))' \ln(x^2) - \tan(e^x) \left(\ln(x^2)\right)'}{\left(\ln(x^2)\right)^2},$$

so the result is

$$F'(x) = \frac{\frac{e^x}{\cos^2(e^x)} \ln(x^2) - \tan(e^x) \frac{2}{x}}{\ln^2(x^2)} = \frac{\frac{e^x \ln(x^2)}{\cos^2(e^x)} - \frac{2\tan(e^x)}{x}}{\ln^2(x^2)}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = \cos(x^3 - 2x^2)\ln(\sin(x))$$

Solution 8.5.7 *Let $M(x)$ the first member of product.*

$$M(x) = \cos(x^3 - 2x^2)$$

First, we calculate the derivative of the numerator using Theorem 5.5 with the usual notation

$$M(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \cos(x) \\ &\quad \textit{and} \\ g(x) &= x^3 - 2x^2. \end{aligned}$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = -\sin(x),$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{x}^3 - 2\mathbf{x}^2) = -\sin(\mathbf{x}^3 - 2\mathbf{x}^2).$$

As

$$g'(x) = 3x^2 - 4x,$$

we get

$$M'(x) = -\sin\left(x^3 - 2x^2\right)\left(3x^2 - 4x\right).$$

Second, we calculate the derivative of the second member. Let $N(x)$

$$N(x) = \ln(\sin(x)),$$

We use Theorem 5.5 again with the usual notation.

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \ln(x) \\ &\quad \textit{and} \\ g(x) &= \sin(x). \end{aligned}$$

With the same method as above, all the derivations can be evaluated.

$$N'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \frac{1}{x},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\sin(\mathbf{x})) = \frac{1}{\sin(x)}.$$

As

$$g'(x) = \cos(x),$$

we get

$$N'(x) = \frac{1}{\sin(x)} \cdot \cos(x) = \frac{\cos(x)}{\sin(x)}.$$

From Theorem 5.3, we have

$$\begin{aligned} F'(x) &= \left(\cos(x^3 - 2x^2)\ln(\sin(x))\right)' = \\ &= \left(\cos(x^3 - 2x^2)\right)'\ln(\sin(x)) + \cos(x^3 - 2x^2)\left(\ln(\sin(x))\right)', \end{aligned}$$

so the result is

$$F'(x) = -\sin\left(x^3 - 2x^2\right)\left(3x^2 - 4x\right)\ln(\sin(x)) + \cos(x^3 - 2x^2)\frac{\cos(x)}{\sin(x)}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = \frac{3^{5x+2}}{\ln(x^2 + x)}$$

Solution 8.5.8 *Let $N(x)$ the numerator.*

$$N(x) = 3^{5x+2}$$

First, we calculate the derivative of the numerator using Theorem 5.5 with the usual notation

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= 3^x \\ & \quad \textit{and} \\ g(x) &= 5x + 2. \end{aligned}$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all *the derivations can be evaluated.*

$$N'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = 3^x \ln(3),$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'((\mathbf{5x}+\mathbf{2}))\ln(3) = \mathbf{3}^{(\mathbf{5x}+\mathbf{2})}\ln(3).$$

As

$$g'(x) = 5,$$

we get

$$N'(x) = 3^{(5x+2)}\ln(3)\cdot 5 = 5\ln(3)3^{(5x+2)}.$$

Second, we calculate the derivative of the denominator. Let $M(x)$ the denominator.

$$M(x) = \ln(x^2 + x),$$

We use Theorem 5.5 again with the usual notation.

$$M(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \ln(x) \\ &\textit{and} \\ g(x) &= x^2 + x. \end{aligned}$$

With the same method as above, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \frac{1}{x},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{x}^2 + \mathbf{x}) = \frac{1}{x^2 + x}.$$

As

$$g'(x) = 2x + 1,$$

we get

$$M'(x) = \frac{1}{x^2+x}(2x+1) = \frac{2x+1}{x^2+x}.$$

From Theorem 5.4, we have

$$F'(x) = \left(\frac{3^{(5x+2)}}{\ln(x^2+x)}\right)' = \frac{\left(3^{(5x+2)}\right)'\ln(x^2+x) - 3^{(5x+2)}\left(\ln(x^2+x)\right)'}{\left(\ln(x^2+x)\right)^2},$$

so the result is

$$F'(x)\frac{5\ln(3)3^{(5x+2)}\ln(x^2+x)-\dfrac{3^{(5x+2)}(2x+1)}{x^2+x}}{\ln^2(x^2+x)}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = e^{3\ln(x)}\cot(x^3-5x)$$

Solution 8.5.9 *Let $M(x)$ the first member of product.*

$$M(x) = e^{3\ln(x)}$$

Notice

$$M(x) = e^{3\ln(x)} = (e^{ln(x)})^3 = x^3,$$

becasuse of this

$$M'(x) = 3x^2,$$

Second, we calculate the derivative of the second member. Let $N(x)$

$$N(x) = \cot(x^3-5x),$$

We use Theorem 5.5 again with the usual notation.

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \cot(x) \\ & \quad \text{and} \\ g(x) &= x^3-5x. \end{aligned}$$

With the same method as above, all the derivations can be evaluated.

$$N'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = \frac{-1}{x},$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'((\mathbf{x^3-5x})) = \frac{-1}{\sin^2(x^3-5x)}.$$

As

$$g'(x) = 3x^2-5,$$

we get

$$N'(x) = \frac{-1}{\sin^2(x^3-5x)} \cdot (3x^2-5) = \frac{5-3x^2}{\sin^2(x^3-5x)}.$$

From Theorem 5.3, we have

$$F'(x) = \left(e^{3\ln(x)}\cot(x^3-5x)\right)' = \left(e^{3\ln(x)}\right)'\cot(x^3-5x) - e^{3\ln(x)}\left(\cot(x^3-5x)\right)',$$

so the result is

$$F'(x) = 3x^2\cot(x^3-5x) + e^{3\ln(x)}\frac{5-3x^2}{\sin^2(x^3-5x)} = 3x^2\cot(x^3-5x) + \frac{x^3(5-3x^2)}{\sin^2(x^3-5x)}.$$

Step-by-Step Solution

Derivate the following function.

$$F(x) = (x^2+1)^{1974}\cos(x^5-3x^2)$$

Solution 8.5.10 *Let $M(x)$ the first member of product.*

$$M(x) = (x^2+1)^{1974}$$

First, we calculate the derivative of the numerator using Theorem 5.5 with the usual notation

$$M(x) = f(g(x)),$$

where

$$f(x) = x^{1974}$$

and

$$g(x) = x^2+1.$$

Using the differentiation rules, 5.3 Table of Derivatives and some basic mathematics, all the derivations can be evaluated.

$$M'(x) = f'(g(x)) \cdot g'(x).$$

and

$$f'(x) = 1974\left(x^{1973}\right),$$

so

$$f'(\mathbf{g(x)}) = f'(\mathbf{x^2+1}) = \mathbf{1974}\,(\mathbf{x^2+1})^{\mathbf{1973}}.$$

As

$$g'(x) = 2x,$$

we get

$$M'(x) = 1973(x^2+1)^{1973} \cdot 2x = 3946x(x^2+1)^{1973}.$$

Second, we calculate the derivative of the second member. Let $N(x)$

$$N(x) = \cos(x^5 - 3x^2),$$

We use Theorem 5.5 again with the usual notation.

$$N(x) = f(g(x)),$$

where

$$\begin{aligned} f(x) &= \cos(x) \\ &\quad \textit{and} \\ g(x) &= x^5 - 3x^2. \end{aligned}$$

With the same method as above, all the derivations can be evaluated.

$$N'(x) = f'(g(x)) \cdot g'(x),$$

and

$$f'(x) = -\sin(x),$$

so

$$f'(\mathbf{g}(\mathbf{x})) = f'(\mathbf{x^5 - 3x^2}) = -\sin\left(x^5 - 3x^2\right).$$

As

$$g'(x) = 5x^4 - 6x,$$

we get

$$N'(x) = -\sin(x^5 - 3x^2)(5x^4 - 6x).$$

From Theorem 5.3, we have

$$\begin{aligned} F'(x) &= \left((x^2+1)^{1974}\cos(x^5 - 3x^2)\right)' = \\ &= \left((x^2+1)^{1974}\right)'\cos(x^5 - 3x^2) - (x^2+1)^{1974}\left(\cos(x^5 - 3x^2)\right)', \end{aligned}$$

so the result is

$$F'(x) = 3946x(x^2+1)^{1973}\cos(x^5 - 3x^2) - (x^2+1)^{1974}\sin(x^5 - 3x^2)(5x^4 - 6x).$$

8.5.1 *Application I. - The Tangent Line*

Step-by-Step Solution

Write the equation of the tangent line of $x_0 = 4$ to the graph of the following function.

$$f(x) = 4x - \frac{1}{x^2}.$$

Solution 8.5.11 *First we need to compute a following derivative function*

$$f'(x) = \left(4x - \frac{1}{x^2}\right)' = 4 + \frac{2}{x^3}.$$

Then we need to compute a following values:

$$f(x_0) = f(4) = 4 \cdot 4 - \frac{1}{4^2} = 16 - \frac{1}{16} = \frac{255}{16}$$

$$f'(x_0) = f'(4) = 4 + \frac{2}{4^3} = \frac{4^4 + 2}{4^3} = \frac{258}{64} = \frac{129}{32}$$

Finally write the equation of the tangent line.

$$y = \frac{129}{32}(x-4) + \frac{255}{16}$$

$$y = \frac{129}{32}x - \frac{3}{16}$$

Step-by-Step Solution

Write the equation of the tangent line of $x_0 = e$ to the graph of the following function.

$$f(x) = x\ln(x).$$

Solution 8.5.12 *First we need to compute a following derivative function*

$$f'(x) = (xln(x))' = ln(x) + x\frac{1}{x} = ln(x) + 1.$$

Then we need to compute a following values:

$$f(x_0) = f(e) = eln(e) = e$$

$$f'(x_0) = f'(e) = ln(e) + 1 = 2$$

Finally write the equation of the tangent line.

$$y = 2(x-e) + e$$

$$y = 2x - e$$

Step-by-Step Solution

Write the equation of the tangent line of $x_0 = 3$ to the graph of the following function.

$$f(x) = \sqrt{x+1}.$$

Solution 8.5.13 *First we need to compute a following derivative function*

$$f'(x) = \left(\sqrt{x+1}\right)' = \frac{1}{2\sqrt{x+1}}.$$

Then we need to compute a following values:

$$\begin{aligned} f(x_0) &= f(3) = \sqrt{3+1} = 2 \\ f'(x_0) &= f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{4} \end{aligned}$$

Finally write the equation of the tangent line.

$$\begin{aligned} y &= \frac{1}{4}(x-3)+2 \\ y &= \frac{1}{4}x+\frac{5}{4} \end{aligned}$$

Step-by-Step Solution

Write the equation of the tangent line of $x_0 = 2$ to the graph of the following function.

$$f(x) = \frac{x+2}{x-3}.$$

Solution 8.5.14 *First we need to compute a following derivative function*

$$f'(x) = \left(\frac{x+2}{x-3}\right)' = \frac{(x-3)-(x+2)}{(x-3)^2} = \frac{-5}{(x-3)^2}.$$

Then we need to compute a following values:

$$\begin{aligned} f(x_0) &= f(2) = \frac{2+2}{2-3} = -4 \\ f'(x_0) &= f'(2) = \frac{-5}{(2-3)^2} = -5 \end{aligned}$$

Finally write the equation of the tangent line.

$$y = -5(x-2) - 4y = -5x+6$$

Step-by-Step Solution

Write the equation of the tangent line of $x_0 = 0$ to the graph of the following function.

$$f(x) = 2x - \frac{1}{x+1}.$$

Solution 8.5.15 *First we need to compute a following derivative function*

$$f'(x) = \left(2x - \frac{1}{x+1}\right)' = 2 + \frac{1}{(x+1)^2}.$$

Then we need to compute a following values:

$$f(x_0) = f(0) = 2 \cdot 0 - \frac{1}{0+1} = -1$$
$$f'(x_0) = f'(0) = 2 + \frac{1}{(0+1)^2} = 3$$

Finally write the equation of the tangent line.

$$y = 3(x-0) - 1$$
$$y = 3x - 1$$

8.5.2 *Application II. - Extremal Values of Functions and Monotonicity*

Step-by-Step Solution

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the functions.

$$f(x) = \frac{x+1}{x-3} \quad , x \in \mathbb{R} \setminus \{3\}.$$

Solution 8.5.16 *This function has one breaking point ($x = 3$), so we have to calculate the following limits:*

$$\lim_{x \to -\infty} \frac{x+1}{x-3} = 1$$
$$\lim_{x \to 3^-} \frac{x+1}{x-3} = -\infty$$
$$\lim_{x \to 3^+} \frac{x+1}{x-3} = \infty$$
$$\lim_{x \to \infty} \frac{x+1}{x-3} = 1$$

This follows that there is neither a global maximal nor a global minimal value, just local. Now, we determine the derivate function.

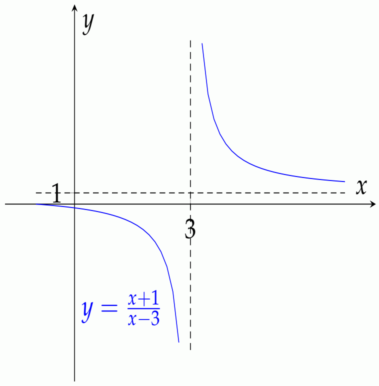
$$f'(x) = \left(\frac{x+1}{x-3}\right)' = \frac{(x-3)-(x+1)}{(x-3)^2} = \frac{-4}{(x-3)^2}$$

We use Theorem 5.7 to determine the intervals, where function f is increasing or decreasing. For this, we determine the intervals where the derivative function f' is negative, the intervals where it is positive and examine where function f' changes sign. Using the derivative function we obtain, that

$$f'(x) = \frac{-4}{(x-3)^2} < 0.$$

We use the following table to summarise our results.

	$x < 3$	3	$3 < x$
$f'(x)$	$-$	×	$-$
$f(x)$	↘	×	↘



Step-by-Step Solution

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the functions.

$$x \in [-1;2], \quad f(x) = x^3 - 6x^2.$$

Solution 8.5.17 *In that case we can get eighter global or local extremal points. Firstly we need to derivate the function.*

$$f'(x) = \left(x^3 - 6x^2\right)' = 3x^2 - 12x$$

Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0$$

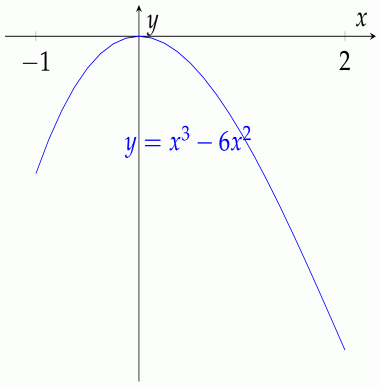
For $x \in [-1;2]$, we have

$$f'(x) = 3x^2 - 12x = 0$$
$$3x(x-4) = 0$$
$$\Downarrow$$
$$x_1 = 0, \quad x_1 \in [-1;2]$$
$$x_2 = 4, \quad x_2 \notin [-1;2]$$

	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$
$f'(x)$		$+$	0	$-$	
$f(x)$	*min*	↗	*max*	↘	*min*

We have one maximal and two minimal values, so we have to decide, which one is global, *and which one is local.*

$$f(-1) = (-1)^3 - 6(-1)^2 = -7, \quad \textit{local minimal value}$$
$$f(0) = 0^3 - 6 \cdot 0^2 = 0, \quad \textit{global maximal value}$$
$$f(2) = 2^3 - 6 \cdot 2^2 = -16, \quad \textit{global minimal value}$$



Step-by-Step Solution

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the functions.

$$f(x) = x^5 + 5x^4, \quad x \in \mathbb{R}$$

Solution 8.5.18 *First, we have to calculate the following limits.*

$$\lim_{x \to -\infty} x^5 + 5x^4 = -\infty$$
$$\lim_{x \to \infty} x^5 + 5x^4 = \infty$$

This follows that there is neither a global maximal nor a global minimal value, just local. Now, we determine the derivate function.

$$f'(x) = \left(x^5 + 5x^4\right)' = 5x^4 + 20x^3$$

Using the derivative function, we can give the extremal points. For this, we solve equation

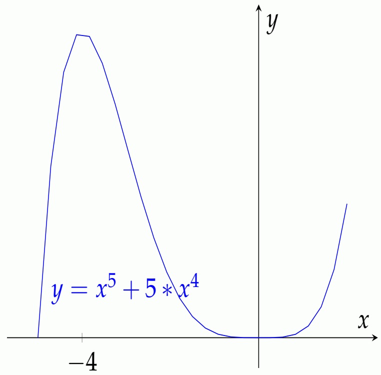
$$f'(x) = 0.$$

$$f'(x) = 5x^4 + 20x^3 = 0$$
$$5x^3(x + 4) = 0$$
$$\Downarrow$$
$$x_1 = 0,$$
$$x_2 = -4,$$

This means that function f can have local extremal value at x_1 and x_2.

We use Theorem 5.7 to determine the intervals, where function f is increasing or decreasing. For this, we determine the intervals where the derivative function f' is negative, the intervals where it is positive and examine whether function f' changes sign at x_1 and x_2. We use the following table to summarise our results.

	$x < -4$	$x = -4$	$-4 < x < 0$	$x = 0$	$0 < x$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	*max*	$\searrow$	*min*	$\nearrow$



Step-by-Step Solution

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the functions.

$$f(x) = f(x) = x \in [-1;1], \quad f(x) = 3x^3 + 9x^2.$$

Solution 8.5.19 *In that case we can get eighter global or local extremal points. Firstly we need to derivate the function.*

$$f'(x) = \left(3x^3 + 9x^2\right)' = 9x^2 + 18x$$

Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0$$

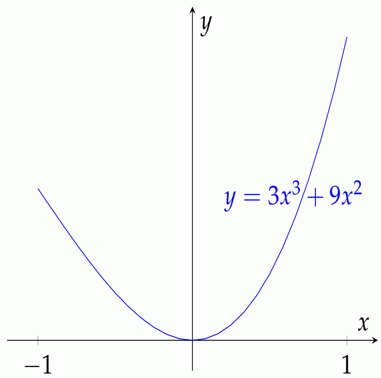
As for $x \in [-1;1]$, we have

$$f'(x) = 9x^2 + 18x = 0$$
$$9x(x+2) = 0$$
$$\Downarrow$$
$$x_1 = -2, \quad x_1 \notin [-1;1]$$
$$x_2 = 0, \quad x_2 \in [-1;1]$$

	$x=-1$	$-1<x<0$	$x=0$	$0<x<1$	$x=1$
$f'(x)$		$-$	0	$+$	
$f(x)$	*max*	$\searrow$	*min*	$\nearrow$	*max*

We have two maximal and one minimal values, so we have to decide, which one is global, a which one is local.

$$f(-1) = 3(-1)^3 + 9(-1)^2 = 6, \quad \textit{local maximal value}$$
$$f(0) = 3 \cdot 0^3 + 9 \cdot 0^2 = 0, \quad \textit{global minimal value}$$
$$f(1) = 3 + 9 = 12, \quad \textit{global maximal value}$$



Step-by-Step Solution

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the functions.

$$f : [0;4] \to \mathbb{R}, \quad f(x) = 2x^3 - 3x^2 - 12x.$$

Solution 8.5.20 *In that case we can get eighter global or local extremal points. Firstly we need to derivate the function.*

$$f'(x) = \left(2x^3 - 3x^2 - 12x\right)' = 6x^2 - 6x - 12 = 6(x^2 - x - 2).$$

Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0$$

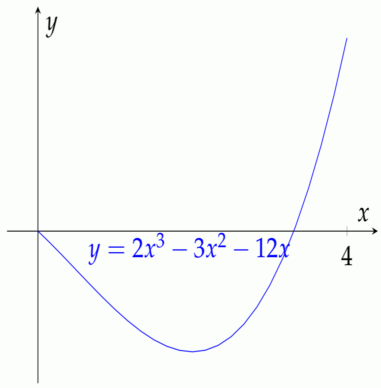
As for $x \in [0;4]$, we have

$$f'(x) = 6(x^2 - x - 2) = 0$$
$$6(x-2)(x+1) = 0$$
$$\Downarrow$$
$$x_1 = 2, \quad x_1 \in [0;4]$$
$$x_2 = -1, \quad x_2 \notin [0;4]$$

	$x=0$	$0<x<2$	$x=2$	$2<x<4$	$x=4$
$f'(x)$		$-$	0	$+$	
$f(x)$	*max*	$\searrow$	*min*	$\nearrow$	*max*

We have two maximal and one minimal values, so we have to decide, which one is global, a which one is local.

$$f(0) = 0, \quad \textit{local maximal value}$$
$$f(2) = 2\cdot 2^3 - 3\cdot^2 - 12\cdot 2 = -8, \quad \textit{global minimal value}$$
$$f(4) = 2\cdot 4^3 - 3\cdot 4^2 - 12\cdot 4 = 32, \quad \textit{global maximal value}$$



Step-by-Step Solution

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the functions.

$$x \in [-2;6], \quad f(x) = 2x^3 - 3x^2 - 120x.$$

Solution 8.5.21 *In that case we can get eighter global or local extremal points. Firstly we need to derivate the function.*

$$f'(x) = \left(2x^3 - 3x^2 - 120x\right)' = 6x^2 - 6x - 120 = 6(x^2 - x - 20).$$

Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0$$

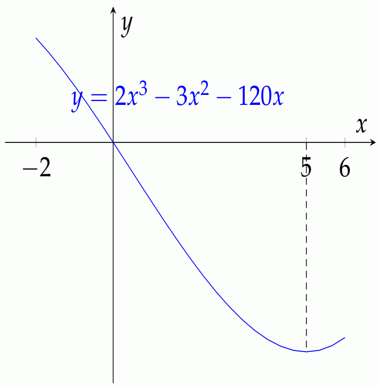
As for $x \in [-2;6]$, we have

$$f'(x) = 6(x^2 - x - 20) = 0$$
$$6(x-5)(x+4) = 0$$
$$\Downarrow$$
$$x_1 = 5, \quad x_1 \in [-2;6]$$
$$x_2 = -4, \quad x_2 \notin [-2;6]$$

	$x = -2$	$-2 < x < 5$	$x = 5$	$5 < x < 6$	$x = 6$
$f'(x)$		$-$	0	$+$	
$f(x)$	*max*	$\searrow$	*min*	$\nearrow$	*max*

We have two maximal and one minimal values, so we have to decide, which one is global, a which one is local.

$$f(-2) = 2x^3 - 3x^2 - 120x = 212, \quad \textit{global maximal value}$$
$$f(5) = 2x^3 - 3x^2 - 120x = -425, \quad \textit{global minimal value}$$
$$f(6) = 2x^3 - 3x^2 - 120x = -396, \quad \textit{local maximal value}$$



Step-by-Step Solution

Calculate the intervals, where the following function is monotone increasing / decreasing and give the extremal points and values of the functions.

$$f(x) = \frac{x^3}{3} + x^2 - 15x, \quad x \in \mathbb{R}.$$

Solution 8.5.22 *First, we have to calculate the following limits.*

$$\lim_{x \to -\infty} \frac{x^3}{3} + x^2 - 15x = -\infty$$
$$\lim_{x \to \infty} \frac{x^3}{3} + x^2 - 15x = \infty$$

This follows that there is neither a global maximal nor a global minimal value, just local. Now, we determine the derivate function.

$$f'(x) = \left(\frac{x^3}{3} + x^2 - 15x\right)' = x^2 + 2x - 15$$

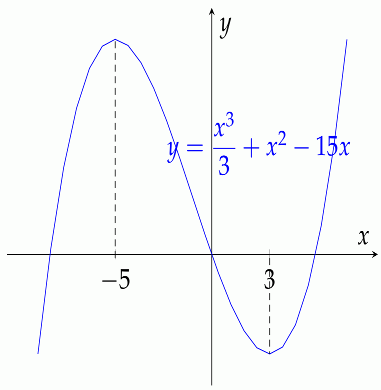
Using the derivative function, we can give the extremal points. For this, we solve equation

$$f'(x) = 0.$$

$$f'(x) = x^2 + 2x - 15 = 0$$
$$(x-3)(x+5) = 0$$
$$\Downarrow$$
$$x_1 = 3,$$
$$x_2 = -5,$$

This means that function f can have local extremal value at x_1 and x_2.
We use Theorem 5.7 to determine the intervals, where function f is increasing or decreasing. For this, we determine the intervals where the derivative function f' is negative, the intervals where it is positive and examine whether function f' changes sign at x_1 and x_2. We use the following table to summarise our results.

	$x < -5$	$x = -5$	$-5 < x < 3$	$x = 3$	$3 < x$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	*max*	$\searrow$	*min*	$\nearrow$



8.5.3 *Application III. - Convexity of Functions and Points of Inflection*

Step-by-Step Solution

Determine all intervals where f is convex / concave and list all inflection points..

$$f(x) = \frac{x+1}{x-3} \quad , x \in \mathbb{R}\backslash\{3\}.$$

Solution 8.5.23 *We use the second derivative function to determine the inflection points. For this, we calculate the first derivate function*

$$f'(x) = \left(\frac{x+1}{x-3}\right)' = \frac{(x-3)-(x+1)}{(x-3)^2} = \frac{-4}{(x-3)^2},$$

and then we give the second derivative function.

$$f''(x) = \left(\frac{x+1}{x-3}\right)'' = \left(\frac{-4}{(x-3)^2}\right)' = \frac{4 \cdot 2(x-3)}{(x-3)^4} = \frac{8}{(x-3)^3}.$$

This means that function f hasn't got any inflection point ($f''(x) \neq 0$). We use Theorem 5.10 to determine the intervals, where function f is convex or concave. For this, we determine the intervals where the derivative function f'' is negative, the intervals where it is positive .Now, we see, that f'' positive, when the denominator of the function is positive, so

$$f''(x) = \frac{8}{(x-3)^3} > 0$$

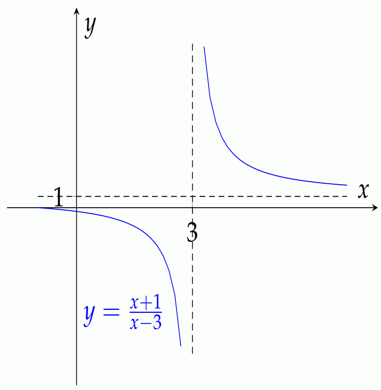
$$(x-3)^3 > 0$$

$$\Downarrow$$

$$x > 3.$$

Similarly the $f'' < 0$ when $x < 3$.
For function f we get the following table.

	$x < 3$	$x = 3$	$3 < x$
$f''(x)$	$-$	×	$+$
$f(x)$	$\cap$	×	$\cup$



Step-by-Step Solution

Determine all intervals where f is convex / concave and list all inflection points..

$$f(x) = \frac{1-x}{e^x} \quad , x \in \mathbb{R}.$$

Solution 8.5.24 *We use the second derivative function to determine the inflection points. For this, we calculate the first derivate function*

$$f'(x) = \left(\frac{1-x}{e^x}\right)' = \frac{-e^x - (1-x)e^x}{(e^x)^2} = \frac{x-2}{e^x},$$

and then we give the second derivative function.

$$f''(x) = \left(\frac{1-x}{e^x}\right)'' = \left(\frac{x-2}{e^x}\right)' = \frac{e^x - (x-2)e^x}{(e^x)^2} = \frac{3-x}{e^x}.$$

Now, we solve the following equation.

$$f''(x) = \frac{3-x}{e^x} = 0$$

As $e^x > 0$, *we have*

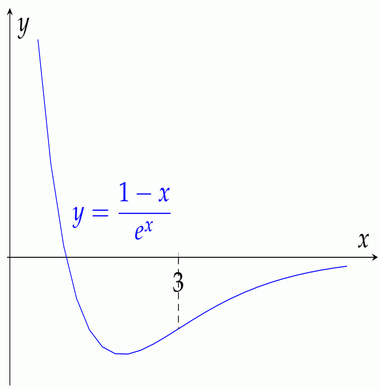
$$3 - x = 0.$$

So the solution is

$$x = 3.$$

This means that function f *can have inflection point at* $x = 3$.
We use Theorem 5.10 to determine the intervals, where function f *is convex or concave. For this, we determine the intervals where the derivative function* f'' *is negative, the intervals where it is positive and examine whether function* f'' *changes sign at* $x = 3$.
For function f *we get the following table.*

	$x < 3$	$x = 3$	$3 < x$
$f''(x)$	$+$	0	$-$
$f(x)$	$\cup$	*inflexion point*	$\cap$



Step-by-Step Solution

Determine all intervals where f is convex / concave and list all inflection points..

$$f(x) = \frac{x+1}{e^x} \quad , x \in \mathbb{R}.$$

Solution 8.5.25 *We use the second derivative function to determine the inflection points. For this, we calculate the first derivate function*

$$f'(x) = \left(\frac{x+1}{e^x}\right)' = \frac{e^x - (x+1)e^x}{(e^x)^2} = \frac{-x}{e^x},$$

and then we give the second derivative function.

$$f''(x) = \left(\frac{x+1}{e^x}\right)'' = \left(\frac{-x}{e^x}\right)' = \frac{-e^x - (-x)e^x}{(e^x)^2} = \frac{x-1}{e^x}.$$

Now, we solve the following equation.

$$f''(x) = \frac{x-1}{e^x} = 0$$

As $e^x > 0$, *we have*

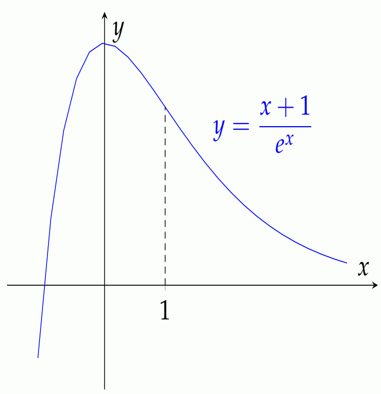
$$x - 1 = 0.$$

So the solution is

$$x = 1.$$

This means that function f can have inflection point at $x = 1$.
We use Theorem 5.10 to determine the intervals, where function f is convex or concave. For this, we determine the intervals where the derivative function f'' is negative, the intervals where it is positive and examine whether function f'' changes sign at $x = 1$. For function f we get the following table.

	$x < 1$	$x = 1$	$1 < x$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$	*inflexion point*	$\cup$



Step-by-Step Solution

Determine all intervals where f is convex / concave and list all inflection points..

$$f(x) = \frac{x^4}{12} + \frac{x^3}{3} - 4x^2 + 6x \quad , x \in \mathbb{R}.$$

Solution 8.5.26 *We use the second derivative function to determine the inflection points. For this, we calculate the first derivate function*

$$f'(x) = \left(\frac{x^4}{12} + \frac{x^3}{3} - 4x^2 + 6x\right)' = \frac{x^3}{3} + x^2 - 8x + 6,$$

and then we give the second derivative function.

$$f''(x) = \left(\frac{x^4}{12} + \frac{x^3}{3} - 4x^2 + 6x\right)'' = \left(\frac{x^3}{3} + x^2 - 8x + 6\right)' = x^2 + 2x - 8.$$

Now, we solve the following equation.

$$f''(x) = x^2 + 2x - 8 = 0$$

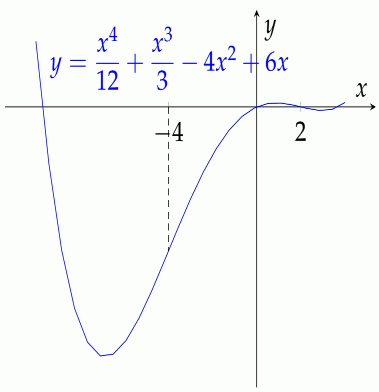
$$(x-2)(x+4) = 0$$

$$\Downarrow$$

$$x_1 = 2, \quad x_2 = -4.$$

This means that function f can have two inflection points at $x_1 = 2$ and $x_2 = -4$.
We use Theorem 5.10 to determine the intervals, where function f is convex or concave. For this, we determine the intervals where the derivative function f'' is negative, the intervals where it is positive and examine whether function f'' changes sign at that points. For function f we get the following table.

	$x < -4$	$x = -4$	$-4 < x < 2$	$x = 2$	$2 < x$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$	*inflexion point*	$\cap$	*inflexion point*	$\cup$



Step-by-Step Solution

Determine all intervals where f is convex / concave and list all inflection points..

$$f(x) = \frac{x^4}{12} + \frac{x^3}{6} - 3x^2 + 12x \quad , x \in \mathbb{R}.$$

Solution 8.5.27 *We use the second derivative function to determine the inflection points. For this, we calculate the first derivate function*

$$f'(x) = \left(\frac{x^4}{12} + \frac{x^3}{6} - 3x^2 + 12x\right)' = \frac{x^3}{3} + \frac{x^2}{2} - 6x + 12,$$

and then we give the second derivative function.

$$f''(x) = \left(\frac{x^4}{12} + \frac{x^3}{6} - 3x^2 + 12x\right)'' = \left(\frac{x^3}{3} + \frac{x^2}{2} - 6x + 12\right)' = x^2 + x - 6.$$

Now, we solve the following equation.

$$f''(x) = x^2 + x - 6 = 0$$

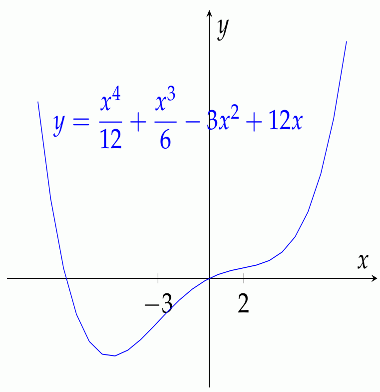
$$(x-2)(x+3) = 0$$

$$\Downarrow$$

$$x_1 = 2, \quad x_2 = -3.$$

This means that function f can have two inflection points at $x_1 = 2$ and $x_2 = -3$. We use Theorem 5.10 to determine the intervals, where function f is convex or concave. For this, we determine the intervals where the derivative function f'' is negative, the intervals where it is positive and examine whether function f'' changes sign at that points. For function f we get the following table.

	$x < -3$	$x = -3$	$-3 < x < 2$	$x = 2$	$2 < x$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$	*inflexion point*	$\cap$	*inflexion point*	$\cup$



8.5.4 *Application IV. - L'Hospital's Rule*

Step-by-Step Solution

Evaluate the following limit.

$$\lim_{x\to 0}\frac{\arctan(x)-x}{1-\cos(x)}$$

Solution 8.5.28 *First, we substitute $x=0$ to the fraction, that is*

$$\frac{\arctan(0)-0}{1-\cos(0)}=\frac{0}{0}$$

We obtain that the limit is type of "$\left(\frac{0}{0}\right)$" and we have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule.

$$\lim_{x\to 0}\frac{\arctan(x)-x}{1-\cos(x)}=\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{\sin(x)}$$

Substitute $x=0$ again.

$$\frac{\frac{1}{1+0^2}-1}{\sin(0)}.$$

We get, that the limit is type of "$\left(\frac{0}{0}\right)$" and we still have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule again.

$$\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{\sin(x)}=\lim_{x\to 0}\frac{\frac{-2x}{(1+x^2)^2}}{\cos(x)}$$

Substiting $x=0$ again, we have

$$\frac{\frac{-2\cdot 0}{(1+0^2)^2}}{\cos(0)}=0,$$

so the result is

$$\lim_{x\to 0}\frac{\arctan(x)-x}{1-\cos(x)}=0.$$

Step-by-Step Solution

Evaluate the following limit.

$$\lim_{x\to 1}\frac{\cos(x-1)+\ln(x)-x}{(x-1)^2},$$

Solution 8.5.29 *First, we substitute* $x = 1$ *to the fraction.*

$$\frac{\cos(1-1)+\ln(1)-1}{(1-1)^2}$$

We obtain, that the limit is type of " $\left(\frac{0}{0}\right)$ *" and we have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule.*

$$\lim_{x\to 1}\frac{\cos(x-1)+\ln(x)-x}{(x-1)^2} = \lim_{x\to 1}\frac{-\sin(x-1)+\frac{1}{x}-1}{2(x-1)}.$$

Substitute $x = 1$ *again.*

$$\frac{-\sin(1-1)+\frac{1}{1}-1}{2(1-1)}.$$

We get, that the limit is type of " $\left(\frac{0}{0}\right)$ *" and we still have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule again.*

$$\lim_{x\to 1}\frac{-\sin(x-1)+\frac{1}{x}-1}{2(x-1)} = \lim_{x\to 1}\frac{-\cos(x-1)-\frac{1}{x^2}}{2}.$$

Substiting $x = 1$ *again, we have*

$$\frac{-\cos(1-1)-\frac{1}{1^2}}{2} = \frac{-2}{2},$$

so the result is

$$\lim_{x\to 1}\frac{\cos(1-1)+\ln(1)-1}{(1-1)^2} = -1.$$

Step-by-Step Solution

Evaluate the following limit.

$$\lim_{x\to 0}\frac{\arctan(x)+2x^2+x}{\cos(x)-1},$$

Solution 8.5.30 *First, we substitute* $x = 0$ *to the fraction.*

$$\frac{\arctan(0)+2\cdot 0^2+0}{\cos(0)-1}$$

We obtain, that the limit is type of " $\left(\frac{0}{0}\right)$ *" and we have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule.*

$$\lim_{x\to 0}\frac{\arctan(x)+2x^2+x}{\cos(x)-1} = \lim_{x\to 0}\frac{\frac{1}{1+x^2}+4x+1}{-\sin(x)}.$$

Substitute $x = 0$ again.

$$\frac{\frac{1}{1+0^2} + 4 \cdot 0 + 1}{-\sin(0)}.$$

We get, that the limit is type of "$\left(\frac{0}{0}\right)$" and we still have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule again.

$$\lim_{x \to 0} \frac{\frac{1}{1+x^2} + 4x + 1}{-\sin(x)} = \lim_{x \to 0} \frac{\frac{-2x}{(1+x^2)^2} + 4}{-\cos(x)}.$$

Substiting $x = 1$ again, we have

$$\frac{\frac{-2}{(1+1^2)^2} + 4}{-\cos(1)} = \frac{2}{-1},$$

so the result is

$$\lim_{x \to 0} \frac{\arctan(x) + 2x^2 + x}{\cos(x) - 1} = -2.$$

Step-by-Step Solution

Evaluate the following limit

$$\lim_{x \to -1^-} \frac{\tan(x+1)}{(x+1)^2}.$$

Solution 8.5.31 *First, we substitute $x = 0$ to the fraction.*

$$\frac{\tan(-1+1)}{(-1+1)^2}$$

We obtain, that the limit is type of "$\left(\frac{0}{0}\right)$" and we have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule.

$$\lim_{x \to -1^-} \frac{\tan(x+1)}{(x+1)^2} = \lim_{x \to -1^-} \frac{\frac{1}{\cos^2(x+1)}}{2(x+1)}.$$

We get the result is

$$\lim_{x \to -1^-} \frac{\frac{1}{\cos^2(x+1)}}{2(x+1)} = \lim_{x \to -1^-} \frac{1}{2\cos^2(x+1)(x+1)} = -\infty.$$

Step-by-Step Solution

Evaluate the following limit.

$$\lim_{x\to 2} \frac{\ln(x-1)-\sin(x-2)}{(x-2)^2}.$$

Solution 8.5.32 *First, we substitute $x = 2$ to the fraction.*

$$\frac{\ln(2-1)-\sin(2-2)}{(2-2)^2}$$

We obtain, that the limit is type of "$\left(\frac{0}{0}\right)$" and we have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule.

$$\lim_{x\to 2} \frac{\ln(x-1)-\sin(x-2)}{(x-2)^2} = \lim_{x\to 2} \frac{\frac{1}{x-1}-\cos(x-2)}{2(x-2)}.$$

Substitute $x = 2$ again.

$$\frac{\frac{1}{2-1}-\cos(2-2)}{2(2-2)}.$$

We get, that the limit is type of "$\left(\frac{0}{0}\right)$" and we still have differentiable functions in the numerator and in the demoninator, so we can apply L'Hospital's rule again.

$$\lim_{x\to 2} \frac{\frac{1}{x-1}-\cos(x-2)}{2(x-2)} = \lim_{x\to 2} \frac{\frac{-1}{(x-1)^2}+\sin(x-2)}{2}.$$

Substiting $x = 2$ again, we have

$$\frac{\frac{-1}{(2-1)^2}+\sin(2-2)}{2} = \frac{-1}{2},$$

so the result is

$$\lim_{x\to 2} \frac{\ln(x-1)-\sin(x-2)}{(x-2)^2} = \frac{-1}{2}.$$

8.6 Antiderivatives and Indefinite Integrals of Real Functions

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \sqrt{x\sqrt{x}}dx.$$

Solution 8.6.1 *Integration is the reverse process of differentiation. We are really just asking what we differentiated to get the given function.*
We use the linear proprety of the indefinite integral (Theorem 6.2) and some basic mathematics.

$$\int \sqrt{x\sqrt{x}}dx = \int \left(x\,(x)^{\frac{1}{2}}\right)^{\frac{1}{2}} dx = \int \left(x^{\frac{3}{2}}\right)^{\frac{1}{2}} dx = \int x^{\frac{3}{4}}dx.$$

Using the 6.2 Table of Standard Indefinite Integrals, we get

$$\int \sqrt{x\sqrt{x}}dx = \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + C = \frac{x^{\frac{7}{4}}}{\frac{7}{4}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{x^2+5}{x}dx.$$

Solution 8.6.2 *Integration is the reverse process of differentiation. We are really just asking what we differentiated to get the given function.*
We use the linear proprety of the indefinite integral (Theorem 6.2) and some basic mathematics.

$$\int \frac{x^2+5}{x}dx = \int \left(x + \frac{5}{x}\right) dx = \int xdx + 5\int \frac{1}{x}dx.$$

Using the 6.2 Table of Standard Indefinite Integrals, we get

$$\int \frac{x^2+5}{x}dx = \frac{x^2}{2} + 5\ln|x| + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \left(2^x - \frac{5}{x^2+1}\right) dx.$$

Solution 8.6.3 *Integration is the reverse process of differentiation. We are really just asking what we differentiated to get the given function.*
We use the linear proprety of the indefinite integral (Theorem 6.2) and some basic mathematics.

$$\int \left(2^x - \frac{5}{x^2+1}\right) dx = \int 2^x dx + 5 \int \frac{1}{x^2+1} dx.$$

Using the 6.2 Table of Standard Indefinite Integrals, we get

$$\int \left(2^x - \frac{5}{x^2+1}\right) dx = \frac{2^x}{\ln(2)} + 5 \arctan(x) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int e^{2x+3} dx.$$

Solution 8.6.4 *Integration is the reverse process of differentiation. We are really just asking what we differentiated to get the given function.*
We use the linear proprety of the indefinite integral (Theorem 6.2) and some basic mathematics.

$$\int e^{2x+3} dx = \int \left(e^2\right)^x \cdot e^3 dx = e^3 \int \left(e^2\right)^x dx.$$

Using the 6.2 Table of Standard Indefinite Integrals, we get

$$\int e^{2x+3} dx = e^3 \frac{\left(e^2\right)^x}{\ln(e^2)} + C = e^3 \frac{e^{2x}}{2\ln(e)} + C = \frac{e^{2x+3}}{2} + C.$$

8.6.1 *Integration by Parts*

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int x^2 \sin(x) dx.$$

Solution 8.6.5 *To use this technique we need to identify candidates for functions* $f'(x)$ *and* $g(x)$. *We wish to replace integral* $\int f'g$ *with another* $\left(\int fg'\right)$, *which can be easier to evaluate. Let*

$$f'(x) = \sin(x)$$

and

$$g(x) = x^2.$$

We have to calculate function $f(x)$ *and* $g'(x)$. *For this we determine functions* $f'(x)$ *and* $g(x)$ *and we* ***integrate*** *function* $f'(x)$ *and* ***differentiate*** *function* $g(x)$.

$$f(x) = \int \sin(x)\, dx = -\cos(x),$$

and

$$g'(x) = 2x.$$

or shortly

	Given	*Calculated*
I	$f'(x) = \sin(x)$	$f(x) = -\cos(x)$
D	$g(x) = x^2$	$g'(x) = 2x$

from Theorem 6.4, we obtain

$$\begin{aligned}\int x^2 \sin(x)\, dx &= -\cos(x) \cdot x^2 - \int \Big(-\cos(x)\Big) \cdot 2x\, dx = \\ &= -x^2\cos(x) + 2\int x\cos(x)\, dx.\end{aligned}$$

Note that

$$\int x\cos(x)\, dx$$

is not a standard integral. We calculate this integral by repeated integration by parts. From

	Given	*Calculated*
I	$f'(x) = \cos(x)$	$f(x) = \int \cos(x)\, dx = \sin(x)$
D	$g(x) = x$	$g'(x) = 1$

we obtain

$$\int x\cos(x)\,dx = x\sin(x) - \int 1\cdot\sin(x)\,dx = x\sin(x) - (-\cos(x)).$$

Combining this with the previous result, we get

$$\int x^2\sin(x)\,dx = -x^2\cos(x) + 2\,(x\sin(x)+\cos(x)) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int (x-1)\cos(x)\,dx.$$

Solution 8.6.6 *We wish to replace integral $\int f'g$ with another $\left(\int fg'\right)$, which can be easier to evaluate. First, we need to identify candidates for functions f and g. Let*

$$f'(x) = \cos(x)$$

and

$$g(x) = x-1.$$

Now we determine functions f and g'. For this we ***integrate*** *function f', and* ***differentiate*** *function g.*

$$f(x) = \int \cos(x)\,dx = \sin(x),$$

and

$$g'(x) = 1,$$

or shortly

	Given	Calculated
I	$f'(x) = \cos(x)$	$f(x) = \int\cos(x)\,dx = \sin(x)$
D	$g(x) = x-1$	$g'(x) = 1$

From Theorem 6.4, we obtain

$$\begin{aligned}\int (x-1)\cos(x)\,dx &= \sin(x)\cdot(x-1) - \int 1\cdot\sin(x)\,dx = \\ &= \sin(x)\cdot(x-1) - \int \sin(x)\,dx.\end{aligned}$$

As

$$\int \sin(x)\,dx = -\cos(x) + C,$$

the solution is

$$\int (x-1)\cos(x)\,dx = \sin(x)\cdot(x-1) + \cos(x) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int x \cdot 2^x dx.$$

Solution 8.6.7 *We wish to replace integral* $\int f'g$ *with another* $\left(\int fg'\right)$, *which can be easier to evaluate. To use this technique we need to identify candidates for functions* f *and* g. *Let*

$$f'(x) = 2^x$$

and

$$g(x) = x.$$

Now we determine functions f *and* g'. *For this we* ***integrate*** *function* f', *and* ***differentiate*** *function* g.

$$f(x) = \int 2^x dx = \frac{2^x}{\ln(2)},$$

and

$$g'(x) = 1.$$

or shortly

	Given	Calculated
I	$f'(x) = 2^x$	$f(x) = \int 2^x dx = \frac{2^x}{\ln(2)}$
D	$g(x) = x$	$g'(x) = 1$

from Theorem 6.4, we obtain

$$\begin{aligned}\int x \cdot 2^x dx &= \frac{2^x}{\ln(2)} \cdot x - \int 1 \cdot \frac{2^x}{\ln(2)} dx = \\ &= \frac{2^x}{\ln(2)} \cdot x - \frac{1}{\ln(2)} \int 2^x dx,\end{aligned}$$

so the solution is

$$\int x \cdot 2^x dx = \frac{2^x}{\ln(2)} \cdot x - \frac{1}{\ln^2(2)} 2^x + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \ln(x)\, dx.$$

Solution 8.6.8 *Using the idea of Theorem 6.4, we need production of functions. For this we rewrite the integrand*

$$\int \ln(x)\, dx = \int 1 \cdot \ln(x)\, dx.$$

So let

$$f'(x) = 1$$

and

$$g(x) = \ln(x).$$

Using the idea of Theorem 6.4, we intregrate function f'

$$f(x) = \int 1 dx = x,$$

and differentiate function g

$$g'(x) = \frac{1}{x}.$$

Thus

	Given	Calculated
I	$f'(x) = 1$	$f(x) = \int 1dx = x$
D	$g(x) = \ln(x)$	$g'(x) = \frac{1}{x}$

and from Theorem 6.4, we get

$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}dx.$$

Next, we must simplify $x \cdot \frac{1}{x}$. *That is*

$$x \cdot \frac{1}{x} = 1$$

so we get the original $\int 1dx$ *integral back. Hence*

$$\int \ln(x)\,dx = x\ln(x) - \int 1dx = x\ln(x) - x + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \sqrt{x}\ln(x)\,dx.$$

Solution 8.6.9 *We wish to replace integral* $\int f'g$ *with another* $\left(\int fg'\right)$, *which can be easier to evaluate. To use this technique we need to identify candidates for functions* f *and* g. *Let*

$$f'(x) = \sqrt{x}$$

and

$$g(x) = \ln(x).$$

Using the idea of Theorem 6.4, we intregrate function f'

$$f(x) = \int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}},$$

and differentiate function g

$$g'(x) = \frac{1}{x}.$$

Thus

	Given	*Calculated*
I	$f'(x) = \sqrt{x}$	$f(x) = \int \sqrt{x}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$
D	$g(x) = \ln(x)$	$g'(x) = \frac{1}{x}$

and from Theorem 6.4, we get

$$\int \sqrt{x} \ln (x)\, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \ln (x) - \int \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \cdot \frac{1}{x} dx.$$

Next, we must simplify $\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \cdot \frac{1}{x}$. *That is*

$$\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \cdot \frac{1}{x} = \frac{x^{\frac{1}{2}}}{\frac{3}{2}}$$

so we get the original $\int x^{\frac{1}{2}} dx$ *integral back, but with another coefficient. Hence*

$$\int \sqrt{x} \ln (x)\, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \ln (x) - \int \frac{x^{\frac{1}{2}}}{\frac{3}{2}} dx = \frac{2x^{\frac{3}{2}}}{3} \ln (x) - \frac{2}{3} \int x^{\frac{1}{2}},$$

so the solution is

$$\int \sqrt{x} \ln (x)\, dx = \frac{2x^{\frac{3}{2}}}{3} \ln (x) - \frac{2}{3} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2x^{\frac{3}{2}}}{3} \ln (x) - \left(\frac{2}{3}\right)^2 \cdot x^{\frac{3}{2}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{1}{\sqrt{x}} \ln (x)\, dx.$$

Solution 8.6.10 *We wish to replace integral* $\int f'g$ *with another* $\left(\int fg'\right)$, *which can be easier to evaluate. To use this technique we need to identify candidates for functions* f *and* g. *Let*

$$f'(x) = \frac{1}{\sqrt{x}}$$

and

$$g(x) = \ln (x).$$

Using the idea of Theorem 6.4, we intregrate function f'

$$f(x) = \int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}},$$

and differentiate function g

$$g'(x) = \frac{1}{x}.$$

Thus

	Given	*Calculated*
I	$f'(x) = \frac{1}{\sqrt{x}}$	$f(x) = \int \frac{1}{\sqrt{x}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$
D	$g(x) = \ln(x)$	$g'(x) = \frac{1}{x}$

and from Theorem 6.4, we get

$$\int \frac{1}{\sqrt{x}} \ln(x)\, dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \ln(x) - \int \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \cdot \frac{1}{x} dx.$$

Next, we must simplify $\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \cdot \frac{1}{x}$. *That is*

$$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \cdot \frac{1}{x} = \frac{x^{-\frac{1}{2}}}{\frac{1}{2}}$$

so we get the original $\int x^{-\frac{1}{2}} dx$ *integral back, but with another coefficient. Hence*

$$\int \frac{1}{\sqrt{x}} \ln(x)\, dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \ln(x) - \int \frac{x^{-\frac{1}{2}}}{\frac{1}{2}} dx = 2x^{\frac{1}{2}} \cdot \ln(x) - 2 \int x^{-\frac{1}{2}},$$

so the solution is

$$\int \frac{1}{\sqrt{x}} \ln(x)\, dx = 2x^{\frac{1}{2}} \cdot \ln(x) - 2 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2x^{\frac{1}{2}} \cdot \ln(x) - 2^2 \cdot x^{\frac{1}{2}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \left(x^2 + 1\right) \ln(x)\, dx.$$

Solution 8.6.11 *We wish to replace integral* $\int f'g$ *with another* $\left(\int fg'\right)$, *which can be easier to evaluate. To use this technique we need to identify candidates for functions* f *and* g. *Let*

$$f'(x) = x^2 + 1$$

and

$$g(x) = \ln(x).$$

Using the idea of Theorem 6.4, we intregrate function f'

$$f(x) = \int \left(x^2 + 1\right) dx = \int x^2 dx + \int 1 dx = \frac{x^3}{3} + x,$$

and differentiate function g

$$g'(x) = \frac{1}{x}.$$

Thus

	Given	*Calculated*
I	$f'(x) = \left(x^2+1\right)$	$f(x) = \int \left(x^2+1\right) dx = \frac{x^3}{3} + x$
D	$g(x) = \ln(x)$	$g'(x) = \frac{1}{x}$

and from Theorem 6.4, we get

$$\int \left(x^2+1\right) \ln(x)\, dx = \left(\frac{x^3}{3} + x\right) \ln(x) - \int \left(\frac{x^3}{3} + x\right) \cdot \frac{1}{x} dx.$$

Next, we must simplify $\left(\frac{x^3}{3} + x\right) \cdot \frac{1}{x}$. *That is*

$$\left(\frac{x^3}{3} + x\right) \cdot \frac{1}{x} = \frac{x^2}{3} + 1,$$

so we got the original $x^2 + 1$ integrand back, but with another coefficients. Hence

$$\begin{aligned}\int \left(x^2+1\right) \ln(x)\, dx &= \left(\frac{x^3}{3} + x\right) \ln(x) - \int \left(\frac{x^2}{3} + 1\right) dx = \\ &= \left(\frac{x^3}{3} + x\right) \ln(x) - \left(\frac{1}{3}\int x^2 dx + \int 1 dx\right),\end{aligned}$$

so the solution is

$$\begin{aligned}\int \left(x^2+1\right) \ln(x)\, dx &= \left(\frac{x^3}{3} + x\right) \ln(x) - \left(\frac{1}{3} \cdot \frac{x^3}{3} + x\right) + C = \\ &= \left(\frac{x^3}{3} + x\right) \ln(x) - \left(\frac{1}{3}\right)^2 \cdot x^3 + C.\end{aligned}$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int 2x \log_2(x)\, dx.$$

Solution 8.6.12 *We wish to replace integral* $\int f'g$ *with another* $\left(\int fg'\right)$, *which can be easier to evaluate. To use this technique we need to identify candidates for functions* f *and* g. *Let*

$$f'(x) = 2x$$

and

$$g(x) = \log_2(x).$$

Using the idea of Theorem 6.4, we intregrate function f'

$$f(x) = \int 2x dx = 2\int x dx = 2\frac{x^2}{2} = x^2,$$

and differentiate function g

$$g'(x) = \frac{1}{x\ln(2)}.$$

Thus

	Given	*Calculated*
I	$f'(x) = 2x$	$f(x) = \int 2x dx = x^2$
D	$g(x) = \log_2(x)$	$g'(x) = \dfrac{1}{x\ln(2)}$

and from Theorem 6.4, we get

$$\int 2x\log_2(x)\,dx = x^2\log_2(x) - \int x^2 \cdot \frac{1}{x\ln(2)}dx.$$

Next, we must simplify $x^2 \cdot \frac{1}{x\ln(2)}$. *That is*

$$x^2 \cdot \frac{1}{x\ln(2)} = x \cdot \frac{1}{\ln(2)}$$

so we got the original x *integrand back, but with another coefficient. Hence*

$$\int 2x\log_2(x)\,dx = x^2\log_2(x) - \int x \cdot \frac{1}{\ln(2)}dx = \left(\frac{x^3}{3} + x\right)\ln(x) - \frac{1}{\ln(2)}\int x dx,$$

so the solution is

$$\int 2x \log_2(x)\,dx = x^2 \log_2(x) - \frac{1}{\ln(2)} \cdot \frac{x^2}{2} + C = x^2 \log_2(x) - \frac{1}{2\ln(2)} \cdot x^2 + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int (x+2)\ln(x+2)\,dx.$$

Solution 8.6.13 *Let*

$$f'(x) = 2x+2$$

and

$$g(x) = \ln(x+2).$$

Using the idea of the previous example, we intregrate function f'

$$f(x) = \int (2x+2)\,dx = \int 2xdx + \int 2dx = 2\frac{x^2}{2} + 2x = x^2 + 2x,$$

and differentiate function g

$$g'(x) = \frac{1}{x+2}.$$

Thus

	Given	*Calculated*
I	$f'(x) = 2x+2$	$f(x) = \int (2x+2)\,dx = x^2 + 2x$
D	$g(x) = \ln(x+2)$	$g'(x) = \frac{1}{x+2}$

and from Theorem 6.4, we get

$$\int (2x+2)\ln(x+2)\,dx = \left(x^2+2x\right)\ln(x+2) - \int \left(x^2+2x\right) \cdot \frac{1}{x+2}dx.$$

Next, we simplify $(x^2+2x) \cdot \frac{1}{x+2}$, that is

$$\left(x^2+2x\right) \cdot \frac{1}{x+2} = x(x+2)\frac{1}{x+2} = x.$$

Hence

$$\int (2x+2)\ln(x+2)\,dx = \left(x^2+2x\right)\ln(x+2) - \int xdx.$$

This yields

$$\int (2x+2)\ln(x+2)\,dx = \left(x^2+2x\right)\ln(x+2) - \frac{x^2}{2} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int xe^{2x+3}dx.$$

Solution 8.6.14 *First, we need to identify candidates for functions f and g. So let*

	Given	Calculated
I	$f'(x) = e^{2x+3}$	$f(x) = \int e^{2x+3}dx$
D	$g(x) = x$	$g'(x) = 1$

As

$$\int e^{2x+3}dx = e^3\int\left(e^2\right)^x dx = e^3\frac{\left(e^2\right)^x}{\ln\left(e^2\right)} + C = e^3\frac{e^{2x}}{2\ln(e)} + C = \frac{e^{2x+3}}{2} + C,$$

using Theorem 6.4, we get

$$\int xe^{2x+3}dx = x\cdot\frac{e^{2x+3}}{2} - \int\frac{e^{2x+3}}{2}\cdot 1dx = x\cdot\frac{e^{2x+3}}{2} - \frac{1}{2}\int e^{2x+3}dx.$$

We got the original e^{2x+3} integrand back, but with another coefficient. Hence

$$\begin{aligned}\int xe^{2x+3}dx &= x\cdot\frac{e^{2x+3}}{2} - \frac{1}{2}\int e^{2x+3}dx = \\ &= \frac{xe^{2x+3}}{2} - \frac{1}{2}\cdot\frac{e^{2x+3}}{2} + C = \frac{xe^{2x+3}}{2} - \frac{e^{2x+3}}{4} + C.\end{aligned}$$

8.6.2 *Integration by Substitution*

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int x^2 e^{x^3} dx.$$

Solution 8.6.15 *As*

$$\left(x^3\right)' = 3x^2$$

rewriting the integrand as

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int 3x^2 e^{x^3} dx$$

the integral contains a composite function and the derivative of its inner function. We use Theorem 6.5 and make the substitution

$$x^3 = y.$$

Then

$$3x^2 dx = dy.$$

This allows us to change variable from x to y, that is

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^y dy.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^y dy = \frac{1}{3} e^y + C = \frac{1}{3} e^{x^3} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int x \sin\left(x^2\right) dx.$$

Solution 8.6.16 *As*

$$\left(x^2\right)' = 2x$$

rewriting the integrand as

$$\int x \sin\left(x^2\right) dx = \frac{1}{2} \int 2x \sin\left(x^2\right) dx$$

the integral contains a composite function and the derivative of its inner function. We use Theorem 6.5 and make the substitution

$$x^2 = y.$$

Then

$$2xdx = dy.$$

This allows us to change variable from x to y, that is

$$\int x \sin\left(x^2\right) dx = \frac{1}{2} \int 2x \sin\left(x^2\right) dx = \frac{1}{2} \int \sin(y)\, dy.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int x \sin\left(x^2\right) dxx = \frac{1}{2} \int \sin(y)\, dy = \frac{1}{2}(-\cos(y)) + C = -\frac{1}{2}\cos\left(x^2\right) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx.$$

Solution 8.6.17 *As*

$$(\tan(x))' = \frac{1}{\cos^2(x)}$$

rewriting the integrand as

$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx = \int e^{\tan(x)} \cdot \frac{1}{\cos^2(x)} dx$$

the integral contains a composite function and the derivative of its inner function. We use Theorem 6.5 and make the substitution

$$\tan(x) = y.$$

Then

$$\frac{1}{\cos^2(x)} dx = dy.$$

This allows us to change variable from x to y, that is

$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx = \int e^{\tan(x)} \cdot \frac{1}{\cos^2(x)} dx = \int e^y dy.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx = \int e^y dy = e^y + C = e^{\tan(x)} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \sin\left(\frac{1}{2}x+3\right)dx.$$

Solution 8.6.18 *The integral contains a composite function and its inner function is a linear function. So the result can be obtained by using*

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C = \frac{F(ax+b)}{a}+C.$$

Let

$$f(x) = \sin(x)$$

and

$$a = \frac{1}{2}.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int \sin(x)\,dx = -\cos(x)+C.$$

Combining this with the Theorem 6.6, the result is

$$\int \sin\left(\frac{1}{2}x+3\right)dx = \frac{-\cos\left(\frac{1}{2}x+3\right)}{\frac{1}{2}}+C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int e^{7x+1}dx.$$

Solution 8.6.19 *The integral contains a composite function and its inner function is a linear function. So the result can be obtained by using*

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C = \frac{F(ax+b)}{a}+C.$$

Let

$$f(x) = e^x$$

and

$$a = 7.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int e^x dx = e^x + C.$$

Combining this with the Theorem 6.6, the result is

$$\int e^{7x+1}dx = \frac{e^{7x+1}}{7} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int 5^{3x-9}dx.$$

Solution 8.6.20 *The integral contains a composite function and its inner function is a linear function. So the result can be obtained by using*

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C = \frac{F(ax+b)}{a} + C.$$

Let

$$f(x) = 5^x$$

and

$$a = 3.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int 5^x dx = \frac{5^x}{\ln(5)} + C.$$

Combining this with the Theorem 6.6, the result is

$$\int 5^{3x-9}dx = \frac{\frac{5^{3x-9}}{\ln(5)}}{3} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \sin(2-3x)\,dx.$$

Solution 8.6.21 *The integral contains a composite function and its inner function is a linear function. So the result can be obtained by using*

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C = \frac{F(ax+b)}{a} + C.$$

Let

$$f(x) = \sin(x)$$

and

$$a = -3.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int \sin(x)\,dx = -\cos(x) + C.$$

Combining this with the Theorem 6.6, the result is

$$\int \sin(2-3x)\,dx = \frac{-\cos(2-3x)}{-3} + C = \frac{F(ax+b)}{a} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \cos\left(1 - \frac{1}{2}x\right) dx.$$

Solution 8.6.22 *The integral contains a composite function and its inner function is a linear function. So the result can be obtained by using*

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

Let

$$f(x) = \cos(x)$$

and

$$a = -\frac{1}{2}.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int \cos(x)\,dx = \sin(x) + C.$$

Combining this with the Theorem 6.6, the result is

$$\int \cos\left(1 - \frac{1}{2}x\right) dx = \frac{\sin\left(1 - \frac{1}{2}x\right)}{-\frac{1}{2}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int e^{5-x} dx.$$

Solution 8.6.23 *The integral contains a composite function and its inner function is a linear function. So the result can be obtained by using*

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C = \frac{F(ax+b)}{a} + C.$$

Let

$$f(x) = e^x$$

and

$$a = -1.$$

Using 6.2 Table of Standard Indefinite Integrals, we get

$$\int e^x dx = e^x + C.$$

Combining this with the Theorem 6.6, the result is

$$\int e^{5-x} dx = \frac{e^{5-x}}{-1} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \tan(x)\, dx.$$

Solution 8.6.24 *We need to rewrite the integral*

$$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)} dx = -\int \frac{-\sin(x)}{\cos(x)} dx.$$

As

$$(\cos(x))' = -\sin(x),$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C.$$

Let

$$f(x) = \cos(x),$$

so from Theorem 6.7, we get

$$\int \tan(x)\, dx = -\int \frac{-\sin(x)}{\cos(x)} dx = -\ln(|\cos(x)|) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{e^{2x}}{e^{2x}+4} dx.$$

Solution 8.6.25 *We need to rewrite the integral*

$$\int \frac{e^{2x}}{e^{2x}+4} dx = \frac{1}{2} \int \frac{2e^{2x}}{e^{2x}+4} dx.$$

As

$$\left(e^{2x}+4\right)' = 2e^{2x},$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

Let

$$f(x) = e^{2x} + 4,$$

so from Theorem 6.7, we get

$$\int \frac{e^{2x}}{e^{2x}+4}dx = \frac{1}{2}\int \frac{2e^{2x}}{e^{2x}+4}dx = \frac{1}{2}\ln\left(\left|e^{2x}+4\right|\right) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{\sin(x)}{1+\cos(x)}dx.$$

Solution 8.6.26 *We need to rewrite the integral*

$$\int \frac{\sin(x)}{1+\cos(x)}dx = -\int \frac{-\sin(x)}{1+\cos(x)}dx.$$

As

$$(1+\cos(x))' = -\sin(x),$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

Let

$$f(x) = -\int \frac{-\sin(x)}{1+\cos(x)},$$

so from Theorem 6.7, we get

$$\int \frac{\sin(x)}{1+\cos(x)}dx = -\int \frac{-\sin(x)}{1+\cos(x)}dx = -\ln(|1+\cos(x)|) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{4}{7x+5}dx.$$

Solution 8.6.27 *We need to rewrite the integral*

$$\int \frac{4}{7x+5}dx = \frac{4}{7}\int \frac{7}{7x+5}dx.$$

As

$$(7x+5)' = 7,$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

Let

$$f(x) = 7x+5,$$

so from Theorem 6.7, we get

$$\int \frac{4}{7x+5}dx = \frac{4}{7}\int \frac{7}{7x+5}dx = \frac{4}{7}\ln(|7x+5|) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{x-2}{x^2-4x+1}dx.$$

Solution 8.6.28 *We need to rewrite the integral*

$$\int \frac{x-2}{x^2-4x+1}dx = \frac{1}{2}\int \frac{2(x-2)}{x^2-4x+1}dx.$$

As

$$\left(x^2-4x+1\right)' = 2x-4 = 2(x-2),$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

Let

$$f(x) = x^2-4x+1,$$

so from Theorem 6.7, we get

$$\int \frac{x-2}{x^2-4x+1}dx = \frac{1}{2}\int \frac{2(x-2)}{x^2-4x+1}dx = \frac{1}{2}\ln\left(\left|x^2-4x+1\right|\right) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{x^2+1}{x^3+3x+4}dx.$$

Solution 8.6.29 *We need to rewrite the integral*

$$\int \frac{x^2+1}{x^3+3x+4}dx = \frac{1}{3}\int \frac{3\left(x^2+1\right)}{x^3+3x+4}dx.$$

As

$$\left(x^3+3x+4\right)' = 3x^2+3 = 3\left(x^2+1\right),$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

Let

$$f(x) = x^3+3x+4,$$

so from Theorem 6.7, we get

$$\int \frac{x^2+1}{x^3+3x+4}dx = \frac{1}{3}\int \frac{3\left(x^2+1\right)}{x^3+3x+4}dx = \frac{1}{3}\ln\left(\left|x^3+3x+4\right|\right) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{1}{\tan(x)\cos^2(x)}dx.$$

Solution 8.6.30 *We need to rewrite the integral*

$$\int \frac{1}{\tan(x)\cos^2(x)}dx = \int \frac{\frac{1}{\cos^2(x)}}{\tan(x)}dx.$$

As

$$(\tan(x))' = \frac{1}{\cos^2(x)},$$

the result can be obtained by using

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

Let

$$f(x) = \tan(x),$$

so from Theorem 6.7, we get

$$\int \frac{1}{\tan(x)\cos^2(x)}dx = \int \frac{\frac{1}{\cos^2(x)}}{\tan(x)}dx = \ln(|\tan(x)|) + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{\tan(x)}{\cos^2(x)} dx.$$

Solution 8.6.31 *We need to rewrite the integral*

$$\int \frac{\tan(x)}{\cos^2(x)} dx = \int \tan(x) \cdot \frac{1}{\cos^2(x)} dx.$$

As

$$(\tan(x))' = \frac{1}{\cos^2(x)},$$

the result can be obtained by using

$$\int f(x) \cdot f'(x)\, dx = \frac{f^2(x)}{2} + C.$$

Let

$$f(x) = \tan(x),$$

so from Theorem 6.8, we get

$$\int \frac{\tan(x)}{\cos^2(x)} dx = \frac{\tan^2(x)}{2} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int x\left(x^2+5\right)^{10} dx.$$

Solution 8.6.32 *We need to rewrite the integral*

$$\int x\left(x^2+5\right)^{10} dx = \frac{1}{2}\int \left(x^2+5\right)^{10} \cdot 2x dx.$$

As

$$\left(x^2+5\right)' = 2x,$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = x^2+5,$$

and $\alpha = 10$*, so from Theorem 6.8, we get*

$$\int x\left(x^2+5\right)^{10} dx = \frac{1}{2}\int \left(x^2+5\right)^{10} \cdot 2x dx = \frac{1}{2} \cdot \frac{\left(x^2+5\right)^{11}}{11} + C = \frac{\left(x^2+5\right)^{11}}{22} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \sqrt{(2x+5)^3}dx.$$

Solution 8.6.33 *We need to rewrite the integral*

$$\int \sqrt{(2x+5)^3}dx = \int (2x+5)^{\frac{3}{2}}\,dx = \frac{1}{2}\int (2x+5)^{\frac{3}{2}} \cdot 2dx.$$

As

$$(2x+5)' = 2,$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = 2x+5,$$

and $\alpha = \frac{3}{2}$, *so from Theorem 6.8, we get*

$$\int \sqrt{(2x+5)^3}dx = \frac{1}{2}\int (2x+5)^{\frac{3}{2}} \cdot 2dx = \frac{1}{2} \cdot \frac{(2x+5)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{(2x+5)^{\frac{5}{2}}}{5} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \sin(x)\cos(x)\,dx.$$

Solution 8.6.34 *As*

$$(\sin(x))' = \cos(x),$$

the result can be obtained by using

$$\int f(x) \cdot f'(x)\,dx = \frac{f^2(x)}{2} + C.$$

Let

$$f(x) = \sin(x),$$

so from Theorem 6.8, we get

$$\int \sin(x)\cos(x)\,dx = \frac{\sin^2(x)}{2} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{\sqrt{\ln(x)}}{x} dx.$$

Solution 8.6.35 *We need to rewrite the integral*

$$\int \frac{\sqrt{\ln(x)}}{x} dx = \int \ln^{\frac{1}{2}}(x) \cdot \frac{1}{x} dx.$$

As

$$(\ln(x))' = \frac{1}{x},$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = \ln(x),$$

and $\alpha = \frac{1}{2}$*, so from Theorem 6.8, we get*

$$\int \frac{\sqrt{\ln(x)}}{x} dx = \int \ln^{\frac{1}{2}}(x) \cdot \frac{1}{x} dx = \frac{\ln^{\frac{3}{2}}(x)}{\frac{3}{2}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{1}{x\sqrt{\ln(x)}} dx.$$

Solution 8.6.36 *We need to rewrite the integral*

$$\int \frac{1}{x\sqrt{\ln(x)}} dx = \int \ln^{-\frac{1}{2}}(x) \cdot \frac{1}{x} dx.$$

As

$$(\ln(x))' = \frac{1}{x},$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = \ln(x),$$

and $\alpha = -\frac{1}{2}$*, so from Theorem 6.8, we get*

$$\int \frac{1}{x\sqrt{\ln(x)}} dx = \int \ln^{-\frac{1}{2}}(x) \cdot \frac{1}{x} dx = \frac{\ln^{\frac{1}{2}}(x)}{\frac{1}{2}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{1}{\sqrt[4]{\tan(x)}\cos^2(x)}dx.$$

Solution 8.6.37 *We need to rewrite the integral*

$$\int \frac{1}{\sqrt[4]{\tan(x)}\cos^2(x)}dx = \int \tan^{-\frac{1}{4}}(x) \cdot \frac{1}{\cos^2(x)}dx.$$

As

$$(\tan(x))' = \frac{1}{\cos^2(x)},$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = \tan(x),$$

and $\alpha = -\frac{1}{4}$, *so from Theorem 6.8, we get*

$$\int \frac{1}{\sqrt[4]{\tan(x)}\cos^2(x)}dx = \int \tan^{-\frac{1}{4}}(x) \cdot \frac{1}{\cos^2(x)}dx = \frac{\tan^{\frac{3}{4}}(x)}{\frac{3}{4}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{\cos(x)}{\sqrt[3]{\sin(x)}}dx.$$

Solution 8.6.38 *We need to rewrite the integral*

$$\int \frac{\cos(x)}{\sqrt[3]{\sin(x)}}dx = \int \sin^{-\frac{1}{3}}(x) \cdot \cos(x)\, dx.$$

As

$$(\sin(x))' = \cos(x),$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = \sin(x),$$

and $\alpha = -\frac{1}{3}$, so from Theorem 6.8, we get

$$\int \frac{\cos(x)}{\sqrt[3]{\sin(x)}}dx = \int \sin^{-\frac{1}{3}}(x) \cdot \cos(x)\, dx = \frac{\sin^{\frac{2}{3}}(x)}{\frac{2}{3}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{x}{\sqrt{x^2+1}}dx.$$

Solution 8.6.39 *We need to rewrite the integral*

$$\int \frac{x}{\sqrt{x^2+1}}dx = \frac{1}{2}\int \left(x^2+1\right)^{-\frac{1}{2}} \cdot 2xdx.$$

As

$$\left(x^2+1\right)' = 2x,$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = x^2+1,$$

and $\alpha = -\frac{1}{2}$, so from Theorem 6.8, we get

$$\int \frac{x}{\sqrt{x^2+1}}dx = \frac{1}{2}\int \left(x^2+1\right)^{-\frac{1}{2}} \cdot 2xdx = \frac{1}{2}\frac{\left(x^2+1\right)^{\frac{1}{2}}}{\frac{1}{2}} + C = \left(x^2+1\right)^{\frac{1}{2}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{1}{(3x+1)^2}dx.$$

Solution 8.6.40 *We need to rewrite the integral*

$$\int \frac{1}{(3x+1)^2}dx = \frac{1}{3}\int (3x+1)^{-2} \cdot 3dx.$$

As

$$(3x+1)' = 3,$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = 3x + 1,$$

and $\alpha = -2$, *so from Theorem 6.8, we get*

$$\int \frac{1}{(3x+1)^2} dx = \frac{1}{3} \int (3x+1)^{-2} \cdot 3dx = \frac{1}{3} \frac{(3x+1)^{-1}}{-1} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{1}{\sqrt[4]{3x+1}} dx.$$

Solution 8.6.41 *We need to rewrite the integral*

$$\int \frac{1}{\sqrt[4]{3x+1}} dx = \frac{1}{3} \int (3x+1)^{-\frac{1}{4}} \cdot 3dx.$$

As

$$(3x+1)' = 3,$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = 3x + 1,$$

and $\alpha = -\frac{1}{4}$, *so from Theorem 6.8, we get*

$$\int \frac{1}{\sqrt[4]{3x+1}} dx = \frac{1}{3} \int (3x+1)^{-\frac{1}{4}} \cdot 3dx = \frac{1}{3} \frac{(3x+1)^{\frac{3}{4}}}{\frac{3}{4}} + C.$$

Step-by-Step Solution

Evaluate the following indefinite integral.

$$\int \frac{x^2}{\sqrt[4]{x^3+1}} dx.$$

Solution 8.6.42 *We need to rewrite the integral*

$$\int \frac{x^2}{\sqrt[4]{x^3+1}} dx = \frac{1}{3} \int \left(x^3+1\right)^{-\frac{1}{4}} \cdot 3x^2 dx.$$

As

$$\left(x^3+1\right)' = 3x^2,$$

the result can be obtained by using

$$\int f^{\alpha}(x) \cdot f'(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C.$$

Let

$$f(x) = x^3 + 1,$$

and $\alpha = -\frac{1}{4}$*, so from Theorem 6.8, we get*

$$\int \frac{x^2}{\sqrt[4]{x^3+1}} dx = \frac{1}{3} \int \left(x^3+1\right)^{-\frac{1}{4}} \cdot 3x^2 dx = \frac{1}{3} \frac{\left(x^3+1\right)^{\frac{3}{4}}}{\frac{3}{4}} + C.$$

8.7 Definite Integrals of Real Functions

Step-by-Step Solution

Evaluate the following definite integral.

$$\int_0^1 \sqrt{x\sqrt{x}}dx.$$

Solution 8.7.1 *From Solution 8.6.1, we have*

$$\int \sqrt{x\sqrt{x}}dx = \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + C = \frac{x^{\frac{7}{4}}}{\frac{7}{4}} + C.$$

So from Theorem 7.1 with $F(x) = \frac{x^{\frac{7}{4}}}{\frac{7}{4}}$, *we get*

$$\int_0^1 \sqrt{x\sqrt{x}}dx = [F(x)]_0^1 = \left[\frac{x^{\frac{7}{4}}}{\frac{7}{4}}\right]_0^1 = \frac{1^{\frac{7}{4}}}{\frac{7}{4}} - 0 = \frac{4}{7}.$$

Step-by-Step Solution

Evaluate the following definite integral.

$$\int_0^\pi (x-1)\cos(x)\, dx.$$

Solution 8.7.2 *From Solution 8.6.6, we have*

$$\int (x-1)\cos(x)\, dx = \sin(x)\cdot(x-1) + \cos(x) + C.$$

So from Theorem 7.1 with $F(x) = \sin(x)\cdot(x-1) + \cos(x)$, *we get*

$$\begin{aligned}\int_0^\pi (x-1)\cos(x)\, dx &= [F(x)]_0^\pi = [\sin(x)\cdot(x-1)+\cos(x)]_0^\pi = \\ &= \sin(\pi)\cdot(\pi-1)+\cos(\pi) - (\sin(0)\cdot(0-1)+\cos(0)) = \\ &= -2.\end{aligned}$$

Step-by-Step Solution

Evaluate the following definite integral.

$$\int_0^1 x \cdot 2^x dx.$$

Solution 8.7.3 *From Solution 8.6.7, we have*

$$\int x \cdot 2^x dx = \frac{2^x}{\ln(2)} \cdot x - \frac{1}{\ln^2(2)} 2^x + C.$$

So from Theorem 7.1 with $F(x) = \frac{2^x}{\ln(2)} \cdot x - \frac{1}{\ln^2(2)} 2^x$, *we get*

$$\begin{aligned} \int_0^1 x \cdot 2^x dx &= [F(x)]_0^1 = \left[\frac{2^x}{\ln(2)} \cdot x - \frac{1}{\ln^2(2)} 2^x \right]_0^1 = \\ &= \frac{2^1}{\ln(2)} \cdot 1 - \frac{1}{\ln^2(2)} 2^1 - \left(\frac{2^0}{\ln(2)} \cdot 0 - \frac{1}{\ln^2(2)} 2^0 \right) = \\ &= \frac{2}{\ln(2)} - \frac{2}{\ln^2(2)} + \frac{1}{\ln^2(2)}. \end{aligned}$$

Step-by-Step Solution

Evaluate the following definite integral.

$$\int_1^e \ln(x) dx.$$

Solution 8.7.4 *From Solution 8.6.8, we have*

$$\int \ln(x) dx = x \ln(x) - \int 1 dx = x \ln(x) - x + C.$$

So from Theorem 7.1 with $F(x) = x \ln(x) - x$, *we get*

$$\int_1^e \ln(x) dx = [F(x)]_1^e = [x \ln(x) - x]_1^e = e \ln(e) - e - (1 \ln(1) - 1) = 1.$$

Step-by-Step Solution

Evaluate the following definite integral.

$$\int_{-1}^{0} e^{5-x} dx,$$

Solution 8.7.5 *From Solution 8.6.23, we have*

$$\int e^{5-x} dx = \frac{e^{5-x}}{-1} + C.$$

So from Theorem 7.1 with $F(x) = \frac{e^{5-x}}{-1}$*, we get*

$$\int_{-1}^{0} e^{5-x} dx = [F(x)]_{-1}^{0} = \left[x\frac{e^{5-x}}{-1}\right]_{-1}^{0} = \frac{e^{5-(-1)}}{-1} - (\frac{e^{5-0}}{-1}) = -e^{6} + e^{5}.$$

Step-by-Step Solution

Evaluate the following definite integral.

$$\int_{0}^{\frac{\pi}{4}} \tan(x)\, dx.$$

Solution 8.7.6 *From Solution 8.6.24, we have*

$$\int \tan(x)\, dx = -\int \frac{-\sin(x)}{\cos(x)} dx = -\ln(|\cos(x)|) + C.$$

So from Theorem 7.1 with $F(x) = -\ln(|\cos(x)|)$*, we get*

$$\int_{0}^{\frac{\pi}{4}} \tan(x)\, dx = [F(x)]_{0}^{\frac{\pi}{4}} = [-\ln(|\cos(x)|)]_{0}^{\frac{\pi}{4}} =$$

$$= -\ln\left(\left|\cos\left(\frac{\pi}{4}\right)\right|\right) - (-\ln(|\cos(0)|)) = -\ln(\frac{1}{\sqrt{2}}) + 0.$$

Step-by-Step Solution

Evaluate the following definite integral.

$$\int_{0}^{\pi} \sin(x)\cos(x)\, dx.$$

Solution 8.7.7 *From Solution 8.6.34, we have*

$$\int \sin(x)\cos(x)\, dx = \frac{\sin^2(x)}{2} + C.$$

So from Theorem 7.1 with $F(x) = \frac{\sin^2(x)}{2}$*, we get*

$$\int_0^\pi \sin(x)\cos(x)\,dx = [F(x)]_0^\pi = \left[\frac{\sin^2(x)}{2}\right]_0^\pi = \frac{\sin^2(\pi)}{2} - \frac{\sin^2(0)}{2} = 0.$$

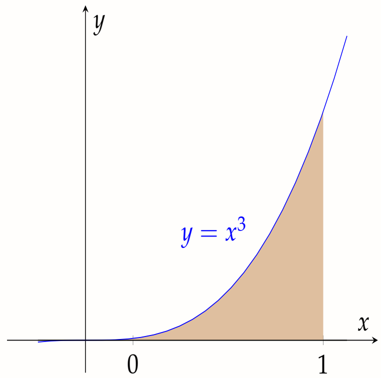
8.7.1 Areas Under, Above and Between Curves

Step-by-Step Solution

Find the area of the indicated region.

Between the vertical lines $x = 0$, $x = 1$, the x -axis and the graph of $f(x) = x^3$.

Solution 8.7.8 *For* $0 \leq x \leq 1$, *we have* $0 \leq x^3 \leq 1$. *Rectangle* $[0,1] \times [0,1]$ *is a suitable enclosing rectangle.*



The required area is

$$A = \int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}.$$

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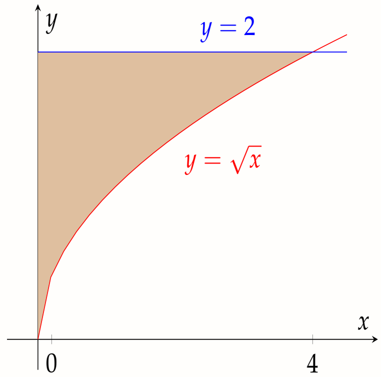
Step-by-Step Solution

Find the area of the indicated region.

Above the graph of $f(x) = \sqrt{x}$, below the line $y = 2$ and between $x = 0$ and $x = 4$.

Solution 8.7.9 *For* $0 \leq x \leq 4$*, we have* $0 \leq \sqrt{x} \leq 2$ *, so the region is above tha graph of function* f*. Rectangle* $[0,4] \times [0,2]$ *is a suitable enclosing rectangle.*



Its area is

$$A = \int_0^4 \left(2 - \sqrt{x}\ \right)\ dx \ = \int_0^4 \left(2 - x^{1/2}\right)\ dx = \left[2x - \frac{x^{3/2}}{3/2}\right]_0^4 =$$

$$= \left(2 \cdot 4 - \frac{2 \cdot \sqrt{4^3}}{3}\right) - \left(2 \cdot 0 - \frac{2 \cdot 0}{3}\right) = \frac{8}{3}.$$

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Step-by-Step Solution

Find the area of the indicated region.

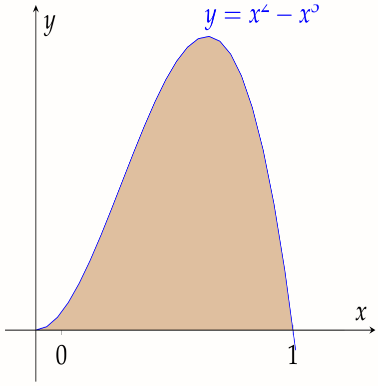
Above the x -axis and below the graph of $f(x) = x^2 - x^3$.

Solution 8.7.10 *Equation*

$$f(x) = x^2 - x^3 = x^2(1-x) = 0$$

has solutions $x_1 = 0$ *and* $x_2 = 1$.

For $0 \le x \le 1$ *each* x^2, x^3 *and* $1-x$ *are between* 0 *and* 1, *so* $f(x) = x^2(1-x)$ *is also between* 0 *and* 1. *This means that rectangle* $[0,1] \times [0,1]$ *is a suitable enclosing rectangle.*



The required area is

$$A = \int_0^1 x^2 - x^3 dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1^3}{3} - \frac{1^4}{4}\right) - \left(\frac{0^3}{3} - \frac{0^4}{4}\right) = \frac{1}{12}\,.$$

NOTE:

$$\begin{aligned}
\frac{d}{dx}\left(x^2 - x^3\right) &= -x \cdot (3x-2) = 0 \\
&\Downarrow \\
x_{\max} &= \frac{2}{3} \\
&\Downarrow \\
f(x_{\max}) &= \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 = \frac{4}{27} \approx 0.14815
\end{aligned}$$

Back to Exercise 7.2 3

Step-by-Step Solution

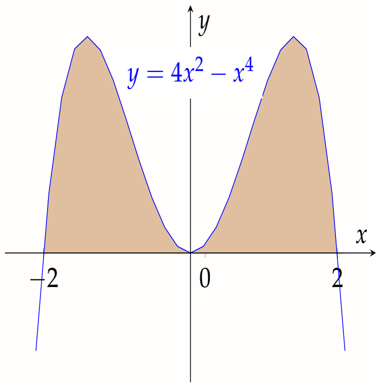
Find the area of the indicated region.

Above the x -axis and below the graph of $f(x) = 4x^2 - x^4$.

Solution 8.7.11 *Equation*

$$f(x) = 4x^2 - x^4 = x^2\left(4 - x^2\right) = 0$$

has solutions $x_1 = -2$, $x_2 = 0$ *and* $x_3 = +2$. *For* $-2 \le x \le 2$, *both* x^2 *and* $4 - x^2$ *are nonnegative, so* $0 \le f(x)$ *for* $-2 \le x \le 2$.



Now the required area is

$$A = \int_{-2}^{2} \left(4x^2 - x^4\right)\, dx = \left[4 \cdot \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} =$$

$$= \left(4 \cdot \frac{2^3}{3} - \frac{2^5}{5}\right) - \left(4 \cdot \frac{(-2)^3}{3} - \frac{(-2)^5}{5}\right) = \frac{128}{15} \approx 8.53\dot{3}.$$

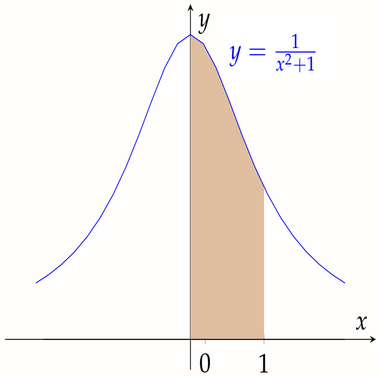
Back to Exercise 7.2 4

Step-by-Step Solution

Find the area of the indicated region.

Above the x -axis and below the graph of $f(x) = \dfrac{1}{1+x^2}$ between $x = 0$ and $x = 1$.

Solution 8.7.12 *Clearly* $0 < f(x)$ *for* $x \in \mathbb{R}$.



So the area is

$$\int_0^1 \frac{1}{1+x^2} = [\arctan(x)]_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

NOTE: $f(x)$ *is an even function, i.e.* $f(-x) = f(x)$ *for all* $x \in dom(f) = \mathbb{R}$.

Back to Exercise 7.2 5

Step-by-Step Solution

Find the area of the indicated region.

The region between the x -axis and the graph of $f(x) = \dfrac{1}{1+x} + \dfrac{x}{2} - 1$.

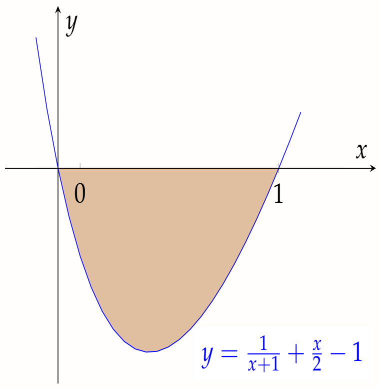
Solution 8.7.13 *First we find the roots of* $f(x) = 0$.

$$f(x) = \frac{1}{1+x} + \frac{x}{2} - 1 \quad = \quad \frac{2 + (1+x)\cdot x - 2\cdot(1+x)}{2(1+x)} = \frac{x(x-1)}{2(1+x)} = 0$$

$$\Updownarrow$$

$$x_1 = 0, \qquad x_2 = 1.$$

For $0 < x < 1$, *this fraction has negative enumerator and positive denumerator, so* $f(x) < 0$.



The signed area is

$$\int_0^1 \left(\frac{1}{1+x} + \frac{x}{2} - 1\right) dx = \left[\ln(x+1) + \frac{x^2}{4} - x\right]_0^1 = \ln(2) - \frac{3}{4} \approx -0.05685.$$

This signed area is negative since $f(x) < 0$ *for* $0 \le x \le 1$, *i.e. (the graph of)* $f(x)$ *is below the x axis. The area is*

$$A = -\int_0^1 \left(\frac{1}{1+x} + \frac{x}{2} - 1\right) dx = -\left(\ln(2) - \frac{3}{4}\right) \approx 0.05685.$$

Back to Exercise 7.2 6

Step-by-Step Solution

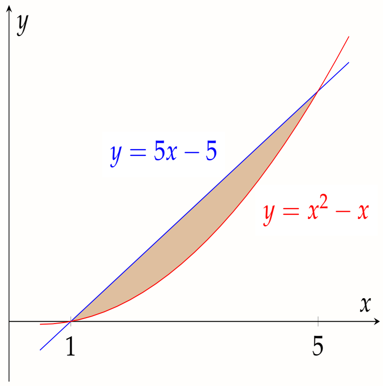
Find the area of the region bounded by the given curves.

$$y = x^2 - x \text{ and } y = 5x - 5.$$

Solution 8.7.14 *The intersection points of the curves are the solutions of*

$$x^2 - x = 5x - 5 \iff x^2 - 6x + 5 = 0,$$

that are $x_1 = 1$ *and* $x_2 = 5$.



Since $f(x) = x^2 - x$ *is convex and* $g(x) = 5x - 5$ *is a straight line, for* $x_1 \leq x \leq x_2$, *we have* $f(x) \leq g(x)$. *From Theorem 7.2, the area is*

$$A = \int_1^5 \left((5x-5) - \left(x^2 - x\right) \right) dx =$$

$$= \int_1^5 \left(-x^2 + 6x - 5d \right) x = \left[\frac{-x^3}{3} + 6 \cdot \frac{x^2}{2} - 5x \right]_1^5 =$$

$$= \left(\frac{-5^3}{3} + 6 \cdot \frac{5^2}{2} - 5 \cdot 5 \right) - \left(\frac{-1^3}{3} + 6 \cdot \frac{1^2}{2} - 5 \cdot 1 \right) = \frac{32}{3} \approx 10.6667\,.$$

Back to Exercise 7.3 1

Step-by-Step Solution

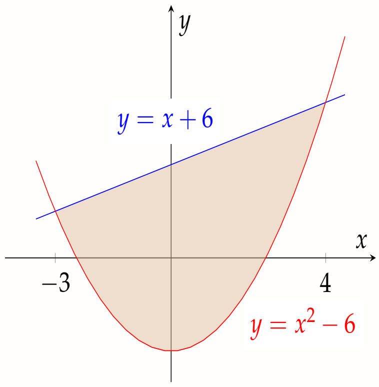
Find the area of the region bounded by the given curves.

$$y = x^2 - 6 \text{ and } y = x + 6.$$

Solution 8.7.15 *The intersection points of the curves are the solutions of*

$$x^2 - 6 = x + 6 \Longleftrightarrow x^2 - x - 12 = 0,$$

that are $x_1 = -3$ *and* $x_2 = 4$.



Since $f(x) = x^2 - 6$ *is convex and* $g(x) = x + 5$ *is a straight line, for* $x_1 \leq x \leq x_2$ *we have* $f(x) \leq g(x)$*. From Theorem 7.2, the area is*

$$\begin{aligned} A &= \int_{-3}^{4} \left((x+6) - \left(x^2 - 6\right) \right) \, dx = \\ &= \int_{-3}^{4} \left(-x^2 + x + 12 \right) \, dx = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 12x \right]_{-3}^{4} = \\ &= \left(-\frac{1}{3} \cdot 4^3 + \frac{1}{2} \cdot 4^2 + 12 \cdot 4 \right) - \left(-\frac{1}{3} \cdot (-3)^3 + \frac{1}{2} \cdot (-3)^2 + 12 \cdot (-3) \right) = \\ &= \frac{343}{6} \approx 57.16667\,. \end{aligned}$$

Back to Exercise 7.3 2

Step-by-Step Solution

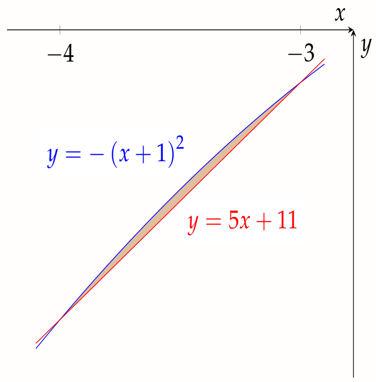
Find the area of the region bounded by the given curves.

$$y = -(x+1)^2 \text{ and } y = 5x + 11.$$

Solution 8.7.16 *The intersection points of the curves are the solutions of*

$$-(x+1)^2 = 5x + 11 \iff -x^2 - 7x - 12 = 0,$$

that are $x_1 = -4$ *and* $x_2 = -3$.



Since $f(x) = -(x+1)^2$ *is concave and* $g(x) = 5x + 11$ *is a straight line, for* $x_1 \leq x \leq x_2$*, we have* $g(x) \leq f(x)$*. From Theorem 7.2, the area is*

$$\begin{aligned} A &= \int_{-4}^{-3} \left(-(x+1)^2 - (5x+11) \right) dx = \int_{-4}^{-3} \left(-x^2 - 7x - 12 \right) dx = \\ &= \left[-\frac{1}{3}x^3 - \frac{7}{2}x^2 - 12x \right]_{-4}^{-3} = \left(-\frac{1}{3} \cdot (-3)^3 - \frac{7}{2} \cdot (-3)^2 - 12 \cdot (-3) \right) - \\ &- \left(-\frac{1}{3} \cdot (-4)^3 - \frac{7}{2} \cdot (-4)^2 - 12 \cdot (-4) \right) = \\ &= \frac{1}{6} \approx 0.16667 \,. \end{aligned}$$

Back to Exercise 7.3 3

Step-by-Step Solution

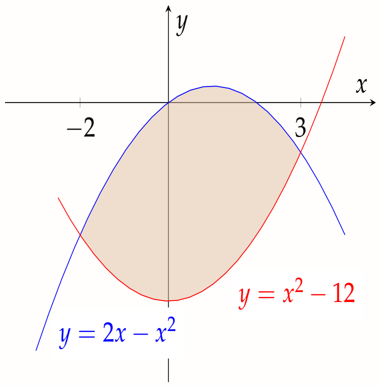
Find the area of the region bounded by the given curves.

$$y = x^2 - 12 \text{ and } y = 2x - x^2.$$

Solution 8.7.17 *The intersection points of the curves are the solutions of*

$$x^2 - 12 = 2x - x^2 \iff 2x^2 - 2x - 12 = 0,$$

that are $x_1 = -2$ *and* $x_2 = 3$.



Since $f(x) = x^2 - 12$ *is convex and* $g(x) = 2x - x^2$ *is concave, for* $x_1 \le x \le x_2$, *we have* $f(x) \le g(x)$. *From Theorem 7.2, the area is*

$$\begin{aligned} A &= \int_{-2}^{3} \left(\left(2x - x^2\right) - \left(x^2 - 12\right) \right) dx = \\ &= \int_{-2}^{3} -2x^2 + 2x + 12 \, dx = \left[-\frac{2}{3}x^3 + x^2 + 12x \right]_{-2}^{3} = \\ &= \left(-\frac{2}{3} \cdot 3^3 + 3^2 + 12 \cdot 3 \right) - \left(-\frac{2}{3} \cdot (-2)^3 + (-2)^2 + 12 \cdot (-2) \right) = \\ &= \frac{125}{3} \approx 41.6667 \,. \end{aligned}$$

Back to Exercise 7.3 4

Step-by-Step Solution

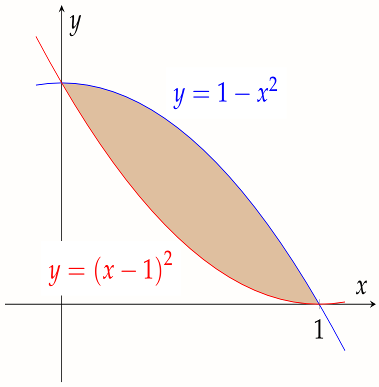
Find the area of the region bounded by the given curves.

$$y = (x-1)^2 \text{ and } y = 1 - x^2.$$

Solution 8.7.18 *The intersection points of the curves are the solutions of*

$$(x-1)^2 = 1 - x^2 \Longleftrightarrow 2x^2 - 2x = 0,$$

that are $x_1 = 0$ *and* $x_2 = 1$.



Since $f(x) = (x-1)^2$ *is convex and* $g(x) = 1 - x^2$ *is concave, for* $x_1 \leq x \leq x_2$ *we have* $f(x) \leq g(x)$. *From Theorem 7.2, the area is*

$$A = \int_0^1 \left(\left(1 - x^2\right) - (x-1)^2\right) dx = \int_0^1 \left(-2x^2\right) dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 =$$

$$= \left(1^2 - \frac{2}{3} \cdot 1^3\right) - 0 = \frac{1}{3} \approx 0.3333\,.$$

Back to Exercise 7.3 5

Step-by-Step Solution

Find the area of the region bounded by the given curves.

$$y^2 = 4x \text{ and } y = 2x.$$

Solution 8.7.19 *The system of equations*

$$\left\{y^2 = 4x \, , \, y = 2x\right\} \Longleftrightarrow 4x = (2x)^2$$

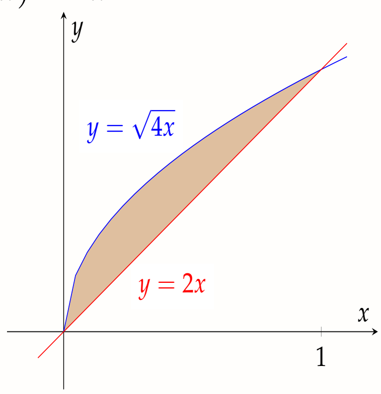
has the solutions $x_1 = 0$, $x_2 = 1$. *However, the implicit equation* $y^2 = 4x$ *hides two functions*

$$f_1(x) = \sqrt{4x},$$

and

$$f_2(x) = -\sqrt{4x}.$$

Equation $y = 2x$ *implies that* $y_1 = 0$ *and* $y_2 = 2$*, so the intersection points are* $P_1 = (0, 0)$ *and* $P_2 = (1, 2)$*. One can easily check that* P_2 *satisfies only* $y = f_1(x)$ *, so the required area is between* $f_1(x)$ *and* $g(x) = 2x$ *.*



Since $f_1(x) = \sqrt{4x} = 2\sqrt{x}$ *is concave and* $g(x)$ *is a straight line, for* $x_1 \leq x \leq x_2$*, we have* $g(x) \leq f(x)$*. From Theorem 7.2, the area is*

$$A = \int_0^1 2\sqrt{x} - 2x \, dx \quad = \quad \left[2 \cdot \frac{x^{3/2}}{3/2} - x^2\right]_0^1 = \left[\frac{4}{3} \cdot \sqrt{x^3} - x^2\right]_0^1 =$$

$$= \quad \left(\frac{4}{3} \cdot 1 - 1^2\right) - 0 = \frac{1}{3} \approx 0.3333 \, .$$

Back to Exercise 7.3 6

Step-by-Step Solution

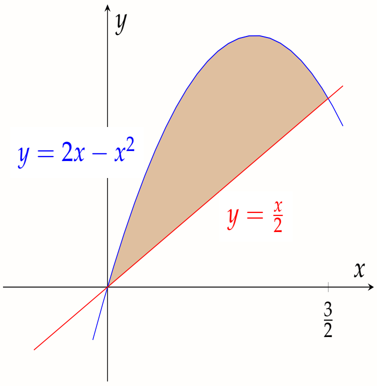
Find the area of the region bounded by the given curves.

$$y = x(2-x) \text{ and } x = 2y.$$

Solution 8.7.20 *The two functions are* $f(x) = x(2-x) = 2x - x^2$ *and* $g(x) = x/2$ *(!). The intersection points of the graphs of the functions are the solutions of*

$$2x - x^2 = x/2 \Longleftrightarrow 3x - 2x^2 = 0,$$

that are $x_1 = 0$ *and* $x_2 = \frac{3}{2}$.



Since $f(x) = 2x - x^2$ *is concave and* $g(x)$ *is a straight line, for* $x_1 \le x \le x_2$, *we have* $g(x) \le f(x)$. *From Theorem 7.2, the area is*

$$A = \int_0^{3/2} \left(\left(2x - x^2\right) - \frac{x}{2} \right) dx = \int_0^{3/2} \left(\frac{3}{2}x - x^2 \right) dx = \left[\frac{3}{4}x^2 - \frac{x^3}{3} \right]_0^{3/2} =$$

$$= \left(\frac{3}{4} \cdot \left(\frac{3}{2}\right)^2 - \frac{\left(\frac{3}{2}\right)^3}{3} \right) - 0 = \frac{9}{16} = 0.5625 \,.$$

Back to Exercise 7.3 7

Step-by-Step Solution

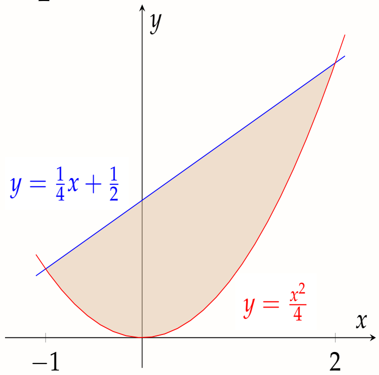
Find the area of the region bounded by the given curves.

$$x^2 = 4y \text{ and } x = 4y - 2.$$

Solution 8.7.21 *Solving the equations $x^2 = 4y$ and $x = 4y - 2$ to y, we get the two functions: $y = f(x) = \frac{x^2}{4}$ and $y = g(x) = \frac{1}{4}x + \frac{1}{2}$.*
Equation

$$\frac{x^2}{4} = \frac{1}{4}x + \frac{1}{2} \Longleftrightarrow x^2 - x - 2 = 0,$$

has the solutions $x_1 = -1$ and $x_2 = 2$.



Since $f(x)$ is convex and $g(x)$ is a straight line, for $x_1 \le x \le x_2$, we have $f(x) \le g(x)$. From Theorem 7.2, the area is

$$\begin{aligned} A &= \int_{-1}^{2} \left(\left(\frac{1}{4}x + \frac{1}{2} \right) - \frac{x^2}{4} \right) dx = \int_{-1}^{2} \left(-\frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{2} \right) dx = \\ &= \left[-\frac{1}{12}x^3 + \frac{1}{8}x^2 + \frac{1}{2}x \right]_{-1}^{2} = \\ &= \left(-\frac{1}{12} \cdot 2^3 + \frac{1}{8} \cdot 2^2 + \frac{1}{2} \cdot 2 \right) - \left(-\frac{1}{12} \cdot (-1)^3 + \frac{1}{8} \cdot (-1)^2 + \frac{1}{2} \cdot (-1) \right) = \\ &= \frac{9}{8} = 1.125\,. \end{aligned}$$

Back to Exercise 7.3 8

Step-by-Step Solution

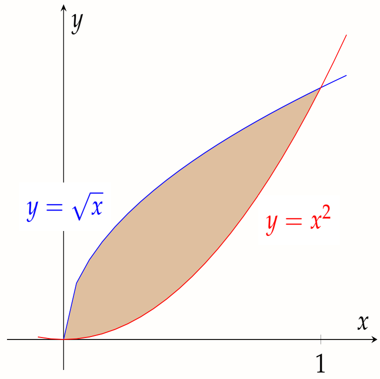
Find the area of the region bounded by the given curves.

$$x = y^2 \text{ and } y = x^2.$$

Solution 8.7.22 *System of equations*

$$\left\{x = y^2\,,\; y = x^2\right\} \Longrightarrow x = x^4 \Longleftrightarrow x\left(x^3 - 1\right) = 0$$

has the (real) solutions $x_1 = 0$, $x_2 = 1$ *and* $y_1 = 0$, $y_2 = 1$, *so the intersection points are* $P_1 = (0,0)$ *and* $P_2 = (1,1)$. *This means, that we have to choose the function* $f(x) = \sqrt{x}$ *from the implicit equation* $y^2 = x$.



Since $f(x)$ *is concave and* $g(x) = x^2$ *is convex, for* $x_1 \le x \le x_2$ *we have* $g(x) \le f(x)$. *From Theorem 7.2, the area is*

$$A = \int_0^1 \left(\sqrt{x} - x^2\right)\,dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1 = \left[\frac{2}{3}\sqrt{x^3} - \frac{x^3}{3}\right]_0^1 =$$

$$= \left(\frac{2}{3}\cdot\sqrt{1^3} - \frac{1^3}{3}\right) - 0 = \frac{1}{3} \approx 0.3333\,.$$

Back to Exercise 7.3 9

Step-by-Step Solution

Find the area of the region bounded by the given curves.

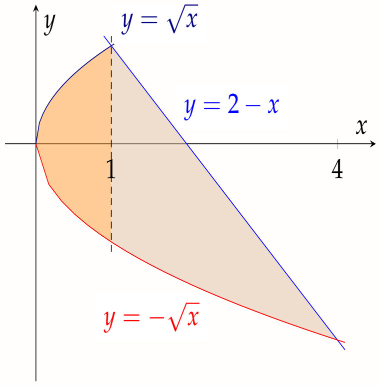
$$y^2 = x \text{ and } x + y = 2.$$

Solution 8.7.23 *System of equations*

$$\left\{y^2 = x\,,\; x + y = 2\right\} \Rightarrow (2 - x)^2 = x \Longleftrightarrow x^2 - 5x + 4 = 0$$

has the (real) solutions $x_1 = 1$, $x_2 = 4$ *and by* $y = 2 - x$, $y_1 = 1$, $y_2 = -2$, *so the intersection points are* $P_1 = (1,1)$ *and* $P_2 = (4,-2)$.
The implicit equation $y^2 = x$ *hides two functions:* $f_1(x) = \sqrt{x}$ *and* $f_2(x) = -\sqrt{x}$.

The problem is, that P_1 *is on the graph of* f_1 *while* P_2 *is on the graph of* f_2 *. This means, that the borders of the closed region we are looking for can not be only the straight line* $x+y=2$ *(i.e.* $g(x)=2-x$*) and only one of the functions* f_1 *,* f_2 *. In other words all of* f_1 *,* f_2 *and* g *limit the region - but how? We must make a drawing (see below).*



Since for the integral we need vertical lines $x=c$ *for some* c *, we have to cut this region into two parts with the vertical line* $x=1$ *(crossing* P_1*), and investigate the convex-concave properties of the functions. From Theorem 7.2, the area is*

$$
\begin{aligned}
A &= \int_0^1 \left(f_1(x)-f_2(x)\right)\,dx+\int_1^4 \left(g(x)-f_2(x)\right)\,dx = \\
&= \int_0^1 \left(\sqrt{x}-(-\sqrt{x})\right)\,dx+\int_1^4 \left((2-x)-(-\sqrt{x})\right)\,dx = \\
&= 2\left[\frac{2}{3}\sqrt{x^3}\right]_0^1+\left[2x-\frac{x^2}{2}+\frac{2}{3}\sqrt{x^3}\right]_1^4 = \\
&= 2\cdot\left(\frac{2}{3}\cdot\sqrt{1^3}-0\right)+\left[\left(2\cdot 4-\frac{4^2}{2}+\frac{2}{3}\cdot\sqrt{4^3}\right)-\left(2\cdot 1-\frac{1^2}{2}+\frac{2}{3}\cdot\sqrt{1^3}\right)\right] = \\
&= \frac{9}{2}=4.5\,.
\end{aligned}
$$

Back to Exercise 7.3 10

Step-by-Step Solution

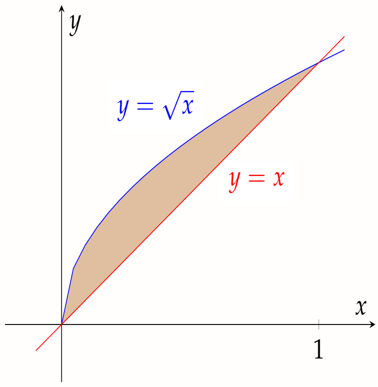
Find the area of the region bounded by the given curves.

$$y = \sqrt{x} \text{ and } y = x.$$

Solution 8.7.24 *The intersection points of the curves are the solutions of*

$$\sqrt{x} = x \iff x^2 = x,$$

that are $x_1 = 0$ *and* $x_2 = 1$.



Since $f(x) = \sqrt{x}$ *is concave and* $g(x) = x$ *is a straight line, for* $x_1 \leq x \leq x_2$, *we have* $g(x) \leq f(x)$. *From Theorem 7.2, the area is*

$$A = \int_0^1 \left(\sqrt{x} - x\right)\, dx = \left[\frac{2}{3}\sqrt{x^3} - \frac{x^2}{2}\right]_0^1 = \left(\frac{2}{3} \cdot \sqrt{1^3} - \frac{1^2}{2}\right) - 0 = \frac{1}{6} \approx 0.16667\,.$$

Back to Exercise 7.3 11

Step-by-Step Solution

Find the area of the region bounded by the given curves.

$$y = x^2 \text{ and } y = 3/(2 + x^2).$$

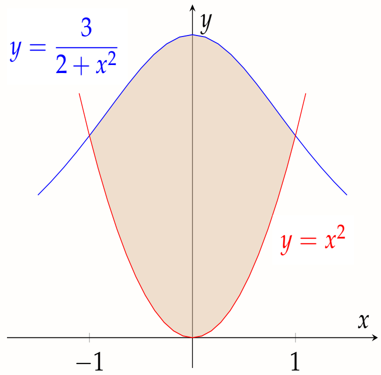
Solution 8.7.25 *The intersection points of the curves are the solutions of*

$$x^2 = \frac{3}{2 + x^2} \iff x^2 \cdot \left(2 + x^2\right) = 3 \iff x^4 + 2x^2 - 3 = 0.$$

Let $a = x^2$, *then*

$$x^4 + 2x^2 - 3 = 0 \iff a^2 + 2a - 3 = 0,$$

and the solutions are $a_1 = 1$ *and* $a_2 = -3$. *As* $a \geq 0$, *we get that* $x_1 = -1$ *and* $x_2 = 1$ *are the solution of equation* $x^2 = \dfrac{3}{2 + x^2}$.



Since $f(x) = x^2$ *is convex and* $g(x) = \dfrac{3}{2+x^2}$ *is concave, for* $x_1 \leq x \leq x_2$ *we have* $f(x) \leq g(x)$. *From Theorem 7.2, the area is*

$$A = \int_{-1}^{1} \left(\frac{3}{2+x^2} - x^2 \right) \, dx$$

The primitive function of g *is (using* $\arctan'(x) = \frac{1}{1+x^2}$*)*

$$\begin{aligned} \int \frac{3}{2+x^2} \, dx &= \frac{3}{2} \int \frac{1}{1+\frac{x^2}{2}} \, dx = \frac{3}{2} \int \frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2} \, dx = \\ &= \frac{3}{2} \cdot \frac{\arctan\left(x/\sqrt{2}\right)}{1/\sqrt{2}} + C = \frac{3\cdot\sqrt{2}}{2} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + C\,, \end{aligned}$$

so the area is

$$\begin{aligned} A &= \int_{-1}^{1} \left(\frac{3}{2+x^2} - x^2 \right) \, dx = \left[\frac{3\cdot\sqrt{2}}{2} \cdot \arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{x^3}{3} \right]_{x=-1}^{x=1} = \\ &= \left(\frac{3\cdot\sqrt{2}}{2} \cdot \arctan\left(\frac{1}{\sqrt{2}}\right) - \frac{1^3}{3} \right) - \left(\frac{3\cdot\sqrt{2}}{2} \cdot \arctan\left(\frac{-1}{\sqrt{2}}\right) - \frac{(-1)^3}{3} \right) = \\ &= 2 \cdot \frac{3\cdot\sqrt{2}}{2} \cdot \arctan\left(\frac{1}{\sqrt{2}}\right) - \frac{2}{3} \approx 1.9446\,. \end{aligned}$$

Note: f *is an even function* ($f(-x) = f(x)$) *so* $\int_{-a}^{+a} f(x)\,dx = 2\int_{0}^{+a} f(x)\,dx$ **.**

Back to Exercise 7.3 12

Step-by-Step Solution

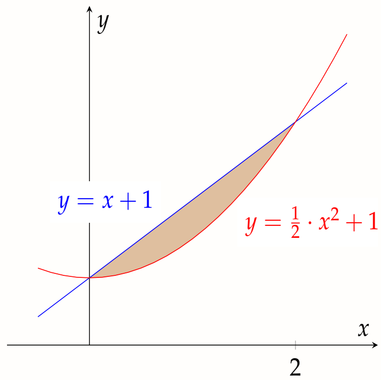
Find the area of the region bounded by the given curves.

$$y = (1/2) \cdot x^2 + 1 \text{ and } y = x + 1.$$

Solution 8.7.26 *The intersection points of the curves are the solutions of*

$$\frac{1}{2} \cdot x^2 + 1 = x + 1 \iff x^2 - 2x = 0.$$

that are $x_1 = 0$ *and* $x_2 = 2$.



Since $f(x) = \frac{1}{2} \cdot x^2 + 1$ *is convex and* $g(x) = x + 1$ *is a straight line, for* $x_1 \leq x \leq x_2$ *we have* $f(x) \leq g(x)$. *From Theorem 7.2, the area is*

$$A = \int_0^2 \left((x+1) - \left(\frac{1}{2} \cdot x^2 + 1 \right) \right) dx = \left[\frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^3}{3} \right]_0^2 = \left(\frac{2^2}{2} - \frac{1}{2} \cdot \frac{2^3}{3} \right) - 0 = \frac{2}{3}.$$

Back to Exercise 7.3 13

Step-by-Step Solution

Find the area of the region bounded by the given curves.

$$y^2 = x \text{ and } x^2 = 16y.$$

Solution 8.7.27 *From*

$$y^2 = x \,,\; x^2 = 16y,$$

we have

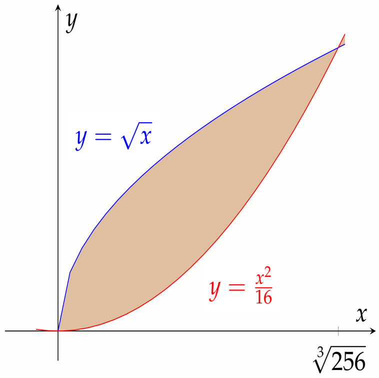
$$y^4 = 16y \iff y \cdot \left(y^3 - 16 \right) = 0.$$

Equation

$$y \cdot \left(y^3 - 16 \right) = 0$$

has (real) solutions $y_1 = 0$, $y_2 = \sqrt[3]{16} \approx 2.5198$. *This follows* $x_1 = 0$ *and* $x_2 = (y_2)^2 = \left(\sqrt[3]{16} \right)^2 = \sqrt[3]{256} \approx 6.3496$.

The the implicit equation $y^2 = x$ means the two functions $y = f_1(x) = \sqrt{x}$ and $y = f_2(x) = -\sqrt{x}$, but the solution $y_2 = \sqrt[3]{16}$ requires the function $y = f_1(x)$. So the region is between $y = f_1(x)$ and $y = g(x) = x^2/16$.



Since $f_1(x)$ is concave and $g(x)$ is convex, for $x_1 \le x \le x_2$ we have $g(x) \le f(x)$, and the area is

$$A = \int_0^{\sqrt[3]{256}} \left(\sqrt{x} - \frac{x^2}{16}\right) dx = \left[\frac{x^{3/2}}{3/2} - \frac{1}{16}\cdot\frac{x^3}{3}\right]_0^{\sqrt[3]{256}} =$$

$$= \left(\frac{\left(\sqrt[3]{256}\right)^{3/2}}{3/2} - \frac{1}{16}\cdot\frac{\left(\sqrt[3]{256}\right)^3}{3}\right) - 0 = \frac{16}{3} \approx 5.3333.$$

Back to Exercise 7.3 14

Step-by-Step Solution

Find the area of the region bounded by the given curves.

$$y^2 = 4ax \text{ and } y = mx.$$

Solution 8.7.28 *We handle the case $0 < a$ and $0 < m$ only, the other similar cases are left to the Reader (both a and m may be negative but nonzero).*
From

$$y^2 = 4ax\ ,\ y = mx,$$

we have

$$(mx)^2 = 4ax \Longleftrightarrow x\cdot\left(m^2x - 4a\right) = 0.$$

Equation

$$x\cdot\left(m^2x - 4a\right) = 0$$

has the solutions $x_1 = 0$ and $x_2 = 4a/m^2$. From this, we get $y_1 = 0$ and $y_2 = mx_2 = 4a/m$, so the intersection points are $P_1(0,0)$ and $P_2\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

Using $0 < a, m$ *we have* $0 < y_2$ *and so the region is between the functions* $y = f(x) = \sqrt{4ax}$ *and* $y = g(x) = mx$. *Since* $f(x)$ *is concave and* $g(x)$ *is a straight line, for* $x_1 \leq x \leq x_2$ *we have* $g(x) \leq f(x)$, *and the area is*

$$A = \int_0^{\frac{4a}{m^2}} \left(\sqrt{4ax} - mx\right) dx = \left[\frac{(4ax)^{3/2}}{4a \cdot 3/2} - m \cdot \frac{x^2}{2}\right]_0^{\frac{4a}{m^2}} =$$

$$= \left(\frac{2}{3} \cdot \frac{\sqrt{\left(4a \cdot \frac{4a}{m^2}\right)^3}}{4a} - m \cdot \frac{\left(\frac{4a}{m^2}\right)^2}{2}\right) - 0 =$$

$$= \frac{2}{12a} \cdot \left(\frac{4a}{m}\right)^3 - \frac{1}{2} \cdot \frac{(4a)^2}{m^3} = \frac{1}{6} \cdot \frac{(4a)^2}{m^3} = \frac{8}{3} \cdot \frac{a^2}{m^3}.$$

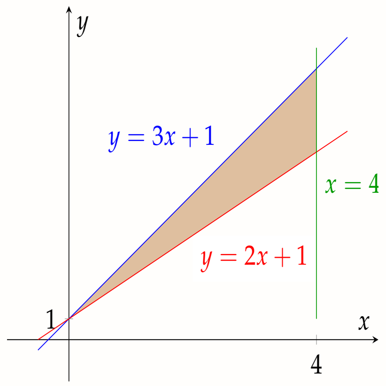
Back to Exercise 7.3 15

Step-by-Step Solution

Use integration to find the area of the triangular region bounded by the given lines:

$$y = 2x + 1, \quad y = 3x + 1 \quad \text{and} \quad x = 4.$$

Solution 8.7.29 *To find the intersection points of lines* $y = 2x + 1$ *and* $y = 3x + 1$, *we need to solve equation* $2x + 1 = 3x + 1$. *The solution is* $x = 0$, *and from this, we get* $y = 1$, *so the intersection point is* $P(0, 1)$. *So, the interval for our region is* $[a, b] = [0, 4]$.



Since the line $y = 3x + 1$ *is above of* $y = 2x + 1$, *the area is*

$$A = \int_0^4 ((3x + 1) - (2x + 1))\, dx = \int_0^4 x dx = \left[\frac{x^2}{2}\right]_0^4 = \frac{4^2}{2} - 0 = 8.$$

Back to Exercise 7.4 1

Step-by-Step Solution

Use integration to find the area of the triangular region bounded by the given lines:

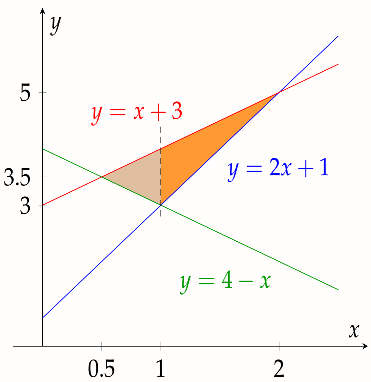
$$y = x + 3, \quad y = 2x + 1 \quad \text{and} \quad andy = 4 - x.$$

Solution 8.7.30 *Call the three lines*

$$\boldsymbol{a} : y = \mathbf{a}(x) = x + 3$$
$$\boldsymbol{b} : y = \mathbf{b}(x) = 2x + 1$$
$$\boldsymbol{c} : y = \mathbf{c}(x) = 4 - x$$

The pairwise intersection points are the following:

$a \cap b$: $y = x + 3 = 2x + 1 \iff x = 2\,, y = 5\,, C = (2, 5)$,
$a \cap c$: $y = x + 3 = 4 - x \iff x = 0.5\,, y = 3.5\,, B = (0.5, 3.5)$,
$b \cap c$: $y = 2x + 1 = 4 - x \iff x = 1\,, y = 3\,, A = (1, 3)$.



Considering the x coordinates of these intersection points we have the order $B_x < A_x < C_x$, which means that the integrals for the area of the region (i.e. the triangle ABC) have the intervals $[B_x\,,\ A_x] = [0.5, 1]$ and $[A_x\,,\ C_x] = [1, 2]$. Now we have to decide, which function (line) is above and below on these intervals.
The interval $[B_x\,,\ A_x]$ starts with $B = \mathbf{a} \cap \mathbf{c}$, so on the interval $[B_x\,,\ A_x]$ lines $\boldsymbol{a}$ and $\boldsymbol{c}$ are present. Since $\mathbf{a}(A_x) = 1 + 3 > \mathbf{c}(A_x) = 4 - 1$, in the whole interval $[B_x\,,\ A_x]$ the relation $\mathbf{a}(x) > \mathbf{c}(x)$ must hold.
The interval $[A_x\,,\ C_x]$ ends with $C = \mathbf{a} \cap \mathbf{b}$, so on this interval $\boldsymbol{a}$ és $\boldsymbol{b}$ are present. Since $\mathbf{a}(A_x) = 1 + 3 > \mathbf{b}(A_x) = 2 \cdot 1 + 1$, in the whole interval $[A_x\,,\ C_x]$ the relation $\mathbf{a}(x) > \mathbf{b}(x)$ must hold.

Now the area is

$$\begin{aligned} A &= \int_{B_x}^{A_x} (\mathbf{a}(x) - \mathbf{c}(x))\, dx + \int_{A_x}^{C_x} (\mathbf{a}(x) - \mathbf{b}(x))\, dx \\ &= \int_{0.5}^{1} ((x+3) - (4-x))\ dx + \int_{1}^{2} ((x+3) - (2x+1))\ dx \\ &= \int_{0.5}^{1} (2x-1)\ dx + \int_{1}^{2} (2-x)\ dx = \left[x^2 - x\right]_{0.5}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{2} = \\ &= \left(1^2 - 1\right) - \left(0.5^2 - 0.5\right) + \left(2 \cdot 2 - \frac{2^2}{2}\right) - \left(2 \cdot 1 - \frac{1^2}{2}\right). \end{aligned}$$

Back to Exercise 7.4 2

Step-by-Step Solution

Find a so that the curves $y = x^2$ and $y = a\cos x$ intersect at the points $(x,y) = \left(\pi/4, \pi^2/16\right)$. Then find the area between these curves.

Solution 8.7.31 *For any function $y = h(x)$ "to meet the point" $P_0 = (x_0, y_0)$ means the equality*

$$y_0 = h(x_0) \ .$$

In our problem $y = f(x) = x^2$, $y = g(x) = a\cos x$ and $P_0 = \left(\pi/4, \pi^2/16\right)$, so we must have

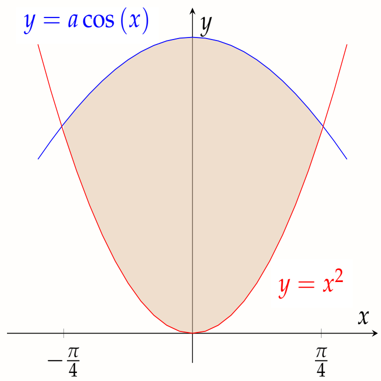
$$\frac{\pi^2}{16} = \left(\frac{\pi}{4}\right)^2 \quad \text{and} \quad \frac{\pi^2}{16} = a \cdot \cos\left(\frac{\pi}{4}\right) = a \cdot \frac{\sqrt{2}}{2}$$

which implies

$$a = \frac{\sqrt{2}}{16}\pi^2 \approx 0.87236.$$

Since both functions are even, the other intersecting point is

$$P_1 = (-x_0, y_0) = \left(-\pi/4, \pi^2/16\right).$$



Since f is convex and g is concave, for $-x_0 \le x \le x_0$ we have $f(x) \le g(x)$, and the area is

$$\begin{aligned} A &= \int_{-\pi/4}^{+\pi/4} \left(a\cos x - x^2\right) dx = \left[a\sin(x) - \frac{x^3}{3}\right]_{-\pi/4}^{+\pi/4} = \\ &= \left(\frac{\sqrt{2}\pi^2}{16} \cdot \sin\left(\frac{\pi}{4}\right) - \frac{\left(\frac{\pi}{4}\right)^3}{3}\right) - \left(\frac{\sqrt{2}\pi^2}{16} \cdot \sin\left(-\frac{\pi}{4}\right) - \frac{\left(-\frac{\pi}{4}\right)^3}{3}\right) \\ &= \frac{\pi^2}{8} - \frac{\pi^3}{96} \approx 0.91072 \ . \end{aligned}$$

Back to Exercise 7.4 3

Step-by-Step Solution

Write a definite integral whose value is the area of the region between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.

Solution 8.7.32 *Circle* $x^2 + y^2 = 1$ *can be drawn with functions*

$$y = f_1(x) = \sqrt{1-x^2}$$

and

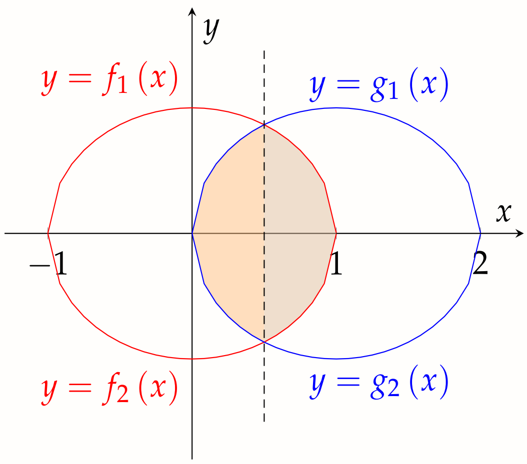
$$y = f_2(x) = -\sqrt{1-x^2}.$$

Similarly $(x-1)^2 + y^2 = 1$ *can be drawn with functions*

$$y = g_1(x) = \sqrt{1-(x-1)^2}$$

and

$$y = g_2(x) = -\sqrt{1-(x-1)^2}.$$



The horizontal endpoints of the region are (see figure above) $a = x_0 = 0$ *and* $b = x_1 = 1$ *, but the limiting function curves change at an intermediate position* x_2 *we have to determine from the system of equations*

$$\begin{cases} x^2 + y^2 = 1 & \text{(I)} \\ (x-1)^2 + y^2 = 1 & \text{(II)} \end{cases}$$

Subtracting (II) from (I) we get $2x - 1 = 0$ *i.e.* $x = 1/2$ *. So, the area is*

$$A = \int_0^{1/2} (g_1(x) - g_2(x))\, dx + \int_{1/2}^{1} (f_1(x) - f_2(x))\, dx.$$

First, we evaluate

$$\int_{1/2}^{1} (f_1(x) - f_2(x))\, dx = \int_{1/2}^{1} 2 \cdot \sqrt{1-x^2}\, dx.$$

For

$$\int \sqrt{1-x^2}\, dx, \quad |x| \le 1,$$

we use substitution

$$x = \sin(t).$$

As

$$\frac{dx}{dt} = \cos(t), \quad |t| \le \frac{\pi}{2},$$

we get

$$\begin{aligned}
\int \sqrt{1-x^2}\, dx &= \int \sqrt{1-(\sin t)^2} \cdot \cos(t)\ dt = \int \cos^2(t)\ dt = \int \frac{1+\cos(2t)}{2} dt = \\
&= \frac{1}{2} \cdot \left(t + \frac{\sin(2t)}{2} \right) + C = \\
&= \frac{1}{2} \cdot \left(\arcsin(x) + \frac{\sin(2 \cdot \arcsin(x))}{2} \right) + C \\
&= \frac{1}{2} \left(\arcsin(x) + x\sqrt{1-x^2} \right) + C.
\end{aligned}$$

So from Theorem 7.1, we get

$$\begin{aligned}
A_1 &= \int_{1/2}^{1} 2 \cdot \sqrt{1-x^2}\, dx = \frac{2}{2} \left[\arcsin(x) + x\sqrt{1-x^2} \right]_{1/2}^{1} = \\
&= \left(\arcsin(1) + 1 \cdot \sqrt{1-1^2} \right) - \left(\arcsin\left(\frac{1}{2}\right) + \frac{1}{2} \cdot \sqrt{1-\left(\frac{1}{2}\right)^2} \right) = \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{4}.
\end{aligned}$$

Now, we evaluate

$$\int_0^{1/2} (g_1(x) - g_2(x))\ dx = \int_0^{1/2} 2 \cdot \sqrt{1-(x-1)^2}\, dx.$$

Similarly, to evaluate

$$\int \sqrt{1-(x-1)^2}\, dx,$$

we use substitution

$$x - 1 = \sin(t).$$

As

$$\frac{dx}{dt} = \cos(t), \quad |t| \le \frac{\pi}{2},$$

we get

$$\int \sqrt{1-(x-1)^2}\,dx = \frac{1}{2}\left(\arcsin(x-1) + (x-1)\cdot\sqrt{1-(x-1)^2}\right) + C.$$

So from Theorem 7.1, we get

$$A_2 = \int_0^{1/2} 2\cdot\sqrt{1-(x-1)^2}\,dx = \frac{2}{2}\left[\arcsin(x-1) + (x-1)\cdot\sqrt{1-(x-1)^2}\right]_0^{1/2} =$$

$$= \left(\arcsin\left(\frac{-1}{2}\right) + \frac{-1}{2}\cdot\sqrt{1-\left(\frac{-1}{2}\right)^2}\right) -$$

$$- \left(\arcsin(-1) + (-1)\cdot\sqrt{1-(-1)^2}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

The area is

$$A = A_1 + A_2 = \frac{2}{3}\pi - \frac{\sqrt{3}}{2} \approx 1.22837.$$

Back to Exercise 7.4 4

8.7.2 *Volume of Revolution*

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = x, \qquad 0 \le x \le 2.$$

Solution 8.7.33

$$V_x = \pi \int_0^2 x^2 dx = \pi \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3}\pi - 0 \approx 8.37758.$$

Back to Exercise 7.5 1

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \sqrt{2-x}, \qquad 0 \le x \le 2.$$

Solution 8.7.34

$$\begin{aligned} V_x &= \pi \int_0^2 (2-x)\, dx = \pi \left[2x - \frac{x^2}{2}\right]_0^2 = \\ &= \pi \left(2 \cdot 2 - \frac{2^2}{2}\right) - 0 = 2\pi \approx 6.283185. \end{aligned}$$

Back to Exercise 7.5 2

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \left(1 + x^2\right)^{-1/2}, \qquad |x| \le 1.$$

Solution 8.7.35

$$\begin{aligned} V_x &= \pi \int_{-1}^{1} \frac{1}{1+x^2} dx = \left[\pi \arctan(x)\,\right]_{-1}^{+1} = \\ &= \pi \cdot (\arctan(1) - \arctan(-1)) = \frac{1}{2}\pi^2 \approx 4.9348. \end{aligned}$$

Back to Exercise 7.5 3

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \sin(x), \qquad 0 \le x \le \pi.$$

Solution 8.7.36

$$V_x = \pi \int_0^\pi \sin^2(x)\,dx = \pi \int_0^\pi \frac{1-\cos(2x)}{2}dx = \frac{\pi}{2}\int_0^\pi (1-\cos(2x))\ dx =$$

$$= \frac{\pi}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^\pi = \frac{\pi}{2}\left(\pi - \frac{\sin(2\pi)}{2} - 0\right) = \frac{1}{2}\pi^2 \approx 4.9348.$$

Back to Exercise 7.5 4

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = 1 - x^2, \qquad |x| \le 1.$$

Solution 8.7.37

$$V_x = \pi \int_{-1}^{+1} \left(1-x^2\right)^2 dx = \pi \int_{-1}^{+1} x^4 - 2x^2 + 1\,dx = \pi\left[\frac{x^5}{5} - 2\frac{x^3}{3} + x\right]_{-1}^{+1} =$$

$$= \pi\left(\frac{1^5}{5} - 2\cdot\frac{1^3}{3} + 1\right) - \pi\left(\frac{(-1)^5}{5} - 2\cdot\frac{(-1)^3}{3} + (-1)\right) = \frac{16}{15}\pi \approx 3.35103.$$

Back to Exercise 7.5 5

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \cos(x), \qquad 0 \le x \le \pi.$$

Solution 8.7.38

$$V_x = \pi \int_0^\pi \cos^2(x)\,dx = \pi \int_0^\pi \frac{1+\cos(2x)}{2}dx = \frac{\pi}{2}\int_0^\pi (1+\cos(2x))\ dx =$$

$$= \frac{\pi}{2}\left[x + \frac{\sin(2x)}{2}\right]_0^\pi = \frac{\pi}{2}\left(\pi + \frac{\sin(2\pi)}{2} - 0\right) = \frac{1}{2}\pi^2 \approx 4.9348.$$

Back to Exercise 7.5 6

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \frac{1}{\cos(x)} \quad , \qquad 0 \leq x \leq \pi/4.$$

Solution 8.7.39

$$V_x = \pi \int_0^{\pi/4} \frac{1}{\cos^2(x)} dx = \pi \left[\tan(x)\right]_0^{\pi/4} = \pi \left(\tan\left(\frac{\pi}{4}\right) - 0\right) =$$

$$= \pi \approx 3.14159.$$

Back to Exercise 7.5 7

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \sqrt{r^2 - x^2}, \qquad 0 \leq x \leq r. \quad \text{(semicircle)}$$

Solution 8.7.40

$$V_x = \pi \int_{-r}^{+r} \left(r^2 - x^2\right) dx = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{+r} =$$

$$= \pi \left(r^2 \cdot r - \frac{r^3}{3}\right) - \pi \left(r^2 \cdot (-r) - \frac{(-r)^3}{3}\right) = \frac{4}{3}\pi r^3 .$$

This is the well known volume of the ***sphere****.*

Back to Exercise 7.5 8

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \sqrt{(5x+1) \cdot e^x}, \qquad 0 \leq x \leq 1.$$

Solution 8.7.41 *First we compute the primitive function, using integration by parts*

$$\begin{aligned}\int f^2(x)\,dx &= \int (5x+1)\cdot e^x dx = e^x\cdot(5x+1) - \int e^x\cdot 5dx = \\ &= e^x\cdot(5x+1) - 5\cdot e^x = e^x\cdot(5x-4)+C,\end{aligned}$$

then the volume is

$$\begin{aligned}V_x &= \pi\cdot\left[e^x\cdot(5x-4)\ \right]_0^1 = \pi\cdot\left[e^1\cdot(5\cdot 1-4) - e^0\cdot(5\cdot 0-4)\right] = \\ &= \pi\,(e+4)\approx 21.1061.\end{aligned}$$

Back to Exercise 7.5 9

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \sqrt{(x+1)\cdot\ln(x)}, \qquad 1\le x\le e.$$

Solution 8.7.42 *First we compute the primitive function, using integration by parts*

$$\begin{aligned}\int f^2(x)\,dx &= \int (x+1)\cdot\ln(x)\ dx = \\ &= \left(\frac{x^2}{2}+x\right)\ln(x) - \int\left(\frac{x^2}{2}+x\right)\cdot\frac{1}{x}\,dx = \\ &= \left(\frac{x^2}{2}+x\right)\ln(x) - \int\left(\frac{x^2}{2}+x\right)\cdot\frac{1}{x}\,dx = \\ &= \left(\frac{x^2}{2}+x\right)\ln(x) - \int\left(\frac{x}{2}+1\right)\ dx = \\ &= \left(\frac{x^2}{2}+x\right)\ln(x) - \left(\frac{x^2}{4}+x\right)+C.\end{aligned}$$

So the volume is

$$\begin{aligned}V_x &= \pi\cdot\left[\left(\frac{x^2}{2}+x\right)\ln(x) - \left(\frac{x^2}{4}+x\right)\right]_1^e = \\ &= \pi\cdot\left[\left(\frac{e^2}{2}+e\right)\ln(e) - \left(\frac{e^2}{4}+e\right) - \left(\frac{1^2}{2}+1\right)\ln(1) + \left(\frac{1^2}{4}+1\right)\right] = \\ &= \pi\cdot\frac{e^2+5}{4}\approx 9.73034.\end{aligned}$$

Back to Exercise 7.5 10

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \sqrt[3]{x}, \qquad 0 \leq x \leq 1.$$

Solution 8.7.43

$$\begin{aligned} V_x &= \pi \int_0^1 \left(\sqrt[3]{x}\right)^2 dx = \pi \int_0^1 x^{2/3} dx = \pi \left[\frac{x^{2/3+1}}{\frac{2}{3}+1}\right]_0^1 = \\ &= \frac{3}{5}\pi \cdot \left(1^{5/3} - 0\right) = \frac{3}{5}\pi \approx 1.884\,95. \end{aligned}$$

Back to Exercise 7.5 11

Step-by-Step Solution

Calculate the volume of the solid over the given interval, when f is revolved around the x -axis.

$$f(x) = \sqrt{\sin(x)}, \qquad 0 \leq x \leq \pi/2.$$

Solution 8.7.44

$$\begin{aligned} V_x &= \pi \int_0^{\pi/2} \left(\sqrt{\sin x}\right)^2 dx = \pi \int_0^{\pi/2} \sin x \, dx = \pi \cdot [-\cos x]_0^{\pi/2} = \\ &= \pi \cdot \left(-\cos \tfrac{\pi}{2} + \cos 0\right) = \pi \approx 3.14159 \end{aligned}$$

Back to Exercise 7.5 12

Step-by-Step Solution

Calculate the volume of the barrel, i.e. when the ellipse below is revolved around the x -axis.

$$\left(\frac{x}{80}\right)^2 + \left(\frac{y}{50}\right)^2 = 1$$

Solution 8.7.45 *Equality*

$$\left(\frac{x}{80}\right)^2 + \left(\frac{y}{50}\right)^2 = 1$$

can be transformed into the functions

$$y = f_{1,2}(x) = \pm 50 \cdot \sqrt{1 - \left(\frac{x}{80}\right)^2},$$

so the volume of the barrel is

$$\begin{aligned} V_x &= \pi \int_{-60}^{+60} 50^2 \left(1 - \left(\frac{x}{80}\right)^2\right)\, dx = 50^2\pi \int_{-60}^{+60} \left(1 - \frac{x^2}{80^2}\right)\, dx = \\ &= 50^2\pi \left[x - \frac{x^3}{3 \cdot 80^2}\right]_{-60}^{+60} = 50^2\pi \cdot 2 \cdot \left(60 - \frac{60^3}{3 \cdot 80^2}\right) = \\ &= 50^2 \cdot 2 \cdot 60 \cdot \pi \cdot \frac{13}{16} \approx 765763.21 cm^3 \approx 765.76\ \ell itres\ . \end{aligned}$$

Back to Exercise 7.5 13

8.7.3 *Improper Integrals over Infinite Interval*

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{-\infty}^{-4} \frac{x+1}{x^2+2x-3}dx.$$

Solution 8.7.46 *Using Definition 7.3 first, we rewrite the improper integral as a limit. That is*

$$\int_{-\infty}^{-4} \frac{x+1}{x^2+2x-3}dx = \lim_{\omega \to -\infty} \left(\int_{\omega}^{-4} \frac{x+1}{x^2+2x-3}dx \right).$$

Next, we evaluate the indedinite integral, that is

$$\int \frac{x+1}{x^2+2x-3}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x-3}dx = \frac{1}{2}\ln\left|x^2+2x-3\right| + C.$$

Now, we evaluate the definite integral. From Theorem 7.1 with $F(x) = \frac{1}{2}\ln\left|x^2+2x-3\right|$*, we get*

$$\begin{aligned}
\int_{\omega}^{-4} \frac{x+1}{x^2+2x-3}dx &= \left[\frac{1}{2}\ln\left|x^2+2x-3\right|\right]_{\omega}^{-4} = \\
&= \frac{1}{2}\left(\ln\left|(-4)^2+2\cdot(-4)-3\right| - \ln\left|(\omega)^2+2\cdot(\omega)-3\right|\right) = \\
&= \frac{1}{2}\left(\ln 5 - \ln\left|(\omega)^2+2\cdot(\omega)-3\right|\right).
\end{aligned}$$

Finally, we evaluate the limit, that is

$$\begin{aligned}
\int_{-\infty}^{-4} \frac{x+1}{x^2+2x-3}dx &= \lim_{\omega\to-\infty} \int_{\omega}^{-4} \frac{x+1}{x^2+2x-3}dx = \\
&= \lim_{\omega\to-\infty} \left(\frac{1}{2}\left(\ln 5 - \ln\left|(\omega)^2+2\cdot(\omega)-3\right|\right)\right) = -\infty.
\end{aligned}$$

So the improper integral is infinite (divergent).

Back to Exercise 7.6 1

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{-\infty}^{-4} \frac{7}{x^2+2x-3} dx.$$

Solution 8.7.47 *As*

$$\frac{7}{x^2+2x-3} = \frac{7}{(x+3)(x-1)} = \frac{7/4}{x-1} - \frac{7/4}{x+3},$$

and

$$\begin{aligned}\int \frac{7}{x^2+2x-3} dx &= \frac{7}{4}\int \frac{1}{x-1} dx - \frac{7}{4}\int \frac{1}{x+3} dx = \\ &= \frac{7}{4}(\ln|x-1| - \ln|x+3|\ ) + C = \frac{7}{4}\ln\left|\frac{x-1}{x+3}\right| + C,\end{aligned}$$

we get

$$\begin{aligned}\int_{-\infty}^{-4} \frac{7}{x^2+2x-3} dx &= \lim_{\omega\to-\infty} \int_{\omega}^{-4} \frac{7}{x^2+2x-3} dx = \\ &= \frac{7}{4}\lim_{\omega\to-\infty}\left[\ln\left|\frac{x-1}{x+3}\right|\right]_{\omega}^{-4} = \\ &= \frac{7}{4}\lim_{\omega\to-\infty}\left(\ln\left|\frac{-4-1}{-4+3}\right| - \ln\left|\frac{\omega-1}{\omega+3}\right|\right) = \\ &= \frac{7}{4}(\ln 5 - \ln 1) = \frac{7}{4}\cdot\ln 5 \approx 2.8165.\end{aligned}$$

See also Excercise 7.9 1.
Back to Exercise 7.6 2

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{\sqrt{2}}^{\infty} \frac{2x-3}{x^2+1} dx.$$

Solution 8.7.48 *As*

$$\int \frac{2x-3}{x^2+1} dx = \int \frac{2x}{x^2+1} dx - \int \frac{3}{x^2+1} dx = \ln\left(x^2+1\right) - 3\arctan(x) + C,$$

we get

$$\int\limits_{\sqrt{2}}^{\infty} \frac{2x-3}{x^2+1}dx = \lim_{\omega\to\infty}\left(\int\limits_{\sqrt{2}}^{\omega} \frac{2x-3}{x^2+1}dx\right) = \lim_{\omega\to\infty}\left[\ln\left(x^2+1\right) - 3\arctan(x)\right]_{\sqrt{2}}^{\omega} =$$

$$= \lim_{\omega\to\infty}\left[\left(\ln\left(\omega^2+1\right) - 3\arctan(\omega)\right)\right] -$$

$$- \lim_{\omega\to\infty}\left[\left(\ln\left(\left(\sqrt{2}\right)^2+1\right) - 3\arctan\left(\sqrt{2}\right)\right)\right] =$$

$$= \infty - 3\cdot\frac{\pi}{2} - \ln(3) + 3\arctan\left(\sqrt{2}\right) = \infty.$$

So the improper integral is infinite (divergent).

Back to Exercise 7.6 3

Step-by-Step Solution

Calculate the following improper integral.

$$\int\limits_{0}^{\infty} e^{-x}\,dx$$

Solution 8.7.49

$$\int\limits_{0}^{\infty} e^{-x}\,dx = \lim_{\omega\to\infty}\left(\int\limits_{0}^{\omega} e^{-x}\,dx\right) = \lim_{\omega\to\infty}\left[\frac{e^{-x}}{-1}\right]_0^{\omega} =$$

$$= -\lim_{\omega\to\infty}\left(e^{-\omega} - e^{-0}\right) = -(0-1) = 1.$$

Back to Exercise 7.6 4

Step-by-Step Solution

Calculate the following improper integral.

$$\int\limits_{0}^{\infty} \frac{2}{e^x+e^{-x}}dx.$$

Solution 8.7.50 *We use Theorem 6.5 and make the substitution*

$$e^x = t.$$

Then

$$x = \ln(t),$$

and

$$\frac{dx}{dt} = \frac{1}{t}.$$

This allows us to change variable from x to t, that is

$$\begin{aligned}\int \frac{2}{e^x + e^{-x}} dx &= \int \frac{2}{t + \frac{1}{t}} \cdot \frac{1}{t} dt = 2 \int \frac{1}{t^2+1} dt = \\ &= 2 \arctan(t) + C = 2 \arctan(e^x) + C.\end{aligned}$$

This follows

$$\begin{aligned}\int_0^{\infty} \frac{2}{e^x + e^{-x}} dx &= \lim_{\omega \to \infty} \int_0^{\omega} \frac{2}{e^x + e^{-x}} dx = 2 \lim_{\omega \to \infty} \left[\arctan(e^x)\right]_0^{\omega} = \\ &= 2\left[\frac{\pi}{2} - \arctan(1)\right] = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \approx 1.5708.\end{aligned}$$

Back to Exercise 7.6 5

Step-by-Step Solution

Calculate the following improper integral.

$$\int_1^{\infty} \frac{1}{x^3} \cdot \exp\left(\frac{-1}{x^2}\right) dx.$$

Solution 8.7.51 *Since*

$$\left(\frac{-1}{x^2}\right)' = \frac{2}{x^3},$$

we have

$$\int \frac{1}{x^3} \cdot \exp\left(\frac{-1}{x^2}\right) dx = \frac{1}{2} \exp\left(\frac{-1}{x^2}\right) + C.$$

This follows

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^3} \cdot \exp\left(\frac{-1}{x^2}\right) dx &= \lim_{\omega \to \infty} \left(\int_1^{\omega} \frac{1}{x^3} \cdot \exp\left(\frac{-1}{x^2}\right) dx\right) = \lim_{\omega \to \infty} \left[\frac{1}{2} \exp\left(\frac{-1}{x^2}\right)\right]_1^{\omega} = \\ &= \frac{1}{2} \lim_{\omega \to \infty} \left[\exp\left(\frac{-1}{\omega^2}\right) - \exp(-1)\right] = \\ &= \frac{1}{2}\left(\exp(0) - \exp(-1)\right) = \frac{1}{2}\left(1 - \frac{1}{e}\right) = \frac{e-1}{2e}.\end{aligned}$$

Back to Exercise 7.6 6

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{1}^{\infty} \frac{dx}{\arctan(x) \cdot (x^2+1)}.$$

Solution 8.7.52 *Since*

$$\arctan'(x) = \frac{1}{x^2+1},$$

we have

$$\int \frac{dx}{\arctan(x) \cdot (x^2+1)} = \int \frac{1}{\arctan(x)} \cdot \frac{1}{x^2+1}\, dx = \ln|\arctan(x)| + C.$$

This follows

$$\begin{aligned}\int_{1}^{\infty} \frac{dx}{\arctan(x) \cdot (x^2+1)} &= \lim_{\omega\to\infty} \int_{1}^{\omega} \frac{dx}{\arctan(x) \cdot (x^2+1)} = \lim_{\omega\to\infty} \left[\ln|\arctan(x)|\right]_1^{\omega} = \\ &= \lim_{\omega\to\infty} \left[\ln|\arctan(\omega)| - \ln|\arctan(1)|\right] = \\ &= \ln\frac{\pi}{2} - \ln\frac{\pi}{4} = \ln 2 \approx 0.69315.\end{aligned}$$

Back to Exercise 7.6 7

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{-\infty}^{7} \frac{1}{x^2+2x+10}\, dx.$$

Solution 8.7.53 *As*

$$\begin{aligned}\int \frac{1}{x^2+2x+10} dx &= \int \frac{1}{(x+1)^2+9} dx = \\ &= \frac{1}{9}\int \frac{1}{\frac{(x+1)^2}{9}+1} dx = \frac{1}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2+1} dx = \\ &= \frac{1}{9} \cdot \frac{\arctan\left(\frac{x+1}{3}\right)}{1/3} + C = \frac{3}{9} \cdot \arctan\left(\frac{x+1}{3}\right) + C,\end{aligned}$$

we have

$$\int_{-\infty}^{7} \frac{1}{x^2+2x+10}\,dx = \lim_{\omega\to-\infty}\int_{\omega}^{7} \frac{1}{x^2+2x+10}\,dx = \frac{1}{3}\lim_{\omega\to-\infty}\left[\arctan\left(\frac{x+1}{3}\right)\right]_{\omega}^{7} =$$

$$= \frac{1}{3}\lim_{\omega\to-\infty}\left[\arctan\left(\frac{8}{3}\right) - \arctan\left(\frac{\omega+1}{3}\right)\right] =$$

$$= \frac{1}{3}\left(\arctan\left(\frac{8}{3}\right) - \left(\frac{-\pi}{2}\right)\right) =$$

$$= \frac{1}{3}\arctan\left(\frac{8}{3}\right) + \frac{\pi}{6} \approx 0.92761.$$

Back to Exercise 7.6 8

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{0}^{\infty} \frac{1}{x^2+4x+6}\,dx.$$

Solution 8.7.54 *As*

$$\int \frac{1}{x^2+4x+6}\,dx = \int \frac{1}{(x+2)^2+2}\,dx = \frac{1}{2}\int \frac{1}{\frac{(x+2)^2}{2}+1}\,dx =$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{x+2}{\sqrt{2}}\right)^2+1}\,dx = \frac{1}{2}\frac{\arctan\left(\frac{x+2}{\sqrt{2}}\right)}{1/\sqrt{2}} + C =$$

$$= \frac{\sqrt{2}}{2}\cdot\arctan\left(\frac{x+2}{\sqrt{2}}\right) + C,$$

we have

$$\int_{0}^{\infty} \frac{1}{x^2+4x+6}\,dx = \lim_{\omega\to\infty}\int_{0}^{\omega} \frac{1}{x^2+4x+6}\,dx = \frac{\sqrt{2}}{2}\lim_{\omega\to\infty}\left[\arctan\left(\frac{x+2}{\sqrt{2}}\right)\right]_{0}^{\omega} =$$

$$= \frac{\sqrt{2}}{2}\lim_{\omega\to\infty}\left(\arctan\left(\frac{\omega+2}{\sqrt{2}}\right) - \arctan\left(\frac{2}{\sqrt{2}}\right)\right) =$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\pi}{2} - \arctan\left(\sqrt{2}\right)\right) \approx 0.43521.$$

Back to Exercise 7.6 9

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4x+8}\,dx,$$

Solution 8.7.55 *As*

$$\begin{aligned}
\int \frac{1}{x^2+4x+8}\,dx &= \int \frac{1}{(x+2)^2+4}\,dx = \frac{1}{4}\int \frac{1}{\frac{(x+2)^2}{4}+1}\,dx = \\
&= \frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx = \frac{1}{4}\frac{\arctan\left(\frac{x+2}{2}\right)}{1/2} + C = \\
&= \frac{2}{4}\arctan\left(\frac{x+2}{2}\right) + C,
\end{aligned}$$

we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^2+4x+8}\,dx &= \int_{-\infty}^{a} \frac{1}{x^2+4x+8}\,dx + \int_{a}^{\infty} \frac{1}{x^2+4x+8}\,dx = \\
&= \lim_{\omega\to-\infty}\int_{\omega}^{a} \frac{1}{x^2+4x+8}\,dx + \lim_{\omega\to\infty}\int_{a}^{\omega} \frac{1}{x^2+4x+8}\,dx = \\
&= \frac{1}{2}\lim_{\omega\to-\infty}\left[\arctan\left(\frac{x+2}{2}\right)\right]_{\omega}^{a} + \frac{1}{2}\lim_{\omega\to\infty}\left[\arctan\left(\frac{x+2}{2}\right)\right]_{a}^{\omega} = \\
&= \frac{1}{2}\lim_{\omega\to-\infty}\left[\arctan\left(\frac{a+2}{2}\right) - \arctan\left(\frac{\omega+2}{2}\right)\right] + \\
&+ \frac{1}{2}\lim_{\omega\to\infty}\left[\arctan\left(\frac{\omega+2}{2}\right) - \arctan\left(\frac{a+2}{2}\right)\right] = \\
&= \frac{1}{2}\arctan\left(\frac{a+2}{2}\right) - \frac{1}{2}\cdot\frac{-\pi}{2} + \\
&+ \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{a+2}{2}\right) = \frac{\pi}{2} \approx 1.5708,
\end{aligned}$$

where $a \in \mathbb{R}$ is any fixed number.

Back to Exercise 7.7 1

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+5},$$

Solution 8.7.56 *First we evaluate the indefinite integral, that is*

$$\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx = \frac{1}{4}\cdot\int \frac{1}{\frac{(x+1)^2}{4}+1}\,dx =$$

$$= \frac{1}{4}\cdot\frac{\arctan\left(\frac{x+1}{2}\right)}{\frac{1}{2}} + C = \frac{1}{2}\cdot\arctan\left(\frac{x+1}{2}\right) + C.$$

From this, we have (choosing $c=3$ in Theorem 7.7.1)

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+5}\,dx = \int_{-\infty}^{3} \frac{1}{x^2+2x+5}\,dx + \int_{3}^{+\infty} \frac{1}{x^2+2x+5}\,dx =$$

$$= \lim_{\omega\to-\infty}\int_{\omega}^{3} \frac{1}{x^2+2x+5}\,dx + \lim_{\omega\to\infty}\int_{3}^{\omega} \frac{1}{x^2+2x+5}\,dx =$$

$$= \frac{1}{2}\lim_{\omega\to-\infty}\left[\arctan\left(\frac{x+1}{2}\right)\right]_{\omega}^{3} + \frac{1}{2}\lim_{\omega\to\infty}\left[\arctan\left(\frac{x+1}{2}\right)\right]_{3}^{\omega} =$$

$$= \frac{1}{2}\lim_{\omega\to-\infty}\left[\arctan\left(\frac{3+1}{2}\right) - \arctan\left(\frac{\omega+1}{2}\right)\right] +$$

$$+ \frac{1}{2}\lim_{\omega\to\infty}\left[\arctan\left(\frac{\omega+1}{2}\right) - \arctan\left(\frac{3+1}{2}\right)\right] =$$

$$= \frac{1}{2}\left(\arctan(2) - \left(\frac{-\pi}{2}\right)\right) + \frac{1}{2}\left(\frac{\pi}{2} - \arctan(2)\right) = \frac{\pi}{2}.$$

Back to Exercise 7.7 2

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{-\infty}^{\infty} \frac{\arctan(x)}{x^2+1} dx,$$

Solution 8.7.57 *Observe, that*

$$\frac{\arctan(x)}{x^2+1} = \arctan(x) \cdot \arctan'(x),$$

so

$$\int \frac{\arctan(x)}{x^2+1} dx = \frac{\arctan^2(x)}{2} + C.$$

Choosing $c = 3$ *in Theorem 7.7.1, we obtain*

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\arctan(x)}{x^2+1} dx &= \int_{-\infty}^{3} \frac{\arctan(x)}{x^2+1} dx + \int_{3}^{+\infty} \frac{\arctan(x)}{x^2+1} dx = \\
&= \lim_{\omega \to -\infty} \int_{\omega}^{3} \frac{\arctan(x)}{x^2+1} dx + \lim_{\omega \to \infty} \int_{3}^{\omega} \frac{\arctan(x)}{x^2+1} dx = \\
&= \frac{1}{2} \lim_{\omega \to -\infty} \left[\arctan^2(x)\right]_{\omega}^{3} + \frac{1}{2} \lim_{\omega \to \infty} \left[\arctan^2(x)\right]_{3}^{\omega} = \\
&= \frac{1}{2}\left[\arctan^2(3) - \left(\frac{-\pi}{2}\right)^2\right] + \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \arctan^2(3)\right] = 0.
\end{aligned}$$

Back to Exercise 7.7 3

Step-by-Step Solution

Calculate the following improper integral.

$$\int_{-\infty}^{+\infty} \frac{8a^3}{x^2+4a^2} dx.$$

Solution 8.7.58 *First we evaluate the indefinite integral, that is*

$$\begin{aligned}
\int \frac{8a^3}{x^2+4a^2} dx &= \frac{8a^3}{4a^2} \int \frac{1}{\left(\frac{x}{2a}\right)^2+1} dx = \frac{8a^3}{4a^2} \cdot \frac{\arctan\left(\frac{x}{2a}\right)}{1/(2a)} + C = \\
&= \frac{8a^3}{4a^2} \cdot 2a \cdot \arctan\left(\frac{x}{2a}\right) + C = 4a^2 \cdot \arctan\left(\frac{x}{2a}\right) + C.
\end{aligned}$$

For each $a \in \mathbb{R}$

$$f_a(x) = \frac{8a^3}{x^2 + 4a^2}$$

is even (symmetric to the y axis, i.e. $f(-x) = f(x)$ *for all* $x \in \mathbb{R}$*). So choosing* $c = 0$ *in Theorem 7.7.1, we obtain*

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{8a^3}{x^2+4a^2}dx &= 2 \cdot \int_{0}^{+\infty} \frac{8a^3}{x^2+4a^2}dx = 2 \cdot \lim_{\omega\to\infty} \int_{0}^{\omega} \frac{8a^3}{x^2+4a^2}dx = \\ &= 2 \cdot 4a^2 \cdot \lim_{\omega\to\infty} \left[\arctan\left(\frac{x}{2a}\right)\right]_0^{\omega} = \\ &= 8a^2 \cdot \lim_{\omega\to\infty} \left[\arctan\left(\frac{\omega}{2a}\right) - \arctan(0)\right] = \\ &= 8a^2 \cdot \left(\frac{\pi}{2} - 0\right) = 4\pi a^2.\end{aligned}$$

Back to Exercise 7.7 4

8.7.4 *Integrating Discontinuous Functions*

Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^1 \frac{2x}{1-x^2}\, dx.$$

Solution 8.7.59 *First, we evaluate the indefinite integral, that is*

$$\int \frac{2x}{1-x^2}dx = -\int \frac{-2x}{1-x^2} = -\ln\left|1-x^2\right| + C.$$

As function

$$f(x) = \frac{2x}{1-x^2}$$

is continuous on all intervals $[a,b] \subsetneqq (-1,1)$*, i.e. for* $-1 < a < b < 1$*, the value of the improper integral is*

$$\begin{aligned}\int_0^1 \frac{2x}{1-x^2}dx &= \lim_{\omega\to 1-} \int_0^{\omega} \frac{2x}{1-x^2}dx = \lim_{\omega\to 1-} \left[-\ln\left|1-x^2\right|\right]_0^{\omega} \\ &= \lim_{\omega\to 1-} \left[-\ln\left|1-\omega^2\right| + \ln|1|\right] = -(-\infty) + 0 = \infty.\end{aligned}$$

Back to Exercise 7.8 1

Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^1 \frac{e^x+1}{e^{2x}-1}dx.$$

Solution 8.7.60 *We use Theorem 6.5 and make the substitution*

$$e^x = t.$$

Then

$$x = \ln(t),$$

and

$$\frac{dx}{dt} = \frac{1}{t}.$$

This allows us to change variable from x to t, that is

$$\begin{aligned}\int \frac{e^x+1}{e^{2x}-1}dx &= \int \frac{t+1}{t^2-1}\cdot\frac{1}{t}dt = \int \frac{t+1}{(t-1)(t+1)}\cdot\frac{1}{t}dt = \int \frac{1}{(t-1)\,t}d = \\ &= \int \frac{1}{t-1}-\frac{1}{t}dt = \ln|t-1| - \ln|t| + C = \ln\left|\frac{t-1}{t}\right| + C = \\ &= \ln\left|1-\frac{1}{t}\right| + C = \ln\left|1-\frac{1}{e^x}\right| + C.\end{aligned}$$

As function

$$f(x) = \frac{e^x+1}{e^{2x}-1}$$

is defined ("meaningful", i.e. can be computed) and continuous for all $0 < x \le 1$, we have

$$\begin{aligned}\int_0^1 \frac{e^x+1}{e^{2x}-1}dx &= \lim_{\omega\to 0+}\int_\omega^1 \frac{e^x+1}{e^{2x}-1}dx = \lim_{\omega\to 0+}\left[\ln\left|1-\frac{1}{e^x}\right|\right]_{x=\omega}^1 \\ &= \lim_{\omega\to 0+}\left[\ln\left|1-\frac{1}{e}\right| - \ln\left|1-\frac{1}{e^\omega}\right|\right] = \left[\ln\left(1-\frac{1}{e}\right) - (-\infty)\right] = \infty,\end{aligned}$$

i.e. the improper integral is divergent.

Back to Exercise 7.8 2

Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^1 \frac{\ln(x)}{2\sqrt{x}}\,dx.$$

Solution 8.7.61 *We use Theorem 6.4 to evaluate the indefinite integral, that is*

$$\begin{aligned}\int \frac{\ln(x)}{2\sqrt{x}}dx &= \int \frac{1}{2\sqrt{x}}\cdot\ln(x)\;dx = \sqrt{x}\cdot\ln(x) - \int \sqrt{x}\cdot\frac{1}{x}\,dx = \\ &= \sqrt{x}\cdot\ln(x) - 2\int\frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\cdot\ln(x) - 2\sqrt{x} + C = \\ &= \sqrt{x}\cdot(\ln(x)-2)+C.\end{aligned}$$

As function

$$f(x) = \frac{\ln(x)}{2\sqrt{x}}$$

is continuous for $0 < x \le 1$, *we get*

$$\begin{aligned}\int_0^1 \frac{\ln(x)}{2\sqrt{x}}dx &= \lim_{\omega\to 0+}\int_\omega^1 \frac{\ln(x)}{2\sqrt{x}}dx = \lim_{\omega\to 0+}\left[\sqrt{x}\cdot(\ln(x)-2)\,\right]_\omega^1 = \\ &= \lim_{\omega\to 0+}\left[\sqrt{1}\cdot(\ln(1)-2) - \sqrt{\omega}\cdot(\ln(\omega)-2)\right] = \\ &= -2 - \lim_{\omega\to 0+}\sqrt{\omega}\cdot\ln(\omega) + 0.\end{aligned}$$

For this latter limit, we use L'Hospital's rule, that is

$$\begin{aligned}\lim_{\omega\to 0+}\sqrt{\omega}\cdot\ln(\omega) &= \lim_{\omega\to 0+}\frac{\ln(\omega)}{\omega^{-1/2}} = \lim_{\omega\to 0+}\frac{\frac{1}{\omega}}{\frac{-1}{2}\omega^{-3/2}} = \\ &= 2\lim_{\omega\to 0+}\omega^{-1+3/2} = -2\lim_{\omega\to 0+}\sqrt{\omega} = 0;\end{aligned}$$

i.e. the final result is

$$\int_0^1 \frac{\ln(x)}{2\sqrt{x}}dx = -2+0 = -2.$$

Back to Exercise 7.8 3

Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^1 \frac{\ln(x)}{x^3}\,dx.$$

Solution 8.7.62 *We use Theorem 6.4 to evaluate the indefinite integral, that is*

$$\begin{aligned}\int \frac{\ln(x)}{x^3}\,dx &= \int x^{-3}\cdot\ln(x)\,\,dx = \frac{x^{-2}}{-2}\cdot\ln(x) - \int \frac{x^{-2}}{-2}\cdot\frac{1}{x}\,dx = \\ &= \frac{x^{-2}}{-2}\cdot\ln(x) + \frac{1}{2}\int x^{-3}\,dx = \frac{x^{-2}}{-2}\cdot\ln(x) + \frac{1}{2}\cdot\frac{x^{-2}}{-2} + C = \\ &= = \frac{x^{-2}}{-2}\cdot\left(\ln(x) + \frac{1}{2}\right) + C.\end{aligned}$$

As function

$$f(x) = \frac{\ln(x)}{x^3}$$

is continuous for $0 < x \leq 1$*, we get*

$$\begin{aligned}\int_0^1 \frac{\ln(x)}{x^3}\,dx &= \lim_{\omega\to 0+}\int_\omega^1 \frac{\ln(x)}{x^3}\,dx = \lim_{\omega\to 0+}\left[\frac{x^{-2}}{-2}\cdot\left(\ln(x)+\frac{1}{2}\right)\right]_\omega^1 = \\ &= \lim_{\omega\to 0+}\left[\frac{1^{-2}}{-2}\cdot\left(\ln(1)+\frac{1}{2}\right) - \frac{\omega^{-2}}{-2}\cdot\left(\ln(\omega)+\frac{1}{2}\right)\right] = \\ &= \frac{-1}{4} + \lim_{\omega\to 0+}\left[-\frac{\omega^{-2}}{-2}\cdot\left(\ln(1)+\frac{1}{2}\right) - \frac{\omega^{-2}}{-2}\cdot\left(\ln(\omega)+\frac{1}{2}\right)\right].\end{aligned}$$

Since

$$\lim_{\omega\to 0+}\omega^{-2} = +\infty$$

and

$$\lim_{\omega\to 0+}\ln(\omega) = -\infty,$$

the final answer is

$$\int_0^1 \frac{\ln(x)}{x^3}\,dx = -\infty.$$

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Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^1 \frac{\ln(x)}{\sqrt[3]{x}}\,dx.$$

Solution 8.7.63 *We use Theorem 6.4 to evaluate the indefinite integral, that is*

$$\begin{aligned}\int \frac{\ln(x)}{\sqrt[3]{x}}\,dx &= \int x^{-1/3}\cdot\ln(x)\;dx = \frac{3}{2}x^{2/3}\cdot\ln(x) - \int \frac{3}{2}x^{2/3}\cdot\frac{1}{x}\,dx = \\ &= \frac{3}{2}x^{2/3}\cdot\ln(x) - \frac{3}{2}\int x^{-1/3}\,dx = \frac{3}{2}x^{2/3}\cdot\ln(x) - \frac{3}{2}\cdot\frac{3}{2}x^{2/3} + C = \\ &= \frac{3}{2}x^{2/3}\cdot\left(\ln(x) - \frac{3}{2}\right) + C.\end{aligned}$$

As function

$$f(x) = \frac{\ln(x)}{\sqrt[3]{x}}$$

is continuous for $0 < x \leq 1$, *we get*

$$\begin{aligned}\int_0^1 \frac{\ln(x)}{\sqrt[3]{x}}\,dx &= \lim_{\omega\to 0+}\int_\omega^1 \frac{\ln(x)}{\sqrt[3]{x}}\,dx = \frac{3}{2}\lim_{\omega\to 0+}\left[x^{2/3}\cdot\left(\ln(x) - \frac{3}{2}\right)\right]_\omega^1 = \\ &= \lim_{\omega\to 0+}\left[\sqrt{1}\cdot(\ln(1) - 2) - \sqrt{\omega}\cdot(\ln(\omega) - 2)\right] = \\ &= \frac{3}{2}\lim_{\omega\to 0+}\left[1^{2/3}\cdot\left(\ln(1) - \frac{3}{2}\right) - \omega^{2/3}\cdot\left(\ln(\omega) - \frac{3}{2}\right)\right] = \\ &= -\left(\frac{3}{2}\right)^2 + \lim_{\omega\to 0+}\left(\omega^{2/3}\cdot\ln(\omega)\right) + 0.\end{aligned}$$

For this latter limit, we use L'Hospital's rule, that is

$$\begin{aligned}\lim_{\omega\to 0+}\omega^{2/3}\cdot\ln(\omega) &= \lim_{\omega\to 0+}\frac{\ln(\omega)}{\omega^{-2/3}} = \lim_{\omega\to 0+}\frac{\frac{1}{\omega}}{\omega^{-5/3}} = \frac{-2}{3}\lim_{\omega\to 0+}\omega^{5/3-1} = \\ &= \frac{-2}{3}\lim_{\omega\to 0+}\omega^{2/3} = 0;\end{aligned}$$

i.e. the final result is

$$\int_0^1 \frac{\ln(x)}{\sqrt[3]{x}}\,dx = -\left(\frac{3}{2}\right)^2 + 0 = -\frac{9}{4}.$$

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Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^1 \frac{1}{x \cdot \ln^2(x)}\, dx.$$

Solution 8.7.64 *Observe, that*

$$\frac{1}{x} = (\ln(x))',$$

so

$$\int \frac{1}{x \cdot \ln^2(x)}\, dx = \int \frac{1}{x} \cdot \ln^{-2}(x)\ dx = \frac{(\ln(x))^{-1}}{-1} + C = \frac{-1}{\ln(x)} + C.$$

Function

$$f(x) = \frac{1}{x \cdot \ln^2(x)}$$

is continuous for all x such that $0 < x < 1$, however this means that we have to calculate limits at both ends *of the interval $(0, 1)$, so we have to cut this interval at an intermediate place c (e.g. $c = \frac{1}{2}$).*

So

$$\int_0^1 \frac{1}{x \cdot \ln^2(x)}\, dx = \int_0^c \frac{1}{x \cdot \ln^2(x)}\, dx + \int_c^1 \frac{1}{x \cdot \ln^2(x)}\, dx.$$

$$\int_0^c \frac{1}{x \cdot \ln^2(x)}\, dx = \lim_{\omega \to 0+} \int_\omega^c \frac{1}{x \cdot \ln^2(x)}\, dx = \lim_{\omega \to 0+} \left[\frac{-1}{\ln(x)}\right]_\omega^c =$$

$$= \lim_{\omega \to 0+} \left[\frac{-1}{\ln(c)} - \frac{-1}{\ln(\omega)}\right] = \frac{-1}{\ln(c)} - 0 = \frac{-1}{\ln(c)}.$$

$$\int_c^1 \frac{1}{x \cdot \ln^2(x)}\, dx = \lim_{\omega \to 1-} \int_c^\omega \frac{1}{x \cdot \ln^2(x)}\, dx = \lim_{\omega \to 1-} \left[\frac{-1}{\ln(x)}\right]_c^\omega =$$

$$= \lim_{\omega \to 1-} \left[\frac{-1}{\ln(\omega)} - \frac{-1}{\ln(c)}\right] = +\infty.$$

This means, that the improper integral

$$\int_0^1 \frac{1}{x \cdot \ln^2(x)}\, dx = \frac{-1}{\ln(c)} + \infty = \infty$$

is divergent.

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Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^1 \frac{1}{\sqrt{x}-1}\,dx.$$

Solution 8.7.65 *First, we evaluate the indefinite integral. For this, we use the substitution*

$$\begin{aligned}\sqrt{x} &= t,\\ x &= t^2,\\ \frac{dx}{dt} &= 2t.\end{aligned}$$

This follows

$$\begin{aligned}\int \frac{1}{\sqrt{x}-1}\,dx &= \int \frac{1}{t-1}\cdot 2t\,dt = 2\int \frac{t-1+1}{t-1}\,dt = 2\int \left(1+\frac{1}{t-1}\right)dt =\\ &= = 2\left(t+\ln|t-1|\right)+C = 2\left(\sqrt{x}+\ln\left|\sqrt{x}-1\right|\right)+C.\end{aligned}$$

As function

$$f(x) = \frac{1}{\sqrt{x}-1}$$

is continuous for all $0 \le x < 1$*, we have to calculate*

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}-1}\,dx &= \lim_{\omega\to 1-}\int_0^\omega \frac{1}{\sqrt{x}-1}\,dx = 2\lim_{\omega\to 1-}\left[\sqrt{x}+\ln\left|\sqrt{x}-1\right|\,\right]_0^\omega =\\ &= 2\lim_{\omega\to 1-}\left[\left(\sqrt{\omega}+\ln\left|\sqrt{\omega}-1\right|\right)-\left(\sqrt{0}+\ln\left|\sqrt{0}-1\right|\right)\,\right] =\\ &= 2\cdot\left[\left(\sqrt{1}-\infty\right)-(0+0)\right] = -\infty,\end{aligned}$$

which means that the improper integral is divergent.

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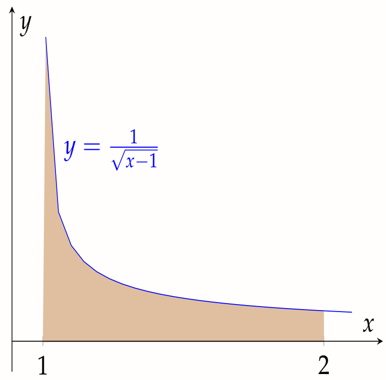
Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx.$$

Solution 8.7.66 *First, we evaluate the indefinite integral.*

$$\int \frac{1}{\sqrt{x-1}}\,dx = 2\int \frac{1}{2\sqrt{x-1}}\,dx = 2\sqrt{x-1}+C.$$



As function

$$f(x) = \frac{1}{\sqrt{x-1}}$$

is continuous for all $1 < x \le 2$*, we have to calculate*

$$\begin{aligned}\int_1^2 \frac{1}{\sqrt{x-1}}\,dx &= \lim_{\omega\to 1+}\int_\omega^2 \frac{1}{\sqrt{x-1}}\,dx = 2\lim_{\omega\to 1+}\left[\sqrt{x-1}\right]_\omega^2 = \\ &= 2\lim_{\omega\to 1+}\left[\sqrt{2-1}-\sqrt{\omega-1}\right] = 2\cdot 1 = 2.\end{aligned}$$

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Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_0^9 \frac{1}{\sqrt{x}\cdot(x-9)}\,dx.$$

Solution 8.7.67 *First, we evaluate the indefinite integral. For this, we use the substitu-*

tion

$$\begin{aligned}\sqrt{x} &= t,\\ x &= t^2,\\ \frac{dx}{dt} &= 2t.\end{aligned}$$

This follows

$$\begin{aligned}\int \frac{1}{\sqrt{x}\cdot(x-9)}\,dx &= \int \frac{2t}{t\cdot(t^2-9)}\,dt = 2\int \frac{1}{t^2-9}\,dt =\\ &= 2\int\left(\frac{1/6}{t-3}-\frac{1/6}{t+3}\right)dt = \frac{2}{6}\left(\ln|t-3|-\ln|t+3|\ \right)+C =\\ &= \frac{2}{6}\left(\ln\left|\sqrt{x}-3\right|-\ln\left|\sqrt{x}+3\right|\ \right)+C.\end{aligned}$$

Function

$$f(x)=\frac{1}{\sqrt{x}\cdot(x-9)}$$

$0<x<9$, *however this means that we have to calculate limits* at both ends *of the interval* $(0,9)$, *so we have to cut this interval at an intermediate place c (e.g.* $c=\frac{9}{2}$*).*

$$\int_0^9 \frac{1}{\sqrt{x}\cdot(x-9)}\,dx = \int_0^c \frac{1}{\sqrt{x}\cdot(x-9)}\,dx + \int_c^9 \frac{1}{\sqrt{x}\cdot(x-9)}\,dx.$$

So

$$\begin{aligned}\int_0^c \frac{1}{\sqrt{x}\cdot(x-9)}\,dx &= \lim_{\omega\to0+}\int_\omega^c \frac{1}{\sqrt{x}\cdot(x-9)}\,dx =\\ &= \frac{2}{6}\lim_{\omega\to0+}\left[\ln\left|\sqrt{x}-3\right|-\ln\left|\sqrt{x}+3\right|\ \right]_\omega^c =\\ &= \frac{2}{6}\lim_{\omega\to0+}\left(\ln\left|\sqrt{c}-3\right|-\ln\left|\sqrt{c}+3\right|\right)-\\ &- \frac{2}{6}\lim_{\omega\to0+}\left(\ln\left|\sqrt{\omega}-3\right|-\ln\left|\sqrt{\omega}+3\right|\right)=\\ &= \frac{2}{6}\left[\left(\ln\left|\sqrt{c}-3\right|-\ln\left|\sqrt{c}+3\right|\right)-(\ln 3-\ln 3)\right]=\\ &= \frac{2}{6}\left(\ln\left|\sqrt{c}-3\right|-\ln\left|\sqrt{c}+3\right|\right),\end{aligned}$$

and

$$\int_c^9 \frac{1}{\sqrt{x}\cdot(x-9)}\,dx = \lim_{\omega\to 9-}\int_c^\omega \frac{1}{\sqrt{x}\cdot(x-9)}\,dx =$$

$$= \frac{2}{6}\lim_{\omega\to 9-}\left[\ln|\sqrt{x}-3| - \ln|\sqrt{x}+3|\,\right]_c^\omega =$$

$$= \frac{2}{6}\lim_{\omega\to 9-}\left(\ln|\sqrt{\omega}-3| - \ln|\sqrt{\omega}+3|\right) -$$

$$- \frac{2}{6}\lim_{\omega\to 9-}\left(\ln|\sqrt{c}-3| - \ln|\sqrt{c}+3|\right) =$$

$$= \frac{2}{6}\left[(-\infty - \ln 6) - \left(\ln|\sqrt{c}-3| - \ln|\sqrt{c}+3|\right)\right] =$$

$$= -\infty.$$

This means, that the improper integral

$$\int_0^9 \frac{1}{\sqrt{x}\cdot(x-9)}\,dx = \frac{2}{6}\left(\ln|\sqrt{c}-3| - \ln|\sqrt{c}+3|\right) - \infty = -\infty$$

is divergent.

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Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_{-2}^0 \frac{-x^2+x-3}{(x^2+5)(x+2)}dx.$$

Solution 8.7.68 *As*

$$\frac{-x^2+x-3}{(x^2+5)(x+2)} = \frac{1}{x^2+5} - \frac{1}{x+2},$$

we get

$$\int \frac{-x^2+x-3}{(x^2+5)(x+2)}dx = \int \left(\frac{1}{x^2+5} - \frac{1}{x+2}\right)\,dx.$$

First we evaluate the indefinite integrals. Since

$$\int \frac{1}{x^2+5}dx = \frac{1}{5}\int \frac{1}{\left(\frac{x}{\sqrt{5}}\right)^2+1}dx = \frac{1}{5}\frac{\arctan\left(x/\sqrt{5}\right)}{1/\sqrt{5}} + C = \frac{\sqrt{5}}{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + C,$$

so

$$\int \left(\frac{1}{x^2+5} - \frac{1}{x+2} \right) dx = \frac{\sqrt{5}}{5} \arctan\left(\frac{x}{\sqrt{5}} \right) - \ln|x+2| + C.$$

As function

$$f(x) = \frac{-x^2+x-3}{(x^2+5)(x+2)}$$

is continuous for $-2 < x \leq 0$*, we get*

$$\begin{aligned}
\int_{-2}^{0} \frac{-x^2+x-3}{(x^2+5)(x+2)} dx &= \lim_{\omega \to -2+} \int_{\omega}^{0} \frac{-x^2+x-3}{(x^2+5)(x+2)} dx = \\
&= \lim_{\omega \to -2+} \left[\frac{\sqrt{5}}{5} \arctan\left(\frac{x}{\sqrt{5}} \right) - \ln|x+2| \right]_{\omega}^{0} = \\
&= \lim_{\omega \to -2+} \left(\frac{\sqrt{5}}{5} \arctan\left(\frac{0}{\sqrt{5}} \right) - \ln|0+2| \right) - \\
&- \lim_{\omega \to -2+} \left(\frac{\sqrt{5}}{5} \arctan\left(\frac{\omega}{\sqrt{5}} \right) - \ln|\omega+2| \right) = \\
&= (0 - \ln 2) - \left(\frac{\sqrt{5}}{5} \arctan\left(\frac{-2}{\sqrt{5}} \right) - (-\infty) \right) = -\infty.
\end{aligned}$$

So the improper integral is divergent.

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Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_{-1}^{0} \frac{x^2-x+1}{(x^2+2)(x+1)} dx.$$

Solution 8.7.69 *Similarly to the previous exercise*

$$\frac{x^2-x+1}{(x^2+2)(x+1)} = \frac{1}{x+1} - \frac{1}{x^2+2},$$

and

$$\int \frac{x^2-x+1}{(x^2+2)(x+1)} dx = \int \left(\frac{1}{x+1} - \frac{1}{x^2+2} \right) dx.$$

This follows

$$\int \left(\frac{1}{x+1} - \frac{1}{x^2+2}\right) dx = \ln|x+1| - \frac{\sqrt{5}}{5} \arctan\left(\frac{x}{\sqrt{5}}\right) + C.$$

As function

$$f(x) = \frac{x^2 - x + 1}{(x^2+2)(x+1)}$$

is continuous for $-1 < x \leq 0$, *we get*

$$\int_{-1}^{0} \frac{x^2 - x + 1}{(x^2+2)(x+1)} dx = \lim_{\omega \to -1+} \int_{\omega}^{0} \frac{x^2 - x + 1}{(x^2+2)(x+1)} dx =$$

$$= \lim_{\omega \to -1+} \left[\ln|x+1| - \frac{\sqrt{5}}{5} \arctan\left(\frac{x}{\sqrt{5}}\right)\right]_{\omega}^{0} =$$

$$= \lim_{\omega \to -1+} \left(\ln|0+1| - \frac{\sqrt{5}}{5} \arctan\left(\frac{0}{\sqrt{5}}\right)\right) -$$

$$- \lim_{\omega \to -1+} \left(\ln|\omega+1| - \frac{\sqrt{5}}{5} \arctan\left(\frac{\omega}{\sqrt{5}}\right)\right) =$$

$$= (0 - 0) - \left(-\infty - \frac{\sqrt{5}}{5} \arctan\left(\frac{-1}{\sqrt{5}}\right)\right) = \infty.$$

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Step-by-Step Solution

Calculate the following improper integral. Find all the points in the interval where the function is discontinuous.

$$\int_{0}^{5} \frac{3x}{x^2 - x - 2} dx.$$

Solution 8.7.70 *As*

$$x^2 - x - 2 = 0 \iff x_1 = -1 \text{ and } x_2 = 2,$$

so

$$\frac{3x}{x^2 - x - 2} = \frac{3x}{(x+1)(x-2)} = \frac{1}{x+1} + \frac{2}{x-2}.$$

This follows

$$\int \frac{3x}{x^2 - x - 2} dx = \int \frac{1}{x+1} + \frac{2}{x-2} dx = \ln|x+1| + 2\ln|x-2| + C.$$

However,

$$f(x) = \frac{3x}{x^2 - x - 2}$$

does not exist at x_2*, which is an internal point of interval* $[0, 3]$.
Since $f(x)$ *is continuous for all other points in* $[0, 3]$*, the improper integral is*

$$\int_0^5 \frac{3x}{x^2 - x - 2}\, dx = \int_0^2 \frac{3x}{x^2 - x - 2}\, dx + \int_2^5 \frac{3x}{x^2 - x - 2}\, dx,$$

where

$$\begin{aligned}
\int_0^2 \frac{3x}{x^2 - x - 2} dx &= \lim_{\omega \to 2-} \int_0^\omega \frac{3x}{x^2 - x - 2} dx = \lim_{\omega \to 2-} \left[\ln|x+1| + 2\ln|x-2|\ \right]_0^\omega = \\
&= \lim_{\omega \to 2-} \left[(\ln|\omega+1| + 2\ln|\omega-2|) - (\ln|0+1| + 2\ln|0-2|)\right] = \\
&= \left[(\ln(3) - \infty) - (0 + 2\ln(2))\right] = -\infty,
\end{aligned}$$

and (though the following, second term is useles to calculate since the first is divergent)

$$\begin{aligned}
\int_2^5 \frac{3x}{x^2 - x - 2}\, dx &= \lim_{\omega \to 2+} \int_\omega^5 \frac{3x}{x^2 - x - 2} dx = \lim_{\omega \to 2+} \left[\ln|x+1| + 2\ln|x-2|\ \right]_\omega^5 = \\
&= \lim_{\omega \to 2+} \left[(\ln|5+1| + 2\ln|5-2|) - (\ln|\omega+1| + 2\ln|\omega-2|)\right] = \\
&= \left[(\ln(6) + 2\ln(3)) - (\ln(3) - \infty)\right] = \infty,
\end{aligned}$$

Since both improper integrals are divergent, the original integral

$$\int_0^5 \frac{3x}{x^2 - x - 2}\, dx$$

is ("twice") divergent.

Remark 8.7.1 *Let us highlight that* $-\infty + \infty \neq 0$ *, so, despite to the above calculations, we can* <u>not</u> *write*

$$\int_0^5 \frac{3x}{x^2 - x - 2}\, dx = \int_0^2 \frac{3x}{x^2 - x - 2}\, dx + \int_2^5 \frac{3x}{x^2 - x - 2}\, dx = -\infty + \infty \neq 0 \quad (\textit{ FALSE! })$$

Moreover, if the disturbating point $x_2 = 2$ *is not detected, the automatic Newton-Leibniz*

Rule would give the ***wrong*** *result*

$$\int_0^5 \frac{3x}{x^2-x-2}\,dx = [\ln|x+1| + 2\ln|x-2|\ ]_0^5 =$$

$$= (\ln|5+1| + 2\ln|5-2|) - (\ln|0+1| + 2\ln|0-2|) =$$

$$= \ln(6) + 2\ln(3) - 2\ln(2) = \ln\left(\frac{6\cdot 3^2}{2^2}\right) \qquad \textbf{(FALSE!)}.$$

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Step-by-Step Solution

Calculate the following improper integrals. Be aware of the critical inner points, too!

$$\int_{-\infty}^{0} \frac{7}{x^2+2x-3}dx.$$

Solution 8.7.71 *Recall from Exercise 7.6 2 that*

$$\int \frac{7}{x^2+2x-3}dx = \frac{7}{4}\ln\left|\frac{x-1}{x+3}\right| + C.$$

Observe that

$$x^2+2x-3=0 \Longleftrightarrow x_1=-3 \text{ and } x_2=1,$$

which means that the integral

$$\int_{-\infty}^{0} \frac{7}{x^2+2x-3}dx$$

has problems at $-\infty$ and at $x_1=-3$. So we have to cut this interval at an intermediate place c (e.g. $c=-4$)

$$\int_{-\infty}^{0} \frac{7}{x^2+2x-3}dx = \int_{-\infty}^{c} \frac{7}{x^2+2x-3}dx + \int_{c}^{-3} \frac{7}{x^2+2x-3}dx + \int_{-3}^{0} \frac{7}{x^2+2x-3}dx.$$

$$\int_{-\infty}^{c} \frac{7}{x^2+2x-3}dx = \lim_{\omega\to-\infty}\int_{\omega}^{c} \frac{7}{x^2+2x-3}dx = \frac{7}{4}\lim_{\omega\to-\infty}\left[\ln\left|\frac{x-1}{x+3}\right|\right]_{\omega}^{c}$$

$$= \frac{7}{4}\lim_{\omega\to-\infty}\left(\ln\left|\frac{c-1}{c+3}\right| - \ln\left|\frac{\omega-1}{\omega+3}\right|\right) =$$

$$= \frac{7}{4}\left(\ln\left|\frac{c-1}{c+3}\right| - \ln(1)\right) = \frac{7}{4}\ln\left|\frac{c-1}{c+3}\right|.$$

$$\int_c^{-3} \frac{7}{x^2+2x-3}dx = \lim_{\omega\to -3-} \int_c^{\omega} \frac{7}{x^2+2x-3}dx = \frac{7}{4} \lim_{\omega\to -3-} \left[\ln\left|\frac{x-1}{x+3}\right|\right]_c^{\omega}$$

$$= \frac{7}{4} \lim_{\omega\to -3-} \left(\ln\left|\frac{\omega-1}{\omega+3}\right| - \ln\left|\frac{c-1}{c+3}\right|\right) =$$

$$= \frac{7}{4} \left(\ln "\infty" - \ln\left|\frac{c-1}{c+3}\right|\right) = \infty.$$

$$\int_{-3}^{0} \frac{7}{x^2+2x-3}dx = \lim_{\omega\to -3+} \int_{\omega}^{0} \frac{7}{x^2+2x-3}dx = \frac{7}{4} \lim_{\omega\to -3+} \left[\ln\left|\frac{x-1}{x+3}\right|\right]_{\omega}^{0}$$

$$= \frac{7}{4} \lim_{\omega\to -3+} \left(\ln\left|\frac{0-1}{0+3}\right| - \ln\left|\frac{\omega-1}{\omega+3}\right|\right) =$$

$$= \frac{7}{4} \lim_{\omega\to -3+} \left(\ln\left(\frac{1}{3}\right) - \ln "\infty"\right) = -\infty.$$

That is, the integral

$$\int_{-\infty}^{0} \frac{7}{x^2+2x-3}dx = \frac{7}{4}\ln\left|\frac{c-1}{c+3}\right| + \infty - \infty$$

is ("twice") divergent.

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Step-by-Step Solution

Calculate the following improper integrals. Be aware of the critical inner points, too!

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x}dx.$$

Solution 8.7.72 *Since*

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$$

the primitive function is

$$\int \frac{1}{x^2+x}dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C.$$

As function

$$f(x) = \frac{1}{x^2+x}$$

has discontinuities at the points $x_1 = -1$ and $x_2 = 0$, we we have to cut this interval at two intermediate places (one can choose e.g. $c_1 = -2$, $c_2 = -0.5$ and $c_3 = 3$).

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x}dx = \int_{-\infty}^{c_1} \frac{1}{x^2+x}dx + \int_{c_1}^{-1} \frac{1}{x^2+x}dx + \int_{-1}^{c_2} \frac{1}{x^2+x}dx +$$

$$+ \int_{c_2}^{0} \frac{1}{x^2+x}dx + \int_{0}^{c_3} \frac{1}{x^2+x}dx + \int_{c_3}^{\infty} \frac{1}{x^2+x}dx.$$

In detail,

$$\int_{-\infty}^{c_1} \frac{1}{x^2+x}dx = \lim_{\omega\to-\infty} \left[\ln\left|\frac{x}{x+1}\right|\right]_{\omega}^{c_1} = \lim_{\omega\to-\infty} \left[\ln\left|\frac{c_1}{c_1+1}\right| - \ln\left|\frac{\omega}{\omega+1}\right|\right] =$$

$$= \ln\left|\frac{c_1}{c_1+1}\right| - \ln(1) = \ln\left|\frac{c_1}{c_1+1}\right|.$$

$$\int_{c_1}^{-1} \frac{1}{x^2+x}dx = \lim_{\omega\to-1-} \left[\ln\left|\frac{x}{x+1}\right|\right]_{c_1}^{\omega} = \lim_{\omega\to-1-} \left[\ln\left|\frac{\omega}{\omega+1}\right| - \ln\left|\frac{c_1}{c_1+1}\right|\right] =$$

$$= \ln{''\infty''} - \ln\left|\frac{c_1}{c_1+1}\right| = \infty.$$

From this point we know, that the original integral $\int_{-\infty}^{\infty} \frac{1}{x^2+x}dx$ *is divergent, but we demonstrate the further calculations, too.*

$$\int_{-1}^{c_2} \frac{1}{x^2+x}dx = \lim_{\omega\to-1+} \left[\ln\left|\frac{x}{x+1}\right|\right]_{\omega}^{c_2} = \lim_{\omega\to-1+} \left[\ln\left|\frac{c_2}{c_2+1}\right| - \ln\left|\frac{\omega}{\omega+1}\right|\right] =$$

$$= \ln\left|\frac{c_2}{c_2+1}\right| - \ln{''\infty''} = -\infty.$$

$$\int_{c_2}^{0} \frac{1}{x^2+x}dx = \lim_{\omega\to0-} \left[\ln\left|\frac{x}{x+1}\right|\right]_{c_2}^{\omega} = \lim_{\omega\to0-} \left[\ln\left|\frac{\omega}{\omega+1}\right| - \ln\left|\frac{c_2}{c_2+1}\right|\right] =$$

$$= \ln{''0''} - \ln\left|\frac{c_2}{c_2+1}\right| = -\infty.$$

$$\int_{0}^{c_3} \frac{1}{x^2+x}dx = \lim_{\omega\to0+} \left[\ln\left|\frac{x}{x+1}\right|\right]_{\omega}^{c_3} = \lim_{\omega\to0+} \left[\ln\left|\frac{c_3}{c_3+1}\right| - \ln\left|\frac{\omega}{\omega+1}\right|\right] =$$

$$= \ln\left|\frac{c_3}{c_3+1}\right| - \ln{''0''} = \infty.$$

$$\int_{c_3}^{\infty} \frac{1}{x^2+x}dx = \lim_{\omega\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_{c_3}^{\omega} = \lim_{\omega\to\infty}\left[\ln\left|\frac{\omega}{\omega+1}\right| - \ln\left|\frac{c_3}{c_3+1}\right|\right] =$$

$$= \ln(1) - \ln\left|\frac{c_3}{c_3+1}\right| = -\ln\left|\frac{c_3}{c_3+1}\right|.$$

The final result in one word is: the integral $\int_{-\infty}^{\infty} \frac{1}{x^2+x}dx$ *is ("four times") divergent.*

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