Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 18 (1—2), 1967, pp. 19—23.

ON A CLASS OF SOLUTIONS OF ALGEBRAIC HOMOGENEOUS LINEAR EQUATIONS

By

Á. PETHŐ (Budapest)

(*Presented by* P. TURÁN)

On solving algebraic homogeneous linear equations by Cramer's rule, solutions can automatically be obtained in which the number of zero elements is maximal in a sense [2]—[3]. In the present communication, these so-called „simple" solutions are defined more simply, in a combinatorial manner, and their properties are formulated more generally. The necessity of introducing simple solutions emerged originally in connection with a chemical problem [2].

§ 1. Definition of simple solutions and several criteria for their existence

Let us consider the set of homogeneous linear equations

$$\sum_{j=1}^{n} x_j a_{ij} = 0, \qquad i = 1, 2, \ldots, m. \tag{1}$$

Introducing the column vectors $\mathbf{a}_j = [a_{1j}, \ldots, a_{mj}]^*$ $(j = 1, 2, \ldots, n)$, instead of (1)

$$\sum_{j=1}^{n} x_j \mathbf{a}_j = 0 \tag{2}$$

can be written. Defining the matrix $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ and the column vector $\mathbf{x} = [x_1, \ldots, x_n]^*$, (1) resp. (2) have the form:

$$\mathbf{A}\mathbf{x} = 0. \tag{3}$$

We will assume $\mathbf{A}$ to have no column and no row consisting of pure zero elements.

DEFINITION 1. In the set of the *solutions* $\mathbf{s} = [s_1, \ldots, s_n]^*$ of (3)

(a) the trivial solution should be disregarded, and

(b) two solutions $\mathbf{s}$ and $\lambda\mathbf{s}$, $\lambda \neq 0$ being a real number, should be considered as a single solution.

So the number of the linearly independent solutions of (3) is $n - r$, where $r = \operatorname{rank} \mathbf{A}$.

DEFINITION 2. Let the *non-zero* elements of the solution $\mathbf{s} = [s_1, \ldots, s_n]^*$ be $s_{j_1}, \ldots, s_{j_q}$, where $C = \{j_1, \ldots, j_q\}$ is a combination of the numbers $1, 2, \ldots, n$ taken $q \leq n$ at a time. Then $\mathbf{s}$ is said a *solution over C*. subset

one

REMARK. Consider now a solution $\mathbf{s}=[s_1, \ldots, s_n]^*$ of the set of equations

$$\sum_{j=1}^{n} x_j \mathbf{a}_j = 0, \quad x_{j_{q+1}} = \ldots = x_{j_n} = 0$$

where $\{j_{q+1}, \ldots, j_n\}$ is the complementary set of C in Definition 2. Let it be agreed that in this case one says, for the sake of shortness, $\mathbf{s}$ to be a solution of the equation

$$\sum_{t=1}^{q} x_{j_t} \mathbf{a}_{j_t} = 0. \tag{4}$$

Consequently, if $\mathbf{s}$ is a solution over $C=\{j_1, \ldots, j_q\}$, $\mathbf{s}$ is a solution of (4).

DEFINITION 3. Let $\mathbf{s}$ be a solution over $C=\{j_1, \ldots, j_q\}$. $\mathbf{s}$ will be said *simple* if it is the *only* solution of (4). Under consideration of Definition 1, $\mathbf{s}$ is simple if and only if

$$\operatorname{rank}[\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_q}] = q-1. \tag{5}$$

THEOREM 1. *For the number of the non-zero elements in a simple solution the inequality holds:*

$$2 \leqq q \leqq r+1, \quad r = \operatorname{rank} \mathbf{A}. \tag{6}$$

PROOF. Since the trivial solution of (3) has been disregarded due to Definition 1, no solution with $q=0$ exists. Nor does a solution exist with $q=1$, $\mathbf{A}$ having no column with only zero elements. Thus, for every solution of (3), consequently for the simple ones as well, $2 \leqq q$ holds. — On the other hand, the inequality

$$\operatorname{rank}[\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_q}] \leqq r$$

is always true, hence, owing to (5):

$$q = \operatorname{rank}[\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_q}] + 1 \leqq r+1.$$

Q. e. d. □

DEFINITION 4. Let $\mathbf{s}^1$ be a solution over C^1 and $\mathbf{s}^2$ be a solution over C^2. The solution $\mathbf{s}^1$ is said *better* than $\mathbf{s}^2$ if C^1 is a proper subset of C^2: $C^1 \subset C^2$.

DEFINITION 5. The solution $\mathbf{s}^1$ is said *just as good as* $\mathbf{s}^2$ if they are solutions over the same C.

THEOREM 2. *A solution is simple if and only if there does not exist any better one.*

PROOF. The condition is trivially necessary on the basis of Definition 3. To show that the condition is sufficient we will prove that, *if a solution is not simple, one can always find a better solution.* Let $\mathbf{s}$ be a not simple solution over $C=\{j_1, \ldots, j_q\}$, then according to Definition 3

$$\operatorname{rank}[\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_q}] \leqq q-2.$$

Setting e. g. x_{j_q} in (4) equal to zero, the new equation

$$\sum_{t=1}^{q-1} x_{j_t} \mathbf{a}_{j_t} = 0 \tag{7}$$

KöV simple sz-hoz legkevesebb nem0 eleme van!

becomes such that unchanged

$$\operatorname{rank}[\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_{q-1}}] \leqq q-2.$$

Therefore, (7) will still have a solution $\mathbf{s}'$ over some C', such that $C' \subset C$: so $\mathbf{s}'$ is a better solution. Q. e. d. □

Corollary. The number of the non-zero elements in a simple solution is at least 2 according to (6). Thus, a solution with 2 non-zero elements — if existing — is certainly simple because of the former theorem.

Theorem 3. *A solution is simple if and only if there does not exist any other just as good one.*

Proof. The condition is trivially necessary, on the basis of Definition 3. The sufficiency will be proved in the form that *if a solution is not simple, one can always find another just as good one.* Let $\mathbf{s}$ be a not simple solution over $C = \{j_1, \dots, j_q\}$, then (4) has also another solution, say $\mathbf{s}'$. Let us now form the solution $\mathbf{s} + \varepsilon\mathbf{s}'$, where $\varepsilon > 0$ is a real number. If ε is small enough, the non-zero elements of $\mathbf{s}$ vary hereby only a little, that is, do not become zero. Therefore, $\mathbf{s} + \varepsilon\mathbf{s}'$ will be a solution just as good as $\mathbf{s}$. Q. e. d. □

Theorem 4. *Let a combination $C = \{j_1, \dots, j_q\}$ be given. A simple solution over C exists if and only if the* any *solution of* (4), *say* $\mathbf{s}$, *is unique, moreover*

$$\prod_{t=1}^{q} s_{j_t} \neq 0. \tag{8}$$

This theorem is a trivial consequence of Definitions 2 and 3. □

Theorem 5. *The statement of the previous theorem holds if and only if*

$$\left.\begin{array}{lll}(9) & \operatorname{rank}[\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_q}] = q-1, & \\ (10) & \operatorname{rank}[\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_{t-1}}, \mathbf{a}_{j_{t+1}}, \dots, \mathbf{a}_{j_q}] = q-1, & t = 1, 2, \dots, q.\end{array}\right\}\text{simplex}$$

A system of linearly dependent vectors should be called a *simplex* if, by omitting any of them, the remaining vectors become linearly independent. The statement of Theorem 4 holds consequently if and only if $\{\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_q}\}$ forms a simplex. iff?

Proof. At first we show that, if $\{\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_q}\}$ forms a simplex, the solution of (4) is unique and (8) is also fulfilled. The solution of (4) is unique, the number of the unknowns, q, being by one greater than $\operatorname{rank}[\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_q}] = q-1$ (see (9)). Consider now the (unique) solution of (4): $\mathbf{s} = [s_1, \dots, s_n]^*$. Were any s_{j_t} $(1 \leqq t \leqq q)$ zero here, (4) without the term corresponding to $\mathbf{a}_{j_t}$ would have no solution (see (10) and Definition 1), notwithstanding that $\mathbf{s}$ was the solution of (4). Thus (8) must be true.

Now we show that, if the solution of (4) is unique and (8) also holds, the vectors $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_q}$ form a simplex. Owing to the uniqueness (9) holds. Here, omitting any vector $\mathbf{a}_{j_t}$ $(1 \leqq t \leqq q)$, the remaining ones become linearly independent; otherwise x_{j_t} would namely be uniquely zero in (4),[1] which contradicts (8). Thus (10) holds, too. □

[1] Cf. Appendix.

§ 2. Construction of the simple solutions

In the foregoing it has not yet been mentioned how the simple solutions of (3) can be found. A few theorems with respect to this question will now be proved.

DEFINITION 6. A solution $\mathbf{s}$ of (3) is called a *base solution* if it is a solution of an equation of the form

$$x_{j_1}\mathbf{a}_{j_1}+\ldots+x_{j_r}\mathbf{a}_{j_r}+x_{j_k}\mathbf{a}_{j_k}=0, \tag{11}$$

where $\{\mathbf{a}_{j_1},\ldots,\mathbf{a}_{j_r}\}$ is a basis (i. e. rank $[\mathbf{a}_{j_1},\ldots,\mathbf{a}_{j_r}]=r$) and $k=r+1,\ldots,n$.

THEOREM 6. (11) *determines one and only one solution* $\mathbf{s}$, *for which also* $s_{j_k}\neq 0$.

PROOF. Let us solve (11). The number of its unknowns being equal to $r+1$ and the rank of its matrix $[\mathbf{a}_{j_1},\ldots,\mathbf{a}_{j_r},\mathbf{a}_{j_k}]$ equal to r, its solution is unique, say $\mathbf{s}$ (by virtue of Definition 1). Here, moreover, $s_{j_k}\neq 0$, otherwise $s_{j_1}=\ldots=s_{j_r}=0$ would have to hold because $\{\mathbf{a}_{j_1},\ldots,\mathbf{a}_{j_r}\}$ is a basis: thus (11) would have no solution though $\mathbf{s}$ was one. Q. e. d. $\square$

The base solutions of equation (3) are obtained when solving it by *Cramer's rule*. More exactly there holds the following

THEOREM 7. *Let us solve* (3) *according to Cramer's rule. As known, choosing a basis* $\{\mathbf{a}_{j_1},\ldots,\mathbf{a}_{j_r}\}$, *the general solution becomes*

$$\mathbf{s}=\sum_{k=r+1}^{n} x_{j_k}\mathbf{s}_{j_k}, \tag{12}$$

where the x_{j_k} *are the so-called free variables. Consider now all the general solutions of type* (12) *belonging to the possible bases among* $\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_n$ *and consider the set of the different* $\mathbf{s}_{j_k}$ *in these solutions. As a trivial consequence of Definition* 6 *we may assert that by these* $\mathbf{s}_{j_k}$ *all the base solutions of* (3) *are represented.*

We can now formulate our following fundamental

THEOREM 8. *The simple solutions are identical with the base solutions.*

PROOF. At first we show that the base solutions are simple ones. Consider a base solution, it is, due to Definition 6, a solution of an equation of type (11). Without loss of generality we may assume (11) to be of the following form:

$$x_1\mathbf{a}_1+\ldots+x_r\mathbf{a}_r+x_k\mathbf{a}_k=0, \tag{13}$$

where $\{\mathbf{a}_1,\ldots,\mathbf{a}_r\}$ is a basis. Here, according to Theorem 6, x_k cannot be zero; if, however, any of the unknowns $x_1,\ldots,x_r$ is zero, then it must be uniquely zero. Thus, let the unknowns $x_{j_1},\ldots,x_{j_{q-1}}$ $(2\leqq q\leqq r+1)$ be different from zero, then (13) becomes:

$$x_{j_1}\mathbf{a}_{j_1}+\ldots+x_{j_{q-1}}\mathbf{a}_{j_{q-1}}+x_k\mathbf{a}_k=0, \tag{14}$$

where none of the unknowns can be zero any more. Thus, owing to Theorem 4, the solution of (14), i. e. the base solution considered, will be a simple one over $C=\{j_1,\ldots,j_{q-1},k\}$.

We prove now that a simple solution is a base solution. Consider a simple solution $\mathbf{s}$, without loss of generality assuming it to be of the form $[s_1, \dots, s_q, 0, \dots, 0]^*$. It is, because of Definition 3, the unique solution of the equation

$$x_1 \mathbf{a}_1 + \dots + x_q \mathbf{a}_q = 0. \tag{15}$$

Here, according to Theorem 5, $\{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ constitutes a simplex, where $q-1 \leqq r$ (see (6)). So we may complete the linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_{q-1}$ with $r-(q-1) \geqq 0$ vectors: $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_{r-q+1}}$, the new vector system $\{\mathbf{a}_1, \dots, \mathbf{a}_{q-1}, \mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_{r-q+1}}\}$ becoming hereby a *basis*. Consider now that of the base solutions belonging to this basis which is determined uniquely by the equation (see Definition 6 and Theorem 6):

$$x_1 \mathbf{a}_1 + \dots + x_{q-1} \mathbf{a}_{q-1} + x_{j_1} \mathbf{a}_{j_1} + \dots + x_{j_{r-q+1}} \mathbf{a}_{j_{r-q+1}} + x_q \mathbf{a}_q = 0.$$

This solution is asserted to be $\mathbf{s}$. Namely, on account of the construction, $x_{j_1}, \dots, x_{j_{r-q+1}}$ are identically zero,[2] consequently the equation (15) is left over, whose unique solution is indeed the simple solution $\mathbf{s}$. So $\mathbf{s}$ is a base solution. $\square$

Appendix

The system of homogeneous linear equations (1) has a solution in which the unknown x_{j_1} ($j_1 = 1, 2, \dots, n$) is uniquely (identically) zero if and only if the rank of the matrix of (1) is by one greater than that of the matrix in which the j_1-th column is dropped [1]:

$$\operatorname{rank}[\mathbf{a}_{j_2}, \dots, \mathbf{a}_{j_n}] = \operatorname{rank}[\mathbf{a}_1, \dots, \mathbf{a}_n] - 1 = r - 1.$$

In conclusion, the author wishes to express his indebtedness to Professor P. Turán for his interest in this work.

(Received 13 November 1965)

Literature

[1] Pethő Á., Néhány megjegyzés lineáris algebrai egyenletrendszerek egyértelmű megoldhatóságával kapcsolatban, *Publ. Math. Inst. Hung. Acad. Sci.*, 3 (1958), pp. 101—108.
[2] Á. Pethő, Zur Theorie der Stöchiometrie chemischer Reaktionssysteme, *Wiss. Z. Tech. Hochsch. Chem. Leuna—Merseburg*, 6 (1964), pp. 13—15.
[3] Pethő Á.—Győry K., Homogén lineáris egyenletrendszerek „sok" zérust tartalmazó megoldásairól, *Matematikai Lapok*, 16 (1965), pp. 267—273.

[2] Cf. Appendix.