All Scales x^a on One Slide Rule

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2018.05.05., 15:20'

Abstract

In this article we answer the question "*How is it possible to realize all scales x^α (for each $\alpha\in$R) on one slide rule*?", raised in our previous article[1]. We describe here the general zooming technique with a thoroughful elementary and higher mathematical analysis and explanation, but the Reader is allowed to skip to the final solution in the last section, in case she or he does not want to get familiar with the "soul" of slide rules.

1 The first steps

The principle of any (old or new) slide rule is[1], that we write each value $t\in$R (real number) on a strip at the (geometric) distance $d=f(t)$ from the left hand end of the scale, which we always denote by S_1 , and where $f(t)$ is a strictly monotonic function (increasing or decreasing). A movement of the slide results the geometric distance addition $d_3=d_1+d_2$ which implies

$$f(t_3) = f(t_1) + f(t_2) \,, \qquad (1)$$

assuming that both the stator and the slide contain the same function $f(t)$. Traditional slide rules contain $f(t)=log(t)$ which gives $log(t_3) = log(t_1)+ log(t_2)$ i.e. $t_3=t_1\cdot t_2$ (the multiplication). In our present article $f(t)=t^\alpha$ which results[0]

$$(t_3)^\alpha = (t_1)^\alpha +(t_2)^\alpha \quad i.e. \quad t_3 = \sqrt[\alpha]{t_1^\alpha + t_2^\alpha} \,. \qquad (2)$$

(Our former article[1] contains many applications of (2), and a more general theory of two variable functions on slide rules.)

In the present article we deal with (all) the functions[0] $f(t)=t^\alpha$ for each possible exponent $\alpha\in$R (real number). For $\alpha=1$ we have the ordinary rulers or measuring tapes with an equidistant scale ("inch by inch"), since $d=f(t)=t$. For $\alpha=-1$ and $\alpha=2$ the scales can be seen in our previous article[1] and in the interactive link[3], they are called "reciprocal" and "quadratic" scales[7]. These three examples are different from each other, so the Reader might pick a pencil and paper, and may wish to <u>sketch</u> some more scales, like $t^{1/3} = \sqrt[3]{t}$, $t^{1/2} = \sqrt{t}$, $t^{3/2} = \sqrt{t^3}$, t^3 , etc. Negative exponents are a little bit harder, we have to be more careful when $\alpha<0$. Or, you can use our Javascript program[6].

Now, place these scales on our table. Let they be parallel to the x axis, and put the starting point S_1 of <u>each</u> scale to the y axis at height $y=\alpha$, i.e. at the point $(0,\alpha)$, as it is shown in Figure 1.

Now, follow the <u>orbit</u> or <u>trajectory</u> of a specific value of t, let us say $t=2$. In each scale the mark t has another distance from S_1 (the y axis), this distance is growing when we increase the exponent α. After the scales for <u>all</u> $\alpha\in$R are put together, the orbits of different marked values t form nice curves, as shown in Figure 2. For example, the "orbit" of the value $t=2$ is marked at heigh (on the scales) α = -2, -1.5, -1, +0.5, +1.0, +1.5 and +2.0 in Figure 2. Of course the other exponents α ($\alpha<-2.2$ or $+2.2<\alpha$) could be drawn on this figure, too, depending on the size of the paper. In general it is true, that on a <u>finite</u> paper we can not draw all the exponents $\alpha\in$R, but at least we have drawn <u>infinitely many</u> of them on Figure 2.

Note: do <u>not</u> consider the numbers on the x axis as orbits: these numbers show the <u>geometric</u> distances from the y axis. Look: the scale at $\alpha=1$ is the same as the numbers on the x axis because $d=t^1=t$ for each $t\in$R and $x=d$ on the x axis. The starting point S_1 of <u>each</u> scale is on the y axis, since $d=0^\alpha=0$ for each exponent $\alpha\in$R (excluding $\alpha=0$ since 0^0 is meaningless). Another interesting fact is the point (1;0) of the coordinate system: all orbits go through this point. This fact is clear from the equality $d=t^0=1$ for $t\in$R.

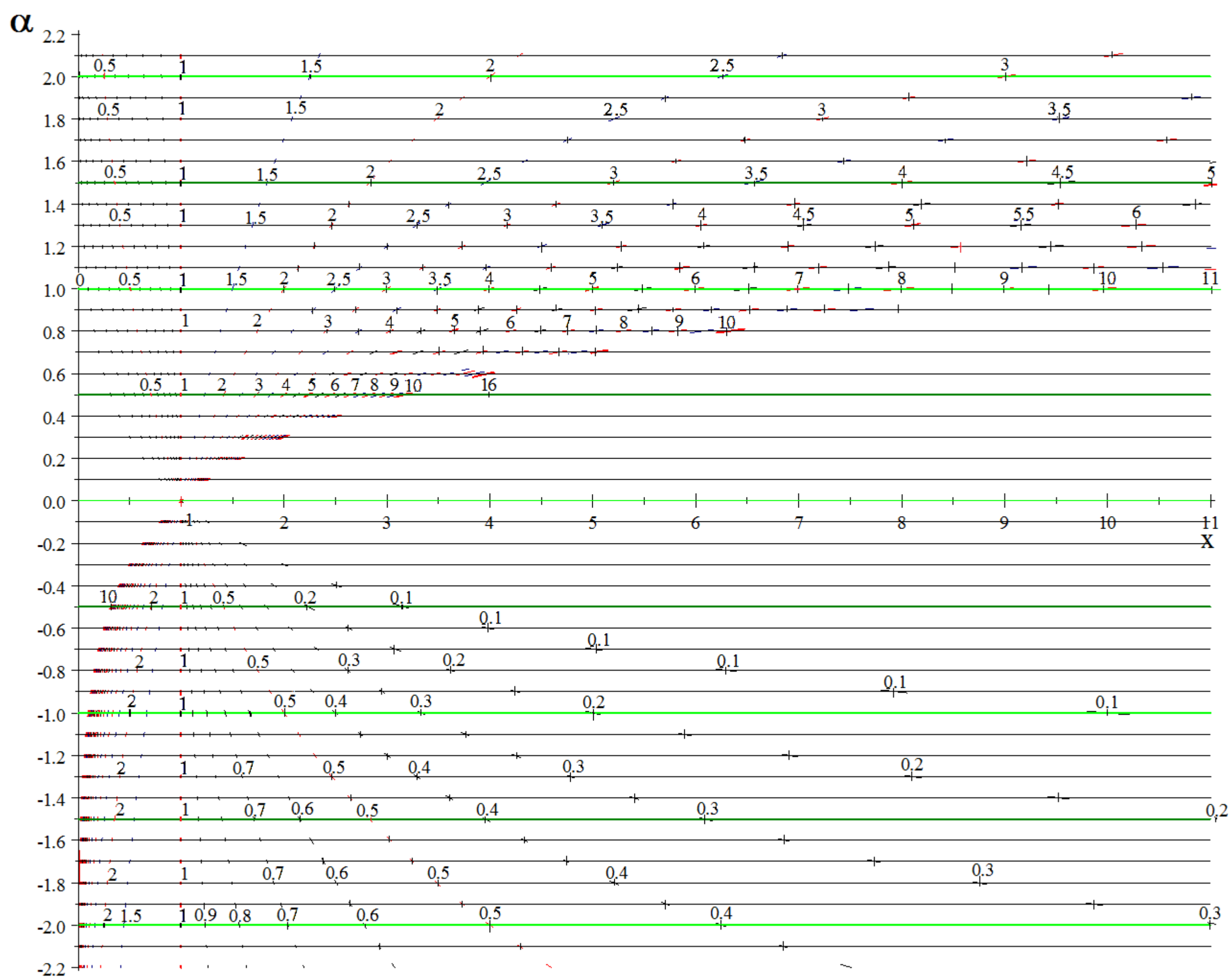


FIGURE 1. Some Scales x^{α} for $-2.2\leq\alpha\leq 2.2$

Now, how to construct and understand these nice orbits on Figure 2 ? They look very similar to the graphs of the functions $y=h_t(x)$ ... hm, let us think ... perhaps logarithms? For understanding this phenomenon, first let us use the letter x instead of d and y instead of α since they are on the axes x and y. Now, the points of the red line, marked by t=2 on Figure 2 satisfy the equality, in general

$$x = 2^{y} \qquad (3)$$

(instead of $d=2^{\alpha}$), and in the usual form (solving for y) we have

$$y = log_2(x) \qquad (4)$$

and similarly for all $t\in$R ($t\neq 1$)

$$y = log_t(x)\ . \qquad (5)$$

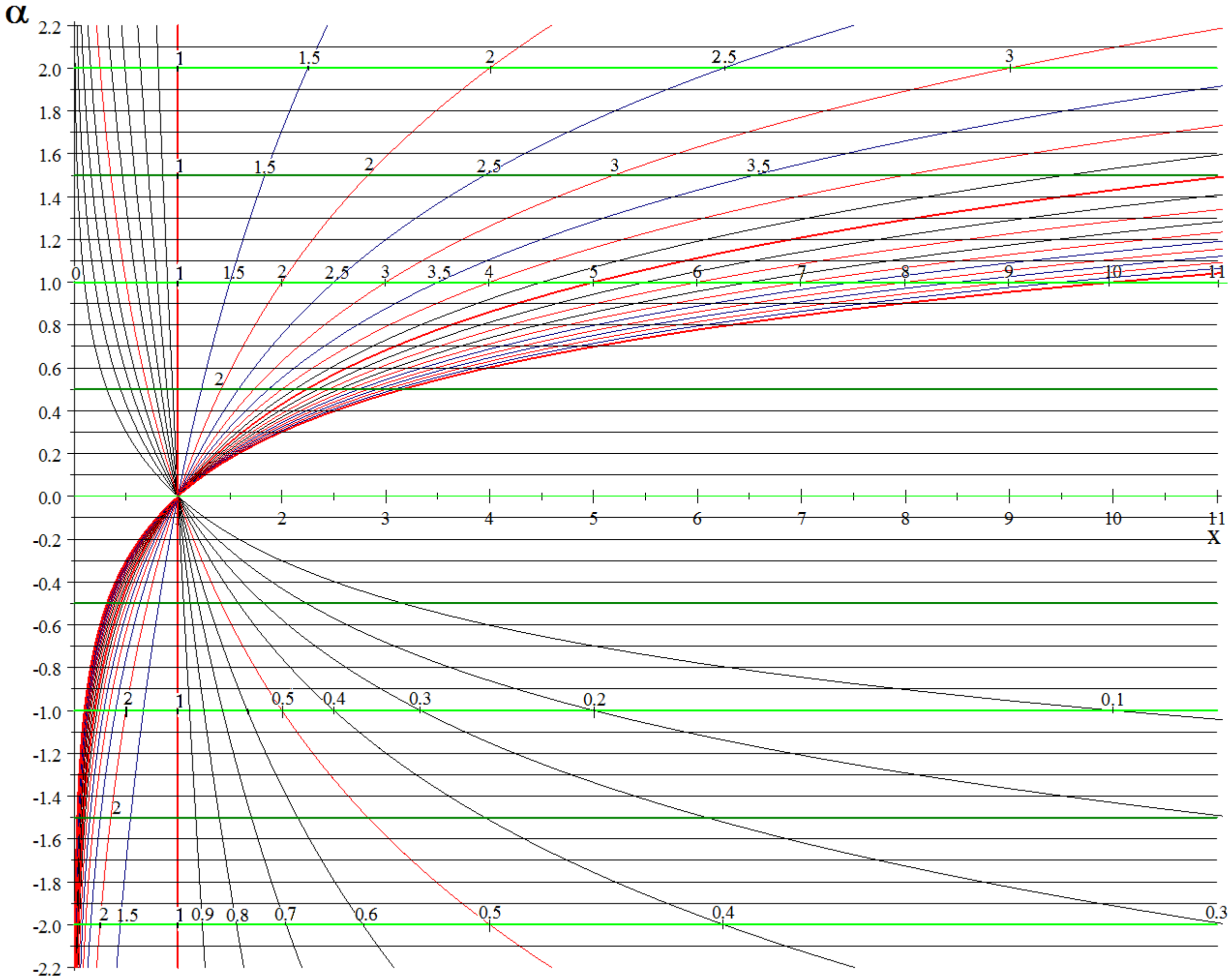


FIGURE 2. All the Scales x^{α} for -2.2≤α≤2.2

Yes, the orbits in Figure 2 are the graphs of the logarithmic functions of different bases! However, intersecting with any horizontal line y=α we get the scale "t^{α}". This also explains, that why are the orbits for negative α reflections of the others for positive α, marked by t and $\frac{1}{t}$: we have $t^{-y} = \left(\frac{1}{t}\right)^{y}$, or, equivalently

$$\log_{\frac{1}{t}}(x) = \frac{\ln(x)}{ln(1/t)} = \frac{\ln(x)}{-ln(t)} = -\log_t(x) \qquad (6)$$

During this long discussion we forget to observe, that our solution has already been done: it can be seen on Figure 2 ! How to use it?

So, each scale t^{α} for any α∈R (-2.2≤α≤+2.2) can be found on Figure 2 if we meet the orbits with the horizontal line y=α . To make a working slide rule, stick one copy of Figure 2 onto the stator and another copy of the same size on the slide. Choose an α and mark the same horizontal lines y=α on both copies of Figure 2, that is, the same scales t^{α} on the stator and the slide. Perhaps two additional horizontal hairlines in the suitable distance would be helpful finding the same scales on the stator and on the slide. Now, use these two scales as usual to compute $t_3^{\alpha} = t_1^{\alpha} + t_2^{\alpha}$, or in other form $t_3 = \sqrt[\alpha]{t_1^{\alpha} + t_2^{\alpha}}$. If you feel the scale on Figure 2 and the resulting slide rule too wide, we may suggest you to stick it onto the surface of two cylinders and to roll them always to pop up above the desired part: corresponding to your α, as it has been realized with Napier's bones[4].

The only matter is, that some scales (for small α) are short, and the numbers on it are too dense, hard to use them with old eyes. Other scales (for large α) contain small numbers only, even medium t values do not fit on them. In the following sections we zoom the scales horizontally in and out for equalizing them. Think on independent rubber strips for each scale t^{α} . Clearly more dense scales will be zoomed by a higher factor. Since we also want to preserve the nice smooth shape of orbits on Figure 2, the zooming factor must be a continuous function of α .

2 Linear transformations

Recall the connection $d=f(t)$ and its role in the equality $f(t_3)=f(t_1)+f(t_2)$ from the first paragraph of the previous section. What happens, if we zoom and translate the scale to

$$d = c\cdot f(t) + b \qquad (7)$$

for some real numbers $c,b\in\mathrm{R}$, $c\neq 0$? Since we move the slide freely, b is unimportant, no difference if $b=0$. (For $b\neq 0$ the scale itself with its starting point S_1 is to be translated to the left or to the right, according to the sign of b.) Next, $d_3=d_1+d_2$ is equivalent to $c\cdot f(t_3)=c\cdot f(t_1)+c\cdot f(t_2)$, i.e. to the original $f(t_3)=f(t_1)+f(t_2)$. Finally: nothing happened! Such modifications of the scale/function $f(t)$ to $c\cdot f(t)+b$ are called <u>linear transformations</u>.

3 Adjusting the scales

As we indicated at the end of Section 1, we want to apply linear transformations for each scales t^α (at height $y=\alpha$). The new scales will be, according to (7)

$$x = c_\alpha \cdot t^\alpha + b_\alpha \qquad (8)$$

since $x=d$ and $f(t)=t^\alpha$. In (8) c_α and b_α are intended to be different for each $\alpha\in\mathrm{R}$. However, for preserving the nice, <u>continuous</u> shape of the orbits t^α on Figure 2, c_α and b_α must be suitable continuous functions of α , but independent of any other variables (as x and t), and we also must have $0<c_\alpha$.

Unlike (3), the equality (8) can not be solved for α, which means, that we can <u>not</u> use the learned explicit form $y=h(x)$ of functions. Instead, we have to switch to <u>parametric plot</u> of form $[x,y]$. For any <u>fixed $t\in\mathrm{R}$</u> the equality (8) means the parametric curve

$$[c_\alpha\cdot t^\alpha+b_\alpha \, , \, \alpha] \, , \quad \alpha\in\mathrm{R} \qquad (9)$$

where $\alpha\in\mathrm{R}$ is the parameter of the curve, running the desired interval, $-2.2\leq\alpha\leq+2.2$ in our case. For each $t\in\mathrm{R}$ we get different curves.

Recall, that we also need c_α be a decreasing function of α, even possibly

$$\lim_{\alpha\to 0} c_\alpha = \infty \quad and \quad \lim_{\alpha\to+\infty} c_\alpha = 0 \, . \qquad (10)$$

and

$$c_\alpha>1 \text{ for } |\alpha|<1 \quad and \quad c_\alpha<1 \text{ for } 1<|\alpha| \qquad (11)$$

In what follows, let us consider the case $0\leq\alpha$ only. (Well, the case $\alpha=0$ will have a further surprise.) Since we want to print <u>all</u> the scales for all α , we are not allowed to plan separate scales for large and small numbers as in our previous article[2].

Let us observe further, that the (left hand) starting point S_1 of our scales remains on the y scale only when $b_\alpha=0$. The S_1 points are important, since we can not use any scale without its own S_1 , marked on the left on the scale, too! However, sticking to $b_\alpha=0$ (equivalently S_1 on the y axis) for any α , only small numbers t will be visible on the scales for small α (e.g. for $\alpha<1$), while for larger α (e.g. $1<\alpha$) we can put only larger numbers t on the scale, by (10) and (11), as it can be seen on Figure 3. Figure 3 has a further speaciality: $c_\alpha = 10^{1-\alpha}$ has been chosen to have the value 10 on all the scales on the <u>same place</u> (the trajectory of 10 is a straight vertical line).

The other possibility is to <u>drop</u> the requirement $b_\alpha=0$, i.e. S_1 on the y axis. For most functions b_α and c_α the orbits may bend in various manner[5], but by suitable b_α and c_α we may fix (any) two trajectories successfully. In Figure 4 we see the functions

$$\left[\frac{t^\alpha-1}{10^\alpha-1}\cdot 9+1\, ,\ \alpha\right], \qquad (12)$$

that is $c_\alpha = \frac{9}{10^\alpha-1}$ and $b_\alpha = 1- c_\alpha = 1-\frac{9}{10^\alpha-1}$. It is a pity that the **brown** trajectory of $0 = S_1$ goes away for $0\leq\alpha\leq{}^1/_2$, so in a short slide rule we can not use these scales. But, on the contrary, for ${}^1/_2\leq\alpha$ we have nice scales. Even, look at the "x axis", the scale for $\alpha=0$: it is "similar" to the logarithmic scales C and D of traditional slide rules! More precisely the scale for $\alpha=0$ is, in fact, a <u>real logarithmic scale</u>, identical to C and D ! This is because of the mathematical theorems

$$\lim_{\alpha\to 0}\frac{t^\alpha-1}{\alpha} = \ln(t) \quad and \quad \lim_{\alpha\to 0}\frac{t^\alpha-1}{10^\alpha-1} = \log_{10}(t) \qquad (13)$$

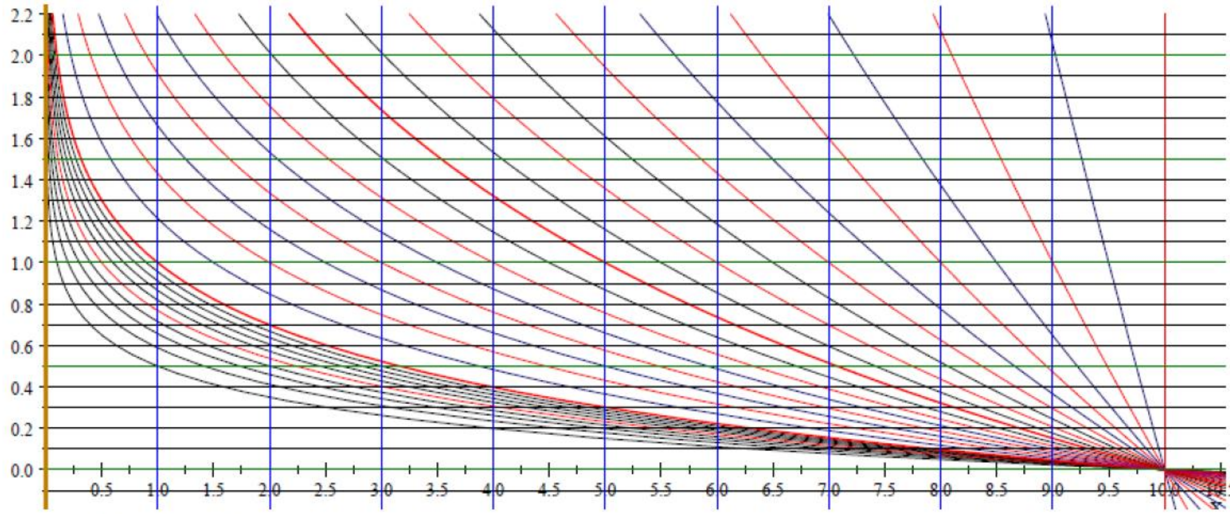
Since for all b_α and c_α the scale for $\alpha=1$ remains an equidistant one, the slide rule on Figure 4 is useful for addition, multiplication (<u>all the four basic operations</u>), and for the new task $t_3 = \sqrt[\alpha]{t_1^\alpha + t_2^\alpha}$.

It is a great advantage to have also multiplication (*log*) scale on our slide rule: for example we can easily calculate the <u>geometric mean of any numbers</u>: first we multiply the numbers $a_1\cdot a_2\cdot\ldots\cdot a_n$, and then the scale $\alpha=1/n$ calculates us the n-th root of this product!

Finally, let us remark, that the requirement "the orbit of 1 remain a straight line " is equivalent to

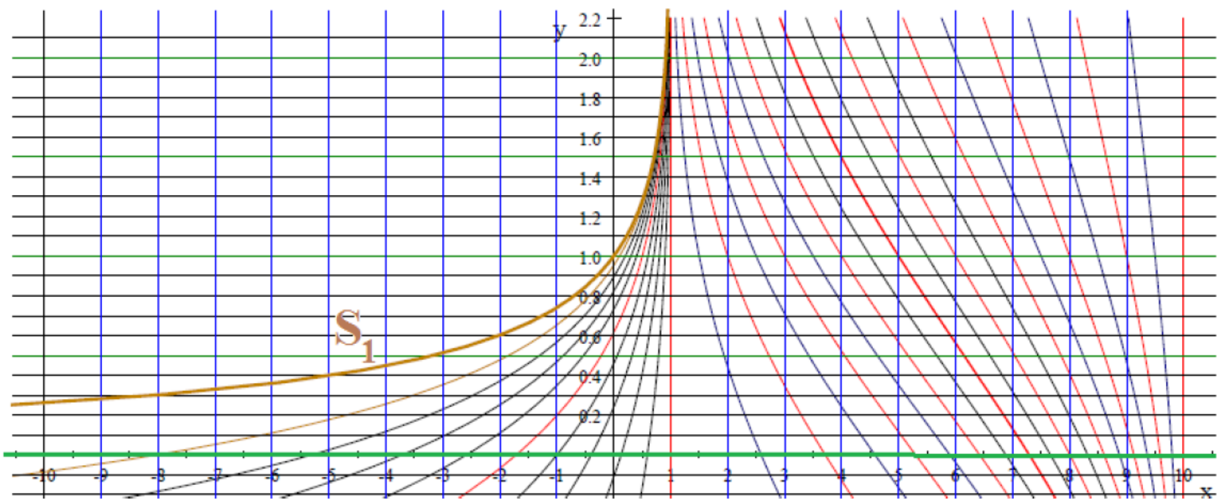
$$b_\alpha = 1- c_\alpha \, . \qquad (14)$$

We have tried out several functions c_α and b_α by theoretical investigations and computer graphic, the Reader is allowed to visit and choose some of our experiments on our webpage[5].



$$\left[z^t \cdot \frac{10}{10^t}\ ,\ t\right] \quad (\mathbf{0} \le z \le 10\ ,\ -2.1 \le t \le 2.1)$$

FIGURE 3. $c_\alpha = 10^{1-\alpha}$, $b_\alpha = 0$



$$\left[\frac{z^t - 1}{t} \cdot \frac{9t}{10^t - 1} + 1\ ,\ t\right] \qquad (0.1 \le z \le 10\ ,\ z = \mathbf{0}\ ,\ -2.1 \le t \le 2.1)$$

FIGURE 4. $c_\alpha = \frac{9}{10^{\alpha}-1}$, $b_\alpha = 1 - c_\alpha$, ### 10-.png, 80% ###

4 Other infinite scale families

Many other scale-families can be drawn on a single slide rule. For example, the distribution functions $F_\lambda(x) = 1\text{-}e^{-\lambda x}$ (see Figure 5) are often used in probability theory. Two copies of the diagram in Figure 5 could be used just as we described in Section 1. Before it some (both theoretical and practical) adjusting are advisable, similar to ones in Sections 2 and 3.

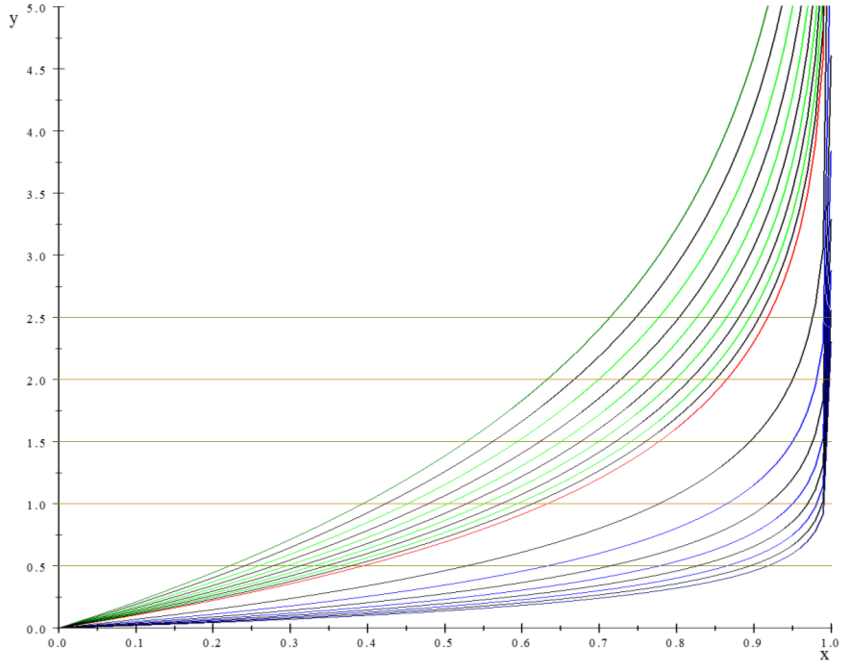


FIGURE 5. The functions $F_\lambda(x) = 1\text{-}e^{-\lambda x}$
OneScal-exp-170524a.png

Notes

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0. Though we talked about "Scales x^α" in the title and in the Abstract, from now we have to change their names to "Scales t^α" because of an other use of the variable x later.

1. Szalkai,I., *General Two-Variable Functions on the Slide Rule*, Journal of the Oughtred Society, 27:1, Spring 2018, pages 14-18.
2. Szalkai,I., Hoffman,A., *Constructing and Understanding New and Old Scales*, Journal of the Oughtred Society, 27:2, Fall 2018.
3. Hoffman,A.,*A Digital Slide Rule*, Journal of the Oughtred Society, 27:1, Spring 2018, pages 19-24.
Animated Digital Slide Rule,
http://www.animatedsoftware.com/elearning/DigitalSlideRule/index.html
http://www.animatedsoftware.com/elearning/DigitalSlideRule/DigitalSlideRule.swf
4. Napier's bones on cylinders: https://en.wikipedia.org/wiki/Napier%27s_bones
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https://upload.wikimedia.org/wikipedia/commons/f/fb/An_18th_century_set_of_Napier%27s_Bones.JPG
5. Szalkai,I., *Examples of different functions* $[c_\alpha \cdot t^\alpha + b_\alpha, \alpha]$,
http://math.uni-pannon.hu/~szalkai/Onescales-examples.pdf
6. Szalkai,I.: *Rubber Band for Scale Experiments*, http://math.uni-pannon.hu/~szalkai/RubberBand.html , in preparation.
7. Editor's comment courtesy Otto van Poelje: Two examples exist of slide rules with the "QuadPlus" (x^2) scales; namely the Dennert & Pape QR41 *Quadratrechner* identified in the ARISTO IM2004 book and the Spanish PY-LO slide rule identified in the IM 2013 Proceedings.