DIFFERENTIABILITY OF SOLUTIONS WITH RESPECT TO PARAMETERS IN DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS

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Contents

Chapter 1

Introduction

1.1 State-dependent delays

The systematic study of differential equations with state-dependent delays (SD-DDEs) started with the work of Driver on the two-body problem of classical electrodynamics in the sixties of the last century [27, 28, 29, 30, 31, 32], and since that it became an active research area. Models with state-dependent delays appear recently in many applications including automatic and remote control, machine cutting, neural networks, population biology, mathematical epidemiology and economics (see, e.g., [1, 2, 9, 10, 18, 19, 33, 35, 36, 37, 64, 65, 66, 69, 87, 88, 91]). For a survey on SD-DDEs we refer to [56], which contains a brief summary of some important applications, general theory and numerical approximation of SD-DDEs, as well as a list of references of about 200 papers on SD-DDEs.

Consider the initial value problem (IVP) associated to a general autonomous functional differential equation

$$\dot{x}(t) = f(x_t), \qquad t \geq 0, \tag{1.1.1}$$

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{1.1.2}$$

Here $r > 0$ is fixed, $f\colon C \to \mathbb{R}^n$, where C is the Banach space of continuous functions $[-r, 0] \to \mathbb{R}^n$ equipped with the supremum norm, $\varphi \in C$, and x_t denotes the segment function defined by

$$x_t\colon [-r, 0] \to \mathbb{R}^n, \qquad x_t(\zeta) := x(t + \zeta).$$

C^1 below will be the space of continuously differentiable functions $\psi\colon [-r, 0] \to \mathbb{R}^n$, where the norm is defined by $|\psi|_{C^1} = \max\{|\psi|_C, |\dot{\psi}|_C\}$.

In (1.1.1) the growth rate of the solution depends on past values of x. The simplest example for this dependence is a linear equation with a single constant delay $\tau \in [0, r]$, i.e., equation

$$\dot{x}(t) = ax(t - \tau).$$

In the case when the delay τ in the previous equation or the selection mechanism of the values of the segment function x_t used in (1.1.1) is not constant, moreover it depends on the segment function x_t itself, we say that in the equation the delay is state-dependent. One of the simplest prototype example of a state-dependent delay equation is the case when f in (1.1.1) has the form $f(\psi) = a\psi(-\tau(\psi(0)))$, and so (1.1.1) reduces to

$$\dot{x}(t) = ax(t - \tau(x(t))). \tag{1.1.3}$$

The form (1.1.1) includes much more general classes of SD-DDEs, see, e.g., [56].

The difficulty in the theory of SD-DDEs can be seen already in the simple SD-DDE (1.1.3): we can't assume even the Lipschitz continuity of the function $f : C \to \mathbb{R}^n$, $f(\psi) = a\psi(-\tau(\psi(0)))$, not even if we assume high order smoothness of the function $\tau : C \to \mathbb{R}$. This makes the basic questions of uniqueness, smooth dependence of the solution on the initial data and other parameters, as well as the principle of linearized stability and other topics interesting and challenging, since the standard methods of the theory of delay equations may not be used, in general, for SD-DDEs (see, e.g., [16, 21, 27, 38, 45, 47, 56, 57, 58, 60, 70, 71, 77, 84, 85, 86, 89, 90]). In particular, C is not suitable as the state-space of solutions in SD-DDEs, but it is not clear what is the best choice to use, especially if we want to have high order smoothness of the solutions on the initial data and on other parameters.

Walter [89, 90] considered the IVP (1.1.1)-(1.1.2), and developed a framework, which is now called frequently as the C^1-framework, where he gave quite general conditions which are satisfied for large classes of SD-DDEs, and which guarantee the existence of a semiflow of continuously differentiable solution operators, the principle of linearized stability, as well as the existence of C^1-smooth local stable and unstable manifolds at hyperbolic stationary points. Using this framework Krisztin showed the existence of C^N-smooth local unstable manifolds and C^1-smooth center manifolds for the semiflow at hyperbolic stationary points [70, 71].

The key assumption of the C^1-framework is that the solutions are restricted to a submanifold of C^1 of codimension n defined by

$$X_f := \{\psi \in C^1 : \dot{\psi}(0) = f(\psi)\}. \tag{1.1.4}$$

In this manuscript we consider two classes of functional differential equations with state-dependent delays. In Chapters 2 and 3 we consider the SD-DDE

$$\dot{x}(t) = f(t, x_t, x(t - \tau(t, x_t, \xi)), \theta), \qquad t \geq 0, \tag{1.1.5}$$

where ξ and θ are parameters in the equation, and the initial condition associated to (1.1.5) is (1.1.2). In Chapter 4 we consider neutral functional differential equations with state-dependent delays (SD-NFDEs) of the form

$$\frac{d}{dt}\Big(x(t) - g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda)\Big) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \theta\Big) \qquad t \geq 0, \tag{1.1.6}$$

where χ and λ are also parameters in the neutral part of the equation. The initial condition associated to (1.1.6) is, again, (1.1.2).

The particular forms of (1.1.5) and (1.1.6) assume that one delay in the retarded and also in the neutral part is time- and state-dependent, and this dependence is described explicitly in (1.1.5) and (1.1.6) by τ and ρ, but we may have other delayed terms in the equation. Here the dependence of f and g on x_t represents all the "non state-dependent" delayed terms, so smooth dependence of f and g on their second variable will be assumed. We note that for simplicity equations (1.1.5) and (1.1.5) contain only one state-dependent term, but all the results can be easily generalized to the case when in the retarded or in the neutral terms there are several state-dependent delays.

In this thesis we use the space of Lipschitz continuous functions $W^{1,\infty}$ (see Section 1.2 for the definition) as the state-space of solutions, and we show existence, uniqueness and continuous dependence of solutions with respect to (wrt) the parameters of the equation for both the SD-DDE (1.1.5) and the SD-NFDE (1.1.6) (see see Sections 2.2 and 4.2, respectively). The main goal of this thesis is to study the differentiability of solutions of (1.1.5) and (1.1.6) wrt the parameters of the IVP. In Chapter 2 we discuss first and second order differentiability of solutions of the SD-DDE (1.1.5) with respect to φ, θ and ξ. In Chapter 3, as an application of the differentiability results, we study a parameter estimation problem associated to (1.1.5), define the quasilinearization method to get approximate solutions, show convergence of the scheme, and give numerical examples to demonstrate the applicability of the method. In Chapter 4 we discuss well-posedness of the IVP associated to the SD-NFDE (1.1.6), and prove a result showing differentiability of the solutions wrt φ, θ, ξ, λ and χ. At the beginning of each chapters a detailed introduction is given to the topic of the chapter.

1.2 Notations and preliminaries

In this section we introduce notations and collect some results will be used throughout this thesis.

$\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive and nonnegative integers, respectively. A fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. C denotes the Banach space of continuous functions $\psi : [-r,0] \to \mathbb{R}^n$ equipped with the norm $|\psi|_C = \max\{|\psi(\zeta)|: \zeta \in [-r,0]\}$. C^1 is the space of continuously differentiable functions $\psi : [-r,0] \to \mathbb{R}^n$ where the norm is defined by $|\psi|_{C^1} = \max\{|\psi|_C, |\dot{\psi}|_C\}$. L^∞ is the space of Lebesgue-measurable functions $\psi\colon [-r,0] \to \mathbb{R}^n$ which are essentially bounded. The norm on L^∞ is denoted by $|\psi|_{L^\infty} = \operatorname{ess\,sup}\{|\psi(\zeta)|: \zeta \in [-r,0]\}$. $W^{1,p}$ denotes the Banach-space of absolutely continuous functions $\psi\colon [-r,0] \to \mathbb{R}^n$ of finite norm defined by

$$|\psi|_{W^{1,p}} := \left(\int_{-r}^{0} |\psi(\zeta)|^p + |\dot{\psi}(\zeta)|^p \, d\zeta \right)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$

$$|\psi|_{W^{1,\infty}} := \max\left\{ |\psi|_C, |\dot{\psi}|_{L^\infty} \right\}.$$

We note that $W^{1,\infty}$ is equal to the space of Lipschitz continuous functions from $[-r, 0]$ to $\mathbb{R}^n$. The subset of $W^{1,\infty}$ consisting of those functions which have absolutely continuous first derivative and essentially bounded second derivative is denoted by $W^{2,\infty}$, where the norm is defined by

$$|\psi|_{W^{2,\infty}} := \max\left\{ |\psi|_C,\ |\dot{\psi}|_C,\ |\ddot{\psi}|_{L^\infty} \right\}.$$

If the domain or the range of the functions is different from $[-r, 0]$ and $\mathbb{R}^n$, respectively, we will use a more detailed notation. E.g., $C(X, Y)$ denotes the space of continuous functions mapping from X to Y. Finally, $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y, where X and Y are normed linear spaces.

An open ball in the normed linear space X centered at a point $x \in X$ with radius δ is denoted by $\mathcal{B}_X(x;\, \delta) := \{y \in Y :\ |x - y| < \delta\}$. The corresponding closed ball is denoted by $\overline{\mathcal{B}}_X(x;\, \delta)$.

Throughout the manuscript $r > 0$ is a fixed constant and $x_t \colon [-r, 0] \to \mathbb{R}^n$, $x_t(\theta) := x(t + \theta)$ is the segment function. To avoid confusion with the notation of the segment function, sequences of functions are denoted using the upper index: x^k.

The derivative of a single variable function $v(t)$ wrt t is denoted by $\dot{v}$. Note that all derivatives we use in this paper are Fréchet derivatives. The partial derivatives of a function $g :\ X_1 \times X_2 \to Y$ wrt the first and second variables will be denoted by $D_1 g$ and $D_2 g$, respectively. The second-order partial derivative wrt its ith and jth variables $(i, j = 1, 2)$ of the function $g :\ X_1 \times X_2 \to Y$ at the point $(x_1, x_2) \in X_1 \times X_2$ is the bounded bilinear operator $A\langle\cdot,\cdot\rangle \colon X_i \times X_j \to Y$, if

$$\lim_{k\to 0} \sup_{h \ne 0} \frac{|D_i g(x_1 + k\delta_{1j}, x_2 + k\delta_{2j})h - D_i g(x_1, x_2)h - A\langle h, k\rangle|_Y}{|h|_{X_i} |k|_{X_1}} = 0, \qquad h \in X_i,\ k \in X_j,$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \ne j$ is the Kronecker-delta. We will use the notation $D_{ij} g(x_1, x_2) = A$. The norm of the bilinear operator $A\langle\cdot,\cdot\rangle :\ X_i \times X_j \to Y$ is defined by

$$|A|_{\mathcal{L}^2(X_i \times X_j, Y)} := \sup\left\{ \frac{|A\langle h, k\rangle|_Y}{|h|_{X_i}|k|_{X_j}} :\ h \in X_i, h \ne 0,\ k \in X_j, k \ne 0 \right\}.$$

In the case when $X_1 = \mathbb{R}$, we simply write $D_1 g(x_1, x_2)$ instead of the more precise notation $D_1 g(x_1, x_2)1$, i.e., here $D_1 g$ denotes the value in Y instead of the linear operator $\mathcal{L}(\mathbb{R}, Y)$. In the case when, let say, $X_2 = \mathbb{R}^n = Y$, then we identify the linear operator $D_2 g(x_1, x_2) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ by an $n \times n$ matrix.

Next we formulate a result which is a simple consequence of the Gronwall's lemma.

Lemma 1.2.1 (see, e.g., [50]) *Suppose* $a > 0$, $b\colon [0,\alpha] \to [0,\infty)$ *and* $u\colon [-r,\alpha] \to \mathbb{R}^n$ *are continuous functions such that* $a \geq |u_0|_C$, *and*

$$|u(t)| \leq a + \int_0^t b(s)|u_s|_C\, ds, \qquad t \in [0,\alpha]. \tag{1.2.1}$$

Then

$$|u(t)| \leq |u_t|_C \leq a e^{\int_0^\alpha b(s)\, ds}, \qquad t \in [0,\alpha]. \tag{1.2.2}$$

The next lemma formalizes a method used frequently in functional inequalities (see, e.g., in [40]) and which will be used in the sequel, as well.

Lemma 1.2.2 ([48]) *Suppose* $h : [0,\alpha] \times [0,\infty)^3 \to [0,\infty)$ *is monotone increasing in all variables, i.e., if* $0 \leq t_i \leq s_i$ *for* $i = 1,2,3,4$, *then* $h(t_1,t_2,t_3,t_4) \leq h(s_1,s_2,s_3,s_4)$; $\eta\colon [0,\alpha] \to [0,r]$ *is such that* $a \leq \eta(t)$ *for* $t \in [0,\alpha]$ *for some* $a > 0$; $u\colon [-r,\alpha] \to [0,\infty)$ *is such that*

$$u(t) \leq h(t, u(t), u(t-\eta(t)), |u_t|_C), \qquad t \in [0,\alpha],$$

and

$$|u_0|_C \leq h(0, u(0), u(-\eta(0)), |u_0|_C).$$

Then

$$v(t) \leq h(t, v(t), v(t-a), v(t)), \qquad t \in [0,\alpha],$$

where $v(t) := \sup\{u(s)\colon\ s \in [-r,t]\}$.

We recall the following results which will be used later.

Lemma 1.2.3 ([40]) *Let* $a > 0$, $b \geq 0$, $r_1 > 0$, $r_2 \geq 0$, $r = \max\{r_1,r_2\}$, *and* $v\colon [0,\alpha] \to [0,\infty)$ *be continuous and nondecreasing. Let* $u\colon [-r,\alpha] \to [0,\infty)$ *be continuous and satisfy the inequality*

$$u(t) \leq v(t) + bu(t-r_1) + a\int_0^t u(s-r_2)\, ds, \qquad t \in [0,\alpha].$$

Then $u(t) \leq d(t)e^{ct}$ *for* $t \in [0,\alpha]$, *where* c *is the unique positive solution of* $cbe^{-cr_1} + ae^{-cr_2} = c$, *and*

$$d(t) := \max\left\{\frac{v(t)}{1-be^{-cr_1}},\ \max_{-r\leq s\leq 0} e^{-cs}u(s)\right\}, \qquad t \in [0,\alpha].$$

Lemma 1.2.4 (see, e.g., [81]) *Suppose that X and Y are normed linear spaces, and U is an open subset of X, and $F : U \to Y$ is differentiable. Let $x, y \in U$ be such that $y + \nu(x - y) \in U$ for $\nu \in [0,1]$. Then*

$$|F(y) - F(x) - F'(x)(y - x)|_Y \le |x - y|_X \sup_{0<\nu<1} |F'(y + \nu(x - y)) - F'(x)|_{\mathcal{L}(X,Y)}.$$

Lemma 1.2.5 *Suppose $\psi \in W^{1,\infty}$. Then*

$$|\psi(b) - \psi(a)| \le |\dot{\psi}|_{L^\infty} |b - a|$$

for every $[a, b] \subset [-r, 0]$.

We recall the following result from [16], which was essential to prove differentiability wrt parameters in SD-DDEs in [21], [50] and [58]. We state the result in a simplified form we need later, it is formulate in a more general form in [16]. Note that the second part of the lemma was stated in [16] under the assumption $|u^k - u|_{W^{1,\infty}([0,\alpha],\mathbb{R})} \to 0$ as $k \to \infty$, but this stronger assumption on the convergence is not needed in the proof. See also the proof of Lemma 4.26 in [44].

Lemma 1.2.6 ([16]) *Let $g \in L^1([c, d], \mathbb{R}^n)$, $\varepsilon > 0$, and $u \in \mathcal{A}(\varepsilon)$, where*

$$\mathcal{A}(\varepsilon) := \{v \in W^{1,\infty}([a, b], [c, d]) \,:\, \dot{v}(s) \ge \varepsilon \text{ for a.e. } s \in [a, b]\}.$$

Then

$$\int_a^b |g(u(s))|\, ds \le \frac{1}{\varepsilon} \int_c^d |g(s)|\, ds. \tag{1.2.3}$$

Moreover, if the sequence $u^k \in \mathcal{A}(\varepsilon)$ is such that $|u^k - u|_{C([a,b],\mathbb{R})} \to 0$ as $k \to \infty$, then

$$\lim_{k\to\infty} \int_a^b \Big| g(u^k(s)) - g(u(s)) \Big|\, ds = 0. \tag{1.2.4}$$

Remark 1.2.7 Changing to the new variable $s = -t$ in the integrals in (1.2.3) and (1.2.4) give easily that the statements of Lemma 1.2.6 hold also in the case when conditions $u, u^k \in \mathcal{A}(\varepsilon)$ are replaced by $-u, -u^k \in \mathcal{A}(\varepsilon)$.

In the next lemma we relax the condition $u \in \mathcal{A}(\varepsilon)$ of the previous lemma.

Lemma 1.2.8 *Suppose $g \in L^\infty([c,d],\mathbb{R})$, and $u\colon [a,b] \to [c,d]$ is an absolutely continuous function, and*

$$\operatorname{ess\,inf}\{\dot u(s)\colon\ s \in [a',b']\} > 0, \qquad \textit{for all}\ \ [a',b'] \subset (a,b). \tag{1.2.5}$$

Then the composite function $g \circ u \in L^\infty([a,b],\mathbb{R})$, and $|g \circ u|_{L^\infty([a,b],\mathbb{R})} \le |g|_{L^\infty([c,d],\mathbb{R})}$.

Proof First note that since u is absolutely continuous, it is a.e. differentiable on $[a,b]$, and condition (1.2.5) yields that u is strictly monotone increasing on $[a,b]$. Let $G := \{v \in [c,d]\colon\ g(v)$ is not defined or $|g(v)| > |g|_{L^\infty([c,d],\mathbb{R})}\}$. Then $meas(G) = 0$. Let $A := \{t \in [a,b]\ :\ g(u(t))$ is not defined or $|g(u(t))| > |g|_{L^\infty([c,d],\mathbb{R})}\}$. Clearly, $A = u^{-1}(G)$. Let $0 < \varepsilon < (b-a)/2$ be fixed. Then let $c' := u(a+\varepsilon)$, $d' := u(b-\varepsilon)$, and let $M := \operatorname{ess\,inf}\{\dot u(s)\colon\ s \in [a+\varepsilon, b-\varepsilon]\}$. Then (1.2.5) yields $M > 0$. Since G is of measure 0, there exist open intervals (c_i,d_i), $i \in \mathbb{N}$ such that

$$G \subset \bigcup_{i=1}^\infty (c_i,d_i) \quad \text{and} \quad \sum_{i=1}^\infty (d_i - c_i) < \varepsilon M.$$

We have

$$A = u^{-1}(G) = u^{-1}\Big(G \cap [c,c']\Big) \cup u^{-1}\Big(G \cap [c',d']\Big) \cup u^{-1}\Big(G \cap [d',d]\Big),$$

and the monotonicity of u yields $u^{-1}\Big(G \cap [c,c']\Big) \subset [a,a+\varepsilon]$, $u^{-1}\Big(G \cap [d',d]\Big) \subset [b-\varepsilon,b]$, and

$$u^{-1}\Big(G \cap [c',d']\Big) \subset u^{-1}\Big([c',d'] \cap \bigcup_{i=1}^\infty [c_i,d_i]\Big) = \bigcup_{i=1}^\infty u^{-1}\Big([c',d'] \cap [c_i,d_i]\Big) = \bigcup_{i=1}^\infty [a_i,b_i],$$

where $a_i := u^{-1}(\max\{c',c_i\})$ and $b_i := u^{-1}(\min\{d',d_i\})$. The definition of M yields

$$d_i - c_i \ge \min\{d',d_i\} - \max\{c',c_i\} = u(b_i) - u(a_i) = \int_{a_i}^{b_i} \dot u(s)\,ds \ge M(b_i - a_i).$$

Therefore $A \subset [a,a+\varepsilon] \cup [b-\varepsilon,b] \cup \bigcup_{i=1}^\infty [a_i,b_i]$, and the sum of the length of the closed intervals covering A is less than 3ε. Since $\varepsilon > 0$ is arbitrary, we get that A is Lebesgue-measurable and $meas(A) = 0$.

We show that $g \circ u$ is Lebesgue-measurable. Let $\kappa \in \mathbb{R}$, and define $G_\kappa := \{v \in [c,d]\colon$ $g(v)$ is defined and $g(v) < \kappa\}$. G_κ is a Lebesgue-measurable set, since $g \in L^\infty([c,d],\mathbb{R})$. Therefore there exists a closed set F_κ such that $F_\kappa \subset G_\kappa$ and $meas(G_\kappa \setminus F_\kappa) = 0$. Since u is continuous, $u^{-1}(F_\kappa)$ is a closed set, and therefore, it is Lebesgue-measurable. Moreover, $u^{-1}(G_\kappa) = u^{-1}(F_\kappa) \cup u^{-1}(G_\kappa \setminus F_\kappa)$, and as in the first part of the proof, we get that $u^{-1}(G_\kappa \setminus F_\kappa)$ is measurable, and so is $u^{-1}(G_\kappa)$. □

Clearly, the statement of the previous Lemma is also valid if (1.2.5) is changed to

$$\operatorname{ess\,sup}\{\dot u(s)\colon\ s\in[a',b']\}<0,\qquad \text{for all}\ \ [a',b']\subset(a,b).$$

We will use the following notation.

Definition 1.2.9 $\mathcal{PM}([a,b],[c,d])$ *denotes the set of absolutely continuous functions* $u\colon [a,b]\to[c,d]$ *which are piecewise stricly monotone on* $[a,b]$ *in the sense that there exists a finite mesh* $a=t_0<t_1<\cdots<t_{m-1}<t_m=b$ *of* $[a,b]$ *such that for all* $i=0,1,\ldots,m-1$ *either*

$$\operatorname{ess\,inf}\{\dot u(s)\colon\ s\in[a',b']\}>0,\qquad \textit{for all}\ \ [a',b']\subset(t_i,t_{i+1})$$

or

$$\operatorname{ess\,sup}\{\dot u(s)\colon\ s\in[a',b']\}<0,\qquad \textit{for all}\ \ [a',b']\subset(t_i,t_{i+1}).$$

Lemma 1.2.8 implies the next result immediately.

Lemma 1.2.10 *Suppose* $g\in L^\infty([c,d],\mathbb{R}^n)$*, and* $u\in\mathcal{PM}([a,b],[c,d])$*. Then the composite function* $g\circ u\in L^\infty([a,b],\mathbb{R}^n)$ *and* $|g\circ u|_{L^\infty([a,b],\,\mathbb{R}^n)}\le|g|_{L^\infty([c,d],\,\mathbb{R}^n)}$.

The next lemma generalizes the convergence property (1.2.4) to the class $\mathcal{PM}$. We comment that to prove the convergence property (1.2.4) for $u,u^k\in\mathcal{PM}([a,b],[c,d])$, we need the stronger assumption $|u^k-u|_{W^{1,\infty}([a,b],\,\mathbb{R})}\to0$ instead of $|u^k-u|_{C([a,b],\,\mathbb{R})}\to0$ what is used in Lemma 1.2.6.

Lemma 1.2.11 *Suppose* $g\in L^\infty([c,d],\mathbb{R}^n)$*, and* $u,u^k\in\mathcal{PM}([a,b],[c,d])$ *(*$k\in\mathbb{N}$*) satisfying*

$$|u^k-u|_{W^{1,\infty}([a,b],\,\mathbb{R})}\to0,\qquad \textit{as}\ k\to\infty. \tag{1.2.6}$$

Then

$$\int_a^b|g(u^k(s))-g(u(s))|\,ds\to0,\qquad \textit{as}\ k\to\infty. \tag{1.2.7}$$

Proof Clearly, it is enough to show (1.2.7) for the case when g is real valued, i.e., $n=1$.

First note that Lemma 1.2.10 yields $g\circ u$, $g\circ u^k\in L^\infty([a,b],\mathbb{R})$. We prove (1.2.7) in three steps.

(i) First suppose that $g\in L^\infty([c,d],\mathbb{R})$ is the characteristic function of an interval $[e,f]\subset[c,d]$, i.e., $g=\chi_{[e,f]}$. Then $|\chi_{[e,f]}(u^k(s))-\chi_{[e,f]}(u(s))|$ is either 0 or 1, hence

$$meas(\{s\in[a,b]\colon\ \chi_{[e,f]}(u^k(s))\ne\chi_{[e,f]}(u(s))\})\le4|u^k-u|_{C([a,b],\mathbb{R})},$$

and so

$$\int_a^b |\chi_{[e,f]}(u^k(s)) - \chi_{[e,f]}(u(s))|\,ds \le 4|u^k - u|_{C([a,b],\mathbb{R})} \to 0, \qquad \text{as } k \to \infty.$$

(ii) Suppose g is a step function, i.e., $g = \sum_{i=1}^m c_i \chi_{A_i}$, where A_i are pairwise disjoint intervals with $\cup_{i=1}^m A_i = [c,d]$. Then

$$\int_a^b |g(u^k(s)) - g(u(s))|\,ds \le \sum_{i=1}^m |c_i| 4 |u^k - u|_{C([a,b],\mathbb{R})} \to 0, \qquad \text{as } k \to \infty.$$

(iii) Let $a = t_0 < t_1 < \cdots < t_m = b$ be the mesh points of u from the Definition 1.2.9, and let $0 < \varepsilon < \min\{t_{i+1} - t_i \colon i = 0, \dots, m-1\}/2$ be fixed, and introduce $t_i' := t_i + \varepsilon$ for $i = 0, \dots, m-1$ and $t_i'' := t_i - \varepsilon$ for $i = 1, \dots, m$, $t_0'' := a$, $t_m' := b$, and let

$$M := \min_{i=0,\dots,m-1} \operatorname*{ess\,inf}_{t \in [t_i', t_{i+1}'']} |\dot u(t)|. \tag{1.2.8}$$

We have $M > 0$, since $u \in \mathcal{PM}([a,b],[c,d])$.

The set of step functions is dense in $L^1([c,d],\mathbb{R})$ (see, e.g., [23]), so for a fixed $g \in L^\infty([c,d],\mathbb{R})$ and $0 < \delta < \varepsilon M/m$ there exists a step function $h\colon [c,d] \to \mathbb{R}$ such that $|g - h|_{L^1([c,d],\mathbb{R})} < \delta$. Let $h = \sum_{i=1}^m c_i \chi_{A_i}$, where A_i are pairwise disjoint intervals with $\cup_{i=1}^m A_i = [c,d]$, and define $h^* := \sum_{i=1}^m c_i^* \chi_{A_i}$, where

$$c_i^* := \begin{cases} c_i, & \text{if } |c_i| \le |g|_{L^\infty([c,d],\mathbb{R})} + 1, \\ |g|_{L^\infty([c,d],\mathbb{R})}, & \text{if } c_i > |g|_{L^\infty([c,d],\mathbb{R})} + 1, \\ -|g|_{L^\infty([c,d],\mathbb{R})}, & \text{if } c_i < -|g|_{L^\infty([c,d],\mathbb{R})} - 1. \end{cases}$$

Then it is easy to check that $|g(v) - h^*(v)| \le 1$ for a.e. $v \in [c,d]$, and

$$\int_c^d |g(v) - h^*(v)|\,dv \le \int_c^d |g(v) - h(v)|\,dv < \delta.$$

We have therefore

$$\begin{aligned}
&\int_a^b |g(u(s)) - h^*(u(s))|\,ds \\
&\quad = \sum_{i=0}^m \int_{t_i''}^{t_i'} |g(u(s)) - h^*(u(s))|\,ds + \sum_{i=0}^{m-1} \int_{t_i'}^{t_{i+1}''} |g(u(s)) - h^*(u(s))|\,ds \\
&\quad \le 2\varepsilon(m+1) + \sum_{i=0}^{m-1} \int_{t_i'}^{t_{i+1}''} |g(u(s)) - h^*(u(s))| \dot u(s) \frac{1}{\dot u(s)}\,ds \\
&\quad \le 2\varepsilon(m+1) + \frac{1}{M} \sum_{i=0}^{m-1} \left| \int_{u(t_i')}^{u(t_{i+1}'')} |g(v) - h^*(v)|\,dv \right| \\
&\quad \le 2\varepsilon(m+1) + \frac{\delta m}{M} \\
&\quad \le (2m+3)\varepsilon.
\end{aligned}$$

Assumption (1.2.6) yields that there exist $k_0 > 0$ such that $|u^k - u|_{W^{1,\infty}([a,b],\mathbb{R})} < \frac{M}{2}$ for $k \geq k_0$. Then for $k \geq k_0$ it follows $|\dot{u}^k(s)| \geq \frac{M}{2}$ for a.e. $s \in [t'_i, t''_{i+1}]$ and $i = 0, \ldots, m-1$. Therefore similarly to the previous estimate we have for $k \geq k_0$

$$\int_a^b |g(u^k(s)) - h^*(u^k(s))|\, ds \leq 2\varepsilon(m+1) + \frac{2\delta m}{M} \leq (2m+4)\varepsilon.$$

Using the above inequalities we get

$$\begin{aligned}
&\int_a^b |g(u^k(s)) - g(u(s))|\, ds \\
&\quad \leq \int_a^b |g(u^k(s)) - h^*(u^k(s))|\, ds + \int_a^b |h^*(u^k(s)) - h^*(u(s))|\, ds \\
&\qquad + \int_a^b |g(u(s)) - h^*(u(s))|\, ds \\
&\quad \leq (4m+7)\varepsilon + \int_a^b |h^*(u^k(s)) - h^*(u(s))|\, ds, \qquad k \geq k_0,
\end{aligned}$$

which yields (1.2.7) using part (ii), since $\varepsilon > 0$ is arbitrary close to 0. □

Lemma 1.2.12 *Suppose $f^{k,h} \in L^\infty([c,d],\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $h \in H$ for some fixed parameter set H,*

$$\lim_{k\to\infty} \sup_{h\in H} \int_c^d |f^{k,h}(s)|\, ds = 0,$$

and there exists $A \geq 0$ such that $|f^{k,h}(s)| \leq A$ for $k \in \mathbb{N}$, $h \in H$ and a.e. $s \in [c,d]$. Let $u, u^k \in \mathcal{PM}([a,b],[c,d])$ ($k \in \mathbb{N}$) be such that (1.2.6) *holds. Then*

$$\lim_{k\to\infty} \sup_{h\in H} \int_a^b |f^{k,h}(u^k(s))|\, ds = 0.$$

Proof Let $a = t_0 < t_1 < \cdots < t_m = b$ be the mesh points of u from the Definition 1.2.9, and let $0 < \varepsilon < \min\{t_{i+1} - t_i : i = 0, \ldots, m-1\}/2$ be fixed, let t'_i and t''_i be defined as in the proof of Lemma 1.2.11, and let M be defined by (1.2.8). Let k_0 be such that $|u^k - u|_{W^{1,\infty}([a,b],\mathbb{R})} \leq M/2$ for $k \geq k_0$. Then for $k \geq k_0$ it follows $|\dot{u}^k(s)| \geq \frac{M}{2}$ for a.e. $s \in [t'_i, t''_{i+1}]$ and $i = 0, \ldots, m-1$. Since $u^k \in \mathcal{PM}([a,b],[c,d])$, it follows from Lemma 1.2.10 that $|f^{k,h}(u^k(s))| \leq A$ for $k \in \mathbb{N}$, $h \in H$ and a.e. $s \in [a,b]$. Therefore for any $k \in \mathbb{N}$ and $h \in H$ we have

$$\begin{aligned}
\int_a^b |f^{k,h}(u^k(s))|\, ds &= \sum_{i=0}^{m} \int_{t''_i}^{t'_i} |f^{k,h}(u^k(s))|\, ds + \sum_{i=0}^{m-1} \int_{t'_i}^{t''_{i+1}} |f^{k,h}(u^k(s))|\, ds \\
&\leq (m+1)A2\varepsilon + \frac{2m}{M} \int_c^d |f^{k,h}(s)|\, ds.
\end{aligned}$$

Then
$$\sup_{h\in H}\int_a^b |f^{k,h}(u^k(s))|\,ds \le (m+1)A2\varepsilon + \sup_{h\in H}\frac{2m}{M}\int_c^d |f^{k,h}(s)|\,ds,$$
which proves the statement, since ε is arbitrarily close to 0. $\square$

Chapter 2

Delay differential equations with state-dependent delays

2.1 Introduction

In this chapter we study the SD-DDE

$$\dot{x}(t) = f(t, x_t, x(t - \tau(t, x_t, \xi)), \theta), \qquad t \in [0, T], \tag{2.1.1}$$

and the corresponding initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.1.2}$$

Let Θ and Ξ be normed linear spaces with norms $|\cdot|_\Theta$ and $|\cdot|_\Xi$, respectively, and suppose $\theta \in \Theta$ and $\xi \in \Xi$.

In this chapter we consider the initial function φ, θ and ξ as parameters in the IVP (2.1.1)-(2.1.2), and we denote the corresponding solution by $x(t, \varphi, \theta, \xi)$. The main goal of this chapter is to discuss the differentiability of $x(t, \varphi, \theta, \xi)$ wrt φ, θ and ξ. By differentiability we always mean Fréchet-differentiability throughout this thesis. Differentiability of solutions wrt parameters is an important qualitative question, but it also has a natural application in the problem of identification of parameters (see [46] and Chapter 3 below). But even for simple constant delay equations this problem leads to technical difficulties if the parameter is the delay [42, 73]. Similar difficulty arises in SD-DDEs.

Theorem 2.2.1 below yields that, under natural assumptions, Lipschitz continuous initial functions generate unique solutions of (2.1.1). As it is common for delay equations, as the time increases, the solution of (2.1.1) gets smoother wrt the time: on the interval $[0, r]$ the solution is C^1, on $[r, 2r]$ it is a C^2 function, etc. But for $t \in [0, r]$ the solution segment function x_t is only Lipschitz continuous. Therefore the linearization of the composite function $x(t - \tau(t, x_t, \xi))$ is not straightforward, which is clearly needed at some point of the proof to obtain differentiability wrt parameters.

To illustrate the difficulty of this problem in the case when we can't assume continuous differentiability of x, we recall a result of Brokate and Colonius [16]. They studied equations of the form

$$x'(t) = f\Big(t, x(t - \tau(t, x(t)))\Big), \qquad t \in [a, b],$$

and investigated differentiability of the composition operator

$$A : W^{1,\infty}([a, b]; \mathbb{R}) \supset \bar{X} \to L^p([a, b]; \mathbb{R}), \qquad A(x)(t) := x(t - \tau(t, x(t))).$$

They assumed that τ is twice continuously differentiable satisfying $a \le t - \tau(t, v) \le b$ for all $t \in [a, b]$ and $v \in \mathbb{R}$, and considered as domain of A the set

$$\bar{X} = \Big\{x \in W^{1,\infty}([a, b]; \mathbb{R}) : \text{ There exists } \varepsilon > 0 \text{ s.t. } \frac{d}{dt}\Big(t - \tau(t, x(t))\Big) \ge \varepsilon \\ \text{for a.e. } t \in [a, b]\Big\}.$$

It was shown in [16] that under these assumptions A is continuously differentiable with the derivative given by

$$(DA(x)u)(t) = -\dot{x}(t - \tau(t, x(t)))D_2\tau(t, x(t))u(t) + u(t - \tau(t, x(t)))$$

for $u \in W^{1,\infty}([a, b], \mathbb{R})$.

Both the strong $W^{1,\infty}$-norm on the domain and the weak L^p-norm on the range, together with the choice of the domain seemed to be necessary to obtain the results in [16]. Note that Manitius in [78] used a similar domain and norm when he studied linearization for a class of SD-DDEs.

Differentiability of solutions wrt parameters for SD-DDEs was studied in [21, 45, 58, 89, 90]. In [45] differentiability of the parameter map was established at parameter values where the compatibility condition

$$\varphi \in C^1, \qquad \dot{\varphi}(0-) = f(0, \varphi, \varphi(-\tau(0, \varphi, \xi)), \theta) \tag{2.1.3}$$

is satisfied. It was proved that the parameter map is differentiable in a pointwise sense, i.e., the map

$$W^{1,\infty} \times \Theta \times \Xi \to \mathbb{R}^n, \qquad (\varphi, \theta, \xi) \mapsto x(t, \varphi, \theta, \xi) \tag{2.1.4}$$

is differentiable for every fixed t from the domain of the solution. Moreover, it was shown that the map

$$W^{1,\infty} \times \Theta \times \Xi \to C, \qquad (\varphi, \theta, \xi) \mapsto x_t(\cdot, \varphi, \theta, \xi), \tag{2.1.5}$$

and, under a little more smoothness assumptions, the map

$$W^{1,\infty} \times \Theta \times \Xi \to W^{1,\infty}, \qquad (\varphi, \theta, \xi) \mapsto x_t(\cdot, \varphi, \theta, \xi) \tag{2.1.6}$$

is also differentiable at fixed parameter values satisfying (2.1.3). Note that condition (1.1.4) used by Walter in [89] and [90] coincides with (2.1.3) for equation (1.1.1). This is the main assumption of the C^1-framework of Walter which was needed to prove the existence of a C^1-smooth solution semiflow for (1.1.1).

In [58] differentiability of the parameter map was proved without assuming the compatibility condition (2.1.3). Instead, it was assumed that the time lag function $t \mapsto t - \tau(t, x_t, \xi)$ corresponding to a fixed solution x is strictly monotone increasing, more precisely,

$$\operatorname*{ess\,inf}_{0\le t\le\alpha} \frac{d}{dt}(t - \tau(t, x_t, \xi)) > 0, \tag{2.1.7}$$

where $\alpha > 0$ is such that the solution exists on $[-r, \alpha]$. Also, instead of a "pointwise" differentiability, the differentiability of the map

$$W^{1,\infty} \times \Theta \times \Xi \to W^{1,p}, \qquad (\varphi, \theta, \xi) \mapsto x_t(\cdot, \varphi, \theta, \xi)$$

was proved in a small neighborhood of the fixed parameter value. Note that here the differentiability was obtained using only a weak norm, the $W^{1,p}$-norm $(1 \le p < \infty)$ on the state-space.

Chen, Hu and Wu in [21] extended the above result to proving second ordered differentiability of the parameter map using the monotonicity condition (2.1.7) of the state-dependent time lag function, the $W^{1,p}$-norm $(1 \le p < \infty)$ on the state space, and the $W^{2,p}$-norm on the space of initial functions. Note that τ was not given explicitly in [21], it was defined through a coupled differential equation, but it satisfied the monotonicity condition (2.1.7).

In [48] the IVP

$$\begin{aligned} \dot{x}(t) &= f(t, x_t, x(t - \tau(t, x_t))), \qquad t \in [\sigma, T], && (2.1.8)\\ x(t) &= \varphi(t - \sigma), \qquad t \in [\sigma - r, \sigma] && (2.1.9) \end{aligned}$$

was considered. In this IVP the parameters θ and ξ were omitted for simplicity, but the initial time σ was considered together with the initial function as parameters in the equation. Combining the techniques of [45] and [58], and assuming the appropriate monotonicity condition (2.1.7), but without assuming the compatibility condition (2.1.3), the continuous differentiability of the parameter maps

$$W^{1,\infty} \to \mathbb{R}^n, \qquad \varphi \mapsto x(t, \sigma, \varphi)$$

and

$$W^{1,\infty} \to C, \qquad \varphi \mapsto x_t(\cdot, \sigma, \varphi)$$

were proved for a fixed t and σ in a neighborhood of a fixed initial function. Note that with this technique similar result can't be given using the $W^{1,\infty}$-norm on the state-space without using the compatibility condition.

Assuming the compatibility condition (2.1.3) it was also shown in [48] that the maps

$$[0,\alpha) \to \mathbb{R}^n, \qquad \sigma \mapsto x(t,\sigma,\varphi)$$

and

$$[0,\alpha) \to C, \qquad \sigma \mapsto x_t(\cdot,\sigma,\varphi)$$

are differentiable for all $t \in [\sigma - r, \alpha]$ and $t \in [\sigma, \alpha]$, respectively, and σ, φ in a neighborhood of a fixed parameter (σ,φ), and where $\alpha > 0$ is a certain constant. Assuming that the functions f and τ have a special form in (2.1.8), i.e., for equations of the form

$$\begin{aligned} \dot{x}(t) \;&=\; \bar{f}\Big(t, x(t-\lambda^1(t)),\ldots,x(t-\lambda^m(t)), \int_{-r}^0 A(t,\theta)x(s+\theta)\,ds, \\ &\qquad x\Big(t - \bar{\tau}\Big[t, x(t-\xi^1(t)),\ldots,x(t-\xi^\ell(t)), \int_{-r}^0 B(t,\theta)x(s+\theta)\,ds\Big]\Big)\Big) \end{aligned}$$

the differentiability of the map

$$[0,\alpha) \to \mathbb{R}^n, \qquad \sigma \mapsto x(t,\sigma,\varphi)$$

was shown in [48] for $t \in [\sigma,\alpha]$ using the monotonicity assumption (2.1.7), but without the compatibility condition (2.1.3). Note that in this case similar result does not hold for the map $\sigma \mapsto x_t(\cdot,\sigma,\varphi)$ using the C-norm, which is not surprising, since it is easy to see [48] that the map $\sigma \mapsto x(t,\sigma,\varphi)$ is differentiable at the point $t = \sigma$ if and only if a compatibility condition similar to (2.1.3) is satisfied.

The organization of this chapter is the following. In Section 2.2 first we list the detailed assumptions on the IVP (2.1.1)-(2.1.2) we will need in our differentiability results later, and formulate a well-posedness result (Theorem 2.2.1) concerning the IVP (2.1.1)-(2.1.2), and prove some estimates will be essential later througout this chapter.

In Section 2.3 using and extending the method introduced in [48], we discuss differentiability of the parameter maps associated to the IVP (2.1.1)-(2.1.2). In the main result of this chapter (see Theorem 2.3.9 below) we show the differentiability of the parameter maps (2.1.4) and (2.1.5) without using the compatibility condition (2.1.3), and also relaxing the monotonicity condition (2.1.7) to the condition that the time lag function $t \mapsto t - \tau(t, x_t, \xi)$ is "piecewise strictly monotone" in the sense of Definition 1.2.9. Note that omitting the compatibility condition is essential in the application of this results in Chapter 3, where we prove the convergence of the quasilinearization method in the problem of parameter estimation. Also, in this application the existence of the derivative is needed in this strong, pointwise sence, i.e., the differentiability of the map (2.1.4) will be used in Chapter 3. Note that in Section 2.3 sufficient conditions are given in Lemma 2.3.8 which imply that the detivative of the solution wrt parameters is Lipschitz continuous wrt the parameters. This result is needed for the proof of the quasilinearization method in Chapter 3.

In Section 2.4 the main result is Theorem 2.4.16, which proves twice continuous differentiability of the maps

$$W^{2,\infty} \times \Theta \times \Xi \to \mathbb{R}^n, \qquad (\varphi, \theta, \xi) \mapsto x(t, \varphi, \theta, \xi)$$

and

$$W^{2,\infty} \times \Theta \times \Xi \to C, \qquad (\varphi, \theta, \xi) \mapsto x_t(\cdot, \varphi, \theta, \xi)$$

at a parameter value (φ, θ, ξ) satisfying the compatibility condition (2.1.3) and such that the corresponding time lag function $t \mapsto \tau(t, x_t, \xi)$ is piecewise strictly monotone in the sense of Definition 1.2.9. Under some additional condition, the continuity of the second derivative wrt the parameters is obtained in a certain sense. Note that this result shows the existence of the second derivative in a pontwise sense, at each t. The only result known in the literature for the existence of a second derivative wrt the parameters is the result of Chen, Hu and Wu [21], where the second order differentiability is proved only using a weak $W^{1,p}$-norm on the state-space.

2.2 Well-posedness and continuous dependence on parameters

In this section we list all the assumptions we need later on the IVP (2.1.1)-(2.1.2), and show some basic results including the well-posedness of the IVP and Lipschitz continuous dependence of the solutions on the parameters φ, θ and γ.

Suppose $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset \Xi$ are open subsets of the respective spaces. $T > 0$ is finite or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$.

We assume

(A1) (i) $f : \mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuous;

(ii) $f(t, \psi, u, \theta)$ is locally Lipschitz continuous in ψ, u and θ, i.e., for every finite $\alpha \in (0, T]$, for every closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, and closed and bounded subset $M_3 \subset \Omega_3$ of Θ there exists a constant $L_1 = L_1(\alpha, M_1, M_2, M_3)$ such that

$$|f(t, \psi, u, \theta) - f(t, \bar{\psi}, \bar{u}, \bar{\theta})| \leq L_1 \Big(|\psi - \bar{\psi}|_C + |u - \bar{u}| + |\theta - \bar{\theta}|_\Theta \Big),$$

for $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_1$, $u, \bar{u} \in M_2$ and $\theta, \bar{\theta} \in M_3$;

(iii) $f : \mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuously differentiable wrt its second, third and fourth arguments;

(iv) $f(t,\psi,u,\theta)$ is locally Lipschitz continuous wrt t, i.e., for every finite $\alpha \in (0,T]$, for every closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, and closed and bounded subset $M_3 \subset \Omega_3$ of Θ there exists a constant $L_1 = L_1(\alpha, M_1, M_2, M_3)$ such that

$$|f(t,\psi,u,\theta) - f(\bar{t},\psi,u,\theta)| \le L_1|t-\bar{t}|$$

for $t,\bar{t} \in [0,\alpha]$, $\psi \in M_1$, $u \in M_2$ and $\theta \in M_3$;

(v) $D_2 f$, $D_3 f$ and $D_4 f$ are locally Lipschitz continuous wrt all of their arguments, i.e., for every finite $\alpha \in (0,T]$, for every closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, and closed and bounded subset $M_3 \subset \Omega_3$ of Θ there exists $L_3 = L_3(\alpha, M_1, M_2, M_3)$ such that

$$|D_i f(t,\psi,u,\theta) - D_i f(\bar{t},\bar{\psi},\bar{u},\bar{\theta})|_{\mathcal{L}(Y_i,\mathbb{R}^n)} \le L_3\Big(|t-\bar{t}| + |\psi-\bar{\psi}|_C + |u-\bar{u}| + |\theta-\bar{\theta}|_\Theta\Big)$$

for $i = 2,3,4$, $t,\bar{t} \in [0,\alpha]$, $\psi,\bar{\psi} \in M_1$, $u,\bar{z} \in M_2$ and $\theta,\bar{\theta} \in M_3$, where $Y_2 := C$, $Y_3 := \mathbb{R}^n$ and $Y_4 := \Theta$;

(vi) $D_2 f$, $D_3 f$ and $D_4 f$ are continuously differentiable wrt their second, third and fourth arguments on $[0,T] \times \Omega_1 \times \Omega_2 \times \Omega_3$;

(A2) (i) $\tau : \mathbb{R} \times C \times \Xi \supset [0,T] \times \Omega_1 \times \Omega_4 \to [0,r] \subset \mathbb{R}$ is continuous;

(ii) $\tau(t,\psi,\xi)$ is locally Lipschitz continuous in ψ and ξ in the following sense: for every finite $\alpha \in (0,T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and closed and bounded subset $M_4 \subset \Omega_4$ of Ξ there exists a constant $L_2 = L_2(\alpha, M_1, M_4)$ such that

$$|\tau(t,\psi,\xi) - \tau(t,\bar{\psi},\bar{\xi})| \le L_2\Big(|\psi-\bar{\psi}|_C + |\xi-\bar{\xi}|_\Xi\Big)$$

for $t \in [0,\alpha]$, $\psi,\bar{\psi} \in M_1$, $\xi,\bar{\xi} \in M_4$;

(iii) $\tau : [0,T] \times C \times \Xi \supset [0,T] \times \Omega_1 \times \Omega_4 \to \mathbb{R}$ is continuously differentiable wrt its second and third arguments;

(iv) $\tau(t,\psi,\xi)$ is locally Lipschitz continuous in t, i.e., for every finite $\alpha \in (0,T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and closed and bounded subset $M_4 \subset \Omega_4$ of Ξ there exists a constant $L_2 = L_2(\alpha, M_1, M_4)$ such that

$$|\tau(t,\psi,\xi) - \tau(\bar{t},\psi,\xi)| \le L_2|t-\bar{t}|$$

for $t,\bar{t} \in [0,\alpha]$, $\psi \in M_1$, $\xi \in M_4$;

(v) for every finite $\alpha \in (0,T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and closed and bounded subset $M_4 \subset \Omega_4$ of Ξ there exists $L_4 = L_4(\alpha, M_1, M_4) \geq 0$ such that

$$\Big|\frac{d}{dt}\tau(t, y_t, \xi) - \frac{d}{dt}\tau(t, \bar{y}_t, \bar{\xi})\Big| \leq L_4\Big(|y_t - \bar{y}_t|_{W^{1,\infty}} + |\xi - \bar{\xi}|_\Xi\Big), \quad \text{a.e. } t \in [0,\alpha],$$

where $\xi, \bar{\xi} \in M_4$, and $y, \bar{y} \in W^{1,\infty}([-r,\alpha], \mathbb{R}^n)$ are such that $y_t, \bar{y}_t \in M_1$ for $t \in [0,\alpha]$;

(vi) $D_2\tau$ and $D_3\tau$ are locally Lipschitz continuous wrt all arguments, i.e., for every finite $\alpha \in (0,T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and closed and bounded subset $M_4 \subset \Omega_4$ of Ξ there exists a constant $L_5 = L_5(\alpha, M_1, M_4)$ such that

$$|D_i\tau(t, \psi, \xi) - D_i\tau(\bar{t}, \bar{\psi}, \bar{\xi})|_{\mathcal{L}(Z_i,\mathbb{R})} \leq L_5\Big(|t - \bar{t}| + |\psi - \bar{\psi}|_C + |\xi - \bar{\xi}|_\Xi\Big)$$

for $i = 2, 3$, $t, \bar{t} \in [0,\alpha]$, $\psi, \bar{\psi} \in M_1$, $\xi, \bar{\xi} \in M_4$, where $Z_2 := C$ and $Z_3 := \Xi$;

(vii) $D_2\tau$ and $D_3\tau$ are continuously differentiable wrt their second and third arguments on $[0,T] \times \Omega_1 \times \Omega_4$;

(viii) for every finite $\alpha \in (0,T]$, for every closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, and closed and bounded subsets $M_3 \subset \Omega_3$ of Θ and $M_4 \subset \Omega_4$ of Ξ there exists $L_6 = L_6(\alpha, M_1, M_2, M_3, M_4)$ such that

$$\begin{aligned}
&\Big|\frac{d}{dt}f(t, y_t, y(t - \tau(t, y_t, \xi)), \theta) - \frac{d}{dt}f(t, \bar{y}_t, \bar{y}(t - \tau(t, \bar{y}_t, \bar{\xi})), \bar{\theta})\Big| \\
&\quad \leq \ L_6\Big(|y_t - \bar{y}_t|_{W^{1,\infty}} + |\xi - \bar{\xi}|_\Xi + |\theta - \bar{\theta}|_\Xi\Big), \quad \text{a.e. } t \in [0,\alpha],
\end{aligned}$$

where $\theta, \bar{\theta} \in M_3$, $\xi, \bar{\xi} \in M_4$, and $y, \bar{y} \in W^{1,\infty}([-r,\alpha], \mathbb{R}^n)$ are such that $y_t, \bar{y}_t \in M_1$ for $t \in [0,\alpha]$.

We introduce the parameter space

$$\Gamma := W^{1,\infty} \times \Theta \times \Xi$$

equipped with the product norm $|\gamma|_\Gamma := |\varphi|_{W^{1,\infty}} + |\theta|_\Theta + |\xi|_\Xi$ for $\gamma = (\varphi, \theta, \xi) \in \Gamma$, and the set of admissible parameters

$$\Pi := \Big\{(\varphi, \theta, \xi) \in \Gamma\colon\ \varphi \in \Omega_1,\ \varphi(-\tau(0,\varphi)) \in \Omega_2,\ \theta \in \Omega_3,\ \xi \in \Omega_4\Big\}.$$

The next theorem shows that every admissible parameter $(\hat{\varphi}, \hat{\theta}, \hat{\xi}) \in \Pi$ has a neighborhood P and there exists a constant $\alpha > 0$ such that the IVP (2.1.1)-(2.1.2) has a unique solution on $[-r,\alpha]$ corresponding to all parameters $\gamma=(\varphi,\theta,\xi)\in P$. This solution will be denoted by $x(t,\gamma)$, and its segment function at t is denoted by $x_t(\cdot,\gamma)$.

The well-posedness of several classes of SD-DDEs was studied in many papers (see, e.g., [27, 56, 58, 84]. The next result is a variant of a result from [50] where the initial time is also considered as a parameter, but the parameters θ and ξ were missing in the equation. The proof is similar to that of Theorem 3.1 in [50], and it also follows from the analogous proof of Theorem 4.2.2 of the neutral case, therefore it is omitted here. The notations and estimates introduced in the next theorem will be essential in the following sections.

Theorem 2.2.1 *Assume (A1) (i), (ii), (A2) (i), (ii), and let $\hat\gamma\in\Pi$. Then there exist $\delta>0$ and $0<\alpha\le T$ finite numbers such that*

(i) for all $\gamma=(\varphi,\theta,\xi)\in P:=\mathcal{B}_\Gamma(\hat\gamma;\,\delta)$ the IVP (2.1.1)-(2.1.2) *has a unique solution $x(t,\gamma)$ on $[-r,\alpha]$;*

(ii) there exist a closed subset $M_1\subset C$ which is also a bounded and convex subset of $W^{1,\infty}$, $M_2\subset\mathbb{R}^n$ compact and convex subset and $M_3\subset\Theta$, $M_4\subset\Xi$ closed, bounded and convex subsets of the respective spaces such that $x_t(\cdot,\gamma)\in M_1$, $x(t-\tau(t,x_t(\cdot,\gamma),\xi),\gamma)\in M_2$, $\theta\in M_3$ and $\xi\in M_4$ for $\gamma=(\varphi,\theta,\xi)\in P$ and $t\in[0,\alpha]$; and

(iii) $x_t(\cdot,\gamma)\in W^{1,\infty}$ for $\gamma\in P$ and $t\in[0,\alpha]$, and there exist constants $N=N(\alpha,\delta)$ and $L=L(\alpha,\delta)$ such that

$$|x_t(\cdot,\gamma)|_{W^{1,\infty}}\le N,\qquad \gamma\in P,\ t\in[0,\alpha],\tag{2.2.1}$$

and

$$|x_t(\cdot,\gamma)-x_t(\cdot,\bar\gamma)|_{W^{1,\infty}}\le L|\gamma-\bar\gamma|_\Gamma,\qquad \gamma\in P,\ t\in[0,\alpha].\tag{2.2.2}$$

The following result is obvious.

Remark 2.2.2 *Suppose the conditions of Theorem 2.2.1 hold, P and α are defined by Theorem 2.2.1, and let $\mathcal{P}$ denote the subset of P consisiting of those parameters which satisfy the compatibility condition, i.e.,*

$$\mathcal{P}:=\Big\{(\varphi,\theta,\xi)\in P\colon\ \varphi\in C^1,\quad \dot\varphi(0-)=f(0,\varphi,\varphi(-\tau(0,\varphi,\xi)),\theta)\Big\}.\tag{2.2.3}$$

Then for all parameter values $\gamma\in\mathcal{P}$ the corresponding solution $x(t,\gamma)$ is continuously differentiable wrt t for $t\in[-r,\alpha]$.

Throughout the rest of the chapter we will use the following notations. The parameter $\hat{\gamma} \in \Pi$ is fixed, and the constants $\delta > 0$, $0 < \alpha \le T$ are defined by Theorem 2.2.1, and let $P := \mathcal{B}_\Gamma(\hat{\gamma}; \delta)$. The sets $M_1 \subset C$, $M_2 \subset \mathbb{R}^n$, $M_3 \subset \Theta$ and $M_4 \subset \Xi$ are defined by Theorem 2.2.1 (ii), $L_1 = L_1(\alpha, M_1, M_2, M_3)$, $L_2 = L_2(\alpha, M_1, M_4)$ and $L_4 = L_4(\alpha, M_1, M_4)$ denote the corresponding Lipschitz constants from (A1) (ii), (A2) (ii) and (A2) (iv), respectively, and the constants $N = N(\alpha, \delta)$ and $L = L(\alpha, \delta)$ are defined by Theorem 2.2.1 (iii). We will restrict our attention to the fixed parameter set P, so the sets M_1, M_2, M_3 and M_4, and the constants L_1, L_2, L_4, L and N can be considered to be fixed througout this chapter.

Lemma 2.2.3 *Assume (A1) (i), (ii), (A2) (i),(ii), $\gamma = (\varphi, \xi, \theta) \in P$, $h_k = (h_k^\varphi, h_k^\xi, h_k^\theta) \in \Gamma$ is a sequence such that $\gamma + h_k \in P$ for $k \in \mathbb{N}$ and $|h_k|_\Gamma \to 0$ as $k \to \infty$. Let $x(t) := x(t, \gamma)$, $x^k(t) := x(t, \gamma + h_k)$ be the corresponding solutions of the IVP* (2.1.1)-(2.1.2)*, and $u^k(s) := t - \tau(t, x_t^k, \xi + h_k^\xi)$ and $u(t) := t - \tau(t, x_t, \xi)$. Then there exists $K_0 \ge 0$ such that*

$$|u^k(t) - u(t)| \le K_0 |h_k|_\Gamma, \qquad t \in [0, \alpha], \quad k \in \mathbb{N}. \tag{2.2.4}$$

If in addition (A2) (iv) holds, then $u, u^k \in W^{1,\infty}([0, \alpha], \mathbb{R})$, and moreover, if (A2) (v) is also satisfied, then there exists $K_1 \ge 0$ such that

$$|u^k - u|_{W^{1,\infty}([0,\alpha],\mathbb{R})} \le K_1 |h_k|_\Gamma, \qquad k \in \mathbb{N}. \tag{2.2.5}$$

Proof Assumption (A2) (ii) implies

$$|u^k(t) - u(t)| = |\tau(t, x_t^k, \xi + h_k^\xi) - \tau(t, x_t, \xi)| \le L_2(|x_t^k - x_t|_C + |h_k^\xi|_\Xi), \quad t \in [0, \alpha],$$

so (2.2.2) yields (2.2.4) with $K_0 := L_2(L + 1)$.

Now assume (A2) (iv) also holds. For simplicity of the notation let $h_0 := 0 = (0, 0, 0) \in \Gamma$, and so $x^0 := x$ and $u^0 := u$. Then (A2) (ii), the Mean Value Theorem and (2.2.1) imply for $k \in \mathbb{N}_0$ and $t, \bar{t} \in [0, \alpha]$

$$|\tau(t, x_t^k, \xi + h_k^\xi) - \tau(\bar{t}, x_{\bar{t}}^k, \xi + h_k^\xi)\Big| \le L_2(|t - \bar{t}| + |x_t^k - x_{\bar{t}}^k|_C) \le L_2(1 + N)|t - \bar{t}|. \tag{2.2.6}$$

Hence u^k is Lipschitz continuous, and so it is almost everywhere differentiable on $[0, \alpha]$, and $|\dot{u}^k|_{L^\infty([0,\alpha],\mathbb{R})} \le L_2(1 + N)$. Therefore $u^k \in W^{1,\infty}([0, \alpha], \mathbb{R})$ for $k \in \mathbb{N}_0$.

Let $L_4 = L_4(\alpha, M_1, M_4)$ be defined by (A2) (v). Assumption (A2) (v) and (2.2.2) give

$$|\dot{u}^k(t) - \dot{u}(t)| = \Big|\frac{d}{dt}\tau(t, x_t^k, \xi + h_k^\xi) - \frac{d}{dt}\tau(t, x_t, \xi)\Big| \le L_4(|x_t^k - x_t|_C + |h_k^\xi|_\Xi) \le L_4(L + 1)|h_k|_\Gamma$$

for a.e. $t \in [0, \alpha]$. Therefore (2.2.5) holds with $K_1 := \max\{K_0, L_4(L + 1)\}$. $\square$

We note that (A2) (v) and (viii) hold under natural assumptions for example for functions of the form

$$\tau(t,\psi,\xi) = \bar\tau\Big(t, \psi(-\eta^1(t)),\ldots,\psi(-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)\psi(\zeta)\,d\zeta, \xi(t)\Big)$$

and

$$f(t,\psi,u,\theta) = \bar f\Big(t, \psi(-\nu^1(t)),\ldots,\psi(-\nu^m(t)), \int_{-r}^0 B(t,\zeta)\psi(\zeta)\,d\zeta, \theta(t)\Big).$$

Here $\Theta = W^{1,\infty}([0,T],\mathbb{R})$ and $\Xi = W^{1,\infty}([0,T],\mathbb{R})$ can be used, and then we have, e.g., for τ under straightforward assumptions we have for a.e. $t\in[0,\alpha]$, $y\in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)$

$$\begin{aligned}
\frac{d}{dt}\tau(t,y_t,\xi) \;=\;& D_1\bar\tau\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big)\\
&+\sum_{i=1}^{\ell} D_{i+1}\bar\tau\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big)\\
&\qquad\times\dot y(t-\eta^i(t))(1-\dot\eta^i(t))\\
&+D_{i+2}\bar\tau\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big)\\
&\qquad\times\int_{-r}^0 [D_1A(t,\zeta)y(t+\zeta)+A(t,\zeta)\dot y(t+\zeta)]\,d\zeta\\
&+D_{i+3}\bar\tau\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big)\dot\xi(t).
\end{aligned}$$

Similar formula holds for $\frac{d}{dt}f(t,y_t,y(t-\tau(t,y_t,\xi)),\theta)$. So if $\bar\tau$ and $\bar f$ are continuously differentiable, η^i are continuously differentiable and $\operatorname{ess\,sup}_{t\in[0,T]}(1-\dot\eta^i(t))>0$ for $i=1,\ldots,\ell$, then it is easy to argue that (A2) (v) and (viii) hold. See also Lemma 4.2.1 in Chapter 4 for a related computation.

2.3 First-order differentiability wrt the parameters

In this section we study the differentiability of the solution $x(t,\gamma)$ of the IVP (2.1.1)-(2.1.2) wrt γ. The proof of our differentiability results will be based on the following lemmas.

Lemma 2.3.1 *Let $y \in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)$, $\omega_k \in (0,\infty)$ ($k\in\mathbb{N}$) be a sequence satisfying $\omega_k \to 0$ as $k\to\infty$. Let $u, u^k \in \mathcal{PM}([0,\alpha],[-r,\alpha])$ ($k\in\mathbb{N}$) be such that*

$$|u^k-u|_{W^{1,\infty}([0,\alpha],\mathbb{R})} \le \omega_k, \qquad k\in\mathbb{N}. \tag{2.3.1}$$

Then

$$\lim_{k\to\infty}\frac{1}{\omega_k}\int_0^\alpha |y(u^k(s))-y(u(s))-\dot y(u(s))(u^k(s)-u(s))|\,ds = 0. \tag{2.3.2}$$

Proof Let $0=t_0<t_1<\cdots<t_{m-1}<t_m=\alpha$ be the mesh points of u from the Definition 1.2.9, and let $0<\varepsilon<\min\{t_{i+1}-t_i:\ i=0,\ldots,m-1\}/2$ be fixed, and introduce $t_i' := t_i+\varepsilon$ for $i=0,\ldots,m-1$, $t_i'' := t_i-\varepsilon$ for $i=1,\ldots,m$, $t_0'':=0$, $t_m':=\alpha$, and let

$$M := \min_{i=0,\ldots,m-1}\ \operatorname*{ess\,inf}_{t\in[t_i',t_{i+1}'']}|\dot u(t)|.$$

We have $M>0$, since $u\in\mathcal{PM}([0,\alpha],[-r,\alpha])$. Assumption (2.3.1) yields that there exists $k_0>0$ such that $|u^k-u|_{W^{1,\infty}([0,\alpha],\mathbb{R})}<\frac{M}{2}$ for $k\ge k_0$. Then for $k\ge k_0$ it follows $|\dot u^k(s)|\ge\frac{M}{2}$ and $|\dot u(s)+\nu(\dot u^k(s)-\dot u(s))|\ge\frac{M}{2}$ for a.e. $s\in[t_i',t_{i+1}'']$, $i=0,\ldots,m-1$ and $\nu\in[0,1]$. Let $A:=|y|_{W^{1,\infty}}([-r,\alpha],\mathbb{R}^n)$. Then simple manipulations, (2.3.1) and Fubini's theorem yield

$$\begin{aligned}
&\int_0^\alpha |y(u^k(s))-y(u(s))-\dot y(u(s))(u^k(s)-u(s))|\,ds\\
&\quad\le \sum_{i=0}^{m}\int_{t_i''}^{t_i'}\Big(|y(u^k(s))-y(u(s))|+|\dot y(u(s))||u^k(s)-u(s)|\Big)\,ds\\
&\qquad+\sum_{i=0}^{m-1}\int_{t_i'}^{t_{i+1}''}\Big|\int_{u(s)}^{u^k(s)}\Big(\dot y(v)-\dot y(u(s))\Big)dv\Big|\,ds\\
&\quad\le (m+1)2\varepsilon 2A|u^k-u|_{C([0,\alpha],\mathbb{R})}\\
&\qquad+\sum_{i=0}^{m-1}\int_{t_i'}^{t_{i+1}''}\Big|\int_0^1\Big[\dot y\Big(u(s)+\nu(u^k(s)-u(s))\Big)-\dot y(u(s))\Big](u^k(s)-u(s))\,d\nu\Big|\,ds\\
&\quad\le \omega_k\Big[(m+1)4A\varepsilon+\sum_{i=0}^{m-1}\int_0^1\int_{t_i'}^{t_{i+1}''}\Big|\dot y\Big(u(s)+\nu(u^k(s)-u(s))\Big)-\dot y(u(s))\Big|\,ds\,d\nu\Big].
\end{aligned}$$

It follows from Lemma 1.2.6 and Remark 1.2.7 that for every $\nu \in [0,1]$

$$\lim_{k\to\infty} \int_{t_i'}^{t_{i+1}''} \Big| \dot y\Big(u(s)+\nu(u^k(s)-u(s))\Big) - \dot y(u(s))\Big|\, ds = 0, \qquad i=0,\dots,m-1,$$

hence we get by using the Lebesgue's Dominated Convergence Theorem that

$$\limsup_{k\to\infty} \frac{1}{\omega_k}\int_0^\alpha |y(u^k(s)) - y(u(s)) - \dot y(u(s))(u^k(s)-u(s))|\, ds \le (m+1)4A\varepsilon.$$

This concludes the proof of (2.3.2), since $\varepsilon>0$ can be arbitrary close to 0. $\square$

We introduce the notations

$$\begin{aligned} \omega_f(t,\bar\psi,\bar u,\bar\theta,\psi,u,\theta) &:= f(t,\psi,u,\theta) - f(t,\bar\psi,\bar u,\bar\theta) - D_2 f(t,\bar\psi,\bar u,\bar\theta)(\psi-\bar\psi) \\ &\quad -D_3 f(t,\bar\psi,\bar u,\bar\theta)(u-\bar u) - D_4 f(t,\bar\psi,\bar u,\bar\theta)(\theta-\bar\theta), \qquad (2.3.3)\\ \omega_\tau(t,\bar\psi,\bar\xi,\psi,\xi) &:= \tau(t,\psi,\xi) - \tau(t,\bar\psi,\bar\xi) - D_2\tau(t,\bar\psi,\bar\xi)(\psi-\bar\psi) \\ &\quad -D_3\tau(t,\bar\psi,\bar\xi)(\xi-\bar\xi) \qquad (2.3.4)\end{aligned}$$

for $t\in[0,T]$, $\bar\psi,\psi\in\Omega_1$, $\bar u,u\in\Omega_2$, $\bar\theta,\theta\in\Omega_3$, $\bar\xi,\xi\in\Omega_4$, and

$$\begin{aligned} \Omega_f(\varepsilon) &:= \max_{i=2,3,4}\sup\Big\{ |D_i f(t,\psi,u,\theta) - D_i f(t,\tilde\psi,\tilde u,\tilde\theta)|_{\mathcal{L}(Y_i,\mathbb{R}^n)} : \\ &\qquad |\psi-\tilde\psi|_C + |u-\tilde u| + |\theta-\tilde\theta|_\Theta \le \varepsilon,\ \ t\in[0,\alpha],\ \psi,\tilde\psi\in M_1, \\ &\qquad u,\tilde u\in M_2,\ \theta,\tilde\theta\in M_3\Big\}, \qquad (2.3.5)\\ \Omega_\tau(\varepsilon) &:= \max_{i=2,3}\sup\Big\{ |D_i\tau(t,\psi,\xi) - D_i\tau(t,\bar\psi,\bar\xi)|_{\mathcal{L}(Z_i,\mathbb{R})} :\ |\psi-\bar\psi|_C + |\xi-\bar\xi|_\Xi \le \varepsilon, \\ &\qquad t\in[0,\alpha],\ \psi,\bar\psi\in M_1,\ \xi,\bar\xi\in M_4\Big\}, \qquad (2.3.6)\end{aligned}$$

where $Y_2 := C$, $Y_3 := \mathbb{R}^n$, $Y_4 := \Theta$, $Z_2 := C$ and $Z_3 := \Xi$.

The following result is an easy generalization of Lemma 4.2 of [50] for the IVP (2.1.1)-(2.1.2), therefore we omit its proof here. (See also the related proof of Lemma 2.4.7 below.)

Lemma 2.3.2 (see [50]) *Suppose (A1) (i)–(iii), (A2) (i)–(iii). Let P and $\alpha>0$ be defined by Theorem 2.2.1, let $\gamma=(\varphi,\theta,\xi)\in P$ be fixed, and $h_k=(h_k^\varphi,h_k^\theta,h_k^\xi)\in\Gamma$ ($k\in\mathbb{N}$) be a sequence satisfying $|h_k|_\Gamma\to 0$ as $k\to\infty$, and $\gamma+h_k\in P$ for $k\in\mathbb{N}$. Let $x(t):=x(t,\gamma)$, $x^k(t):=x(t,\gamma+h_k)$, $u(t):=t-\tau(t,x_t,\xi)$ and $u^k(t):=t-\tau(t,x^k,\xi+h_k^\xi)$. Then*

$$\lim_{k\to\infty}\frac{1}{|h_k|_\Gamma}\int_0^\alpha |\omega_f(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta)|\, ds = 0 \qquad (2.3.7)$$

and

$$\lim_{k\to\infty}\frac{1}{|h_k|_\Gamma}\int_0^\alpha |\omega_\tau(s,x_s,\xi,x_s^k,\xi+h_k^\xi)|\,ds=0. \tag{2.3.8}$$

A solution $x(\cdot,\gamma)$ of the IVP (2.1.1)-(2.1.2) for $\gamma\in P$ is, in general, only a $W^{1,\infty}$-function on the interval $[-r,0]$, but it is continuously differentiable for $t\ge 0$. In [58] (see also [50]) a parameter set

$$P_1 := \{\gamma=(\varphi,\theta,\xi)\in P\colon\ x(\cdot,\gamma)\in X(\alpha,\xi)\}$$

was considered, where

$$\begin{aligned} X(\alpha,\xi) \;:=\; \Big\{ & x\in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)\colon\ x_t\in\Omega_1,\ x(t-\tau(t,x_t,\xi))\in\Omega_2 \text{ for } t\in[0,\alpha],\\ & \text{and}\quad \operatorname{ess\,inf}\Big\{\frac{d}{dt}(t-\tau(t,x_t,\xi))\colon\ \text{a.e. } t\in[0,\alpha^*]\Big\}>0\Big\}\end{aligned}$$

and $\alpha^* := \min\{r,\alpha\}$. Then Lemma 1.2.6 yields that the function $t\mapsto \dot x(t-\tau(t,x_t,\xi))$ is well-defined for a.e. $t\in[0,\alpha^*]$ and it is integrable on $[0,\alpha^*]$, and it is well-defined and continuous on $[\alpha^*,\alpha]$. Note that it was shown in [58] (see also [50]) that P_1 is an open subset of the parameter set P. In this section we relax this condition. We define the parameter set

$$\begin{aligned} P_2 \;:=\; \{ & \gamma=(\varphi,\theta,\xi)\in P\colon \text{ the map }\ [0,\alpha^*]\to\mathbb{R},\ t\mapsto t-\tau(t,x_t(\cdot,\gamma),\xi)\\ & \text{belongs to } \mathcal{PM}([0,\alpha^*],[-r,\alpha^*])\}.\end{aligned} \tag{2.3.9}$$

Then we have $P_1\subset P_2\subset P$, and Lemma 1.2.10 yields that for a solution x corresponding to parameter $\gamma\in P_2$ the function $t\mapsto \dot x(t-\tau(t,x_t,\xi))$ is well-defined for a.e. $t\in[0,\alpha^*]$ and it is integrable on $[0,\alpha^*]$. Therefore, as the next discussion will show, the parameter set where the variational equation, and correspondingly the differentiability of the solution wrt the parameters can be obtained is larger than in the previous papers [45, 50, 58].

Let $\gamma=(\varphi,\theta,\xi)\in P_2$ be fixed, and let $x(t):=x(t,\gamma)$. Consider the space $C\times\Theta\times\Xi$ equipped with the product norm $|(h^\varphi,h^\theta,h^\xi)|_{C\times\Theta\times\Xi} := |h^\varphi|_C+|h^\theta|_\Theta+|h^\xi|_\Xi$. Then for a.e. $t\in[0,\alpha]$ we introduce the linear operator $L(t,x)\colon C\times\Theta\times\Xi\to\mathbb{R}^n$ by

$$\begin{aligned} & L(t,x)(h^\varphi,h^\theta,h^\xi)\\ &\quad:= D_2f(t,x_t,x(t-\tau(t,x_t,\xi)),\theta)h^\varphi + D_3f(t,x_t,x(t-\tau(t,x_t,\xi)),\theta)\\ &\qquad\times\Big[-\dot x(t-\tau(t,x_t,\xi))\Big(D_2\tau(t,x_t,\xi)h^\varphi+D_3\tau(t,x_t,\xi)h^\xi\Big)+h^\varphi(-\tau(t,x_t,\xi))\Big]\\ &\qquad+D_4f(t,x_t,x(t-\tau(t,x_t,\xi)),\theta)h^\theta \end{aligned} \tag{2.3.10}$$

for $(h^\varphi,h^\theta,h^\xi)\in C\times\Theta\times\Xi$. We have by (A1) (ii), (A2) (ii) and (2.2.1)

$$\begin{aligned} |L(t,x)(h^\varphi,h^\theta,h^\xi)| &\le L_1|h^\varphi|_C + L_1\Big[N(L_2|h^\varphi|_C+L_2|h^\xi|_\Xi)+|h^\varphi|_C\Big]+L_1|h^\theta|_\Theta\\ &\le L_1N_0|(h^\varphi,h^\theta,h^\xi)|_{C\times\Theta\times\Xi},\qquad \text{a.e. } t\in[0,\alpha], \end{aligned} \tag{2.3.11}$$

where

$$N_0 := NL_2 + 3. \tag{2.3.12}$$

Therefore

$$|L(t,x)|_{\mathcal{L}(C\times\Theta\times\Xi,\mathbb{R}^n)} \le L_1 N_0, \qquad \text{a.e. } t\in[0,\alpha].$$

Hence $L(t,x)$ is a bounded linear operator for all t for which $\dot{x}(t-\tau(t,x_t,\xi))$ exists.

For $\gamma\in P_2$ we define the variational equation associated to $x=x(\cdot,\gamma)$ as

$$\dot{z}(t) = L(t,x)(z_t,h^\theta,h^\xi) \qquad \text{a.e. } t\in[0,\alpha], \tag{2.3.13}$$

$$z(t) = h^\varphi(t), \qquad t\in[-r,0], \tag{2.3.14}$$

where $h=(h^\varphi,h^\theta,h^\xi)\in C\times\Theta\times\Xi$ is fixed. The IVP (2.3.13)-(2.3.14) is a Carathéodory type linear delay equation. By its solution we mean a continuous function $z\colon [-r,\alpha]\to\mathbb{R}^n$, which is absolutely continuous on $[0,\alpha]$, and it satisfies (2.3.13) for a.e. $t\in[0,\alpha]$ and (2.3.14) for all $t\in[-r,0]$. Standard argument ([22], [43]) shows that the IVP (2.3.13)-(2.3.14) has a unique solution $z(t)=z(t,\gamma,h)$ for $t\in[-r,\alpha]$, $\gamma\in P_2$ and $h=(h^\varphi,h^\theta,h^\xi)\in C\times\Theta\times\Xi$.

The following result was proved in [50] for the parameter set P_1 (see Lemma 4.4 in [50]), but the proof is identical for the parameter set P_2, as well.

Lemma 2.3.3 (see [50]) *Assume (A1) (i)–(iii), (A2) (i)–(iii). Let $\gamma\in P_2$, and $x(t):=x(t,\gamma)$ for $t\in[-r,\alpha]$. Let $h\in C\times\Theta\times\Xi$ and let $z(t,\gamma,h)$ be the corresponding solution of the IVP* (2.3.13)-(2.3.14) *on $[-r,\alpha]$. Then*

(i) $z(t,\gamma,\cdot)\in\mathcal{L}(C\times\Theta\times\Xi,\mathbb{R}^n)$, the map $C\times\Theta\times\Xi\to C$, $h\mapsto z_t(\cdot,\gamma,h)$ is in $\mathcal{L}(C\times\Theta\times\Xi,C)$, and

$$|z(t,\gamma,h)|\le|z_t(\cdot,\gamma,h)|_C\le N_1|h|_{C\times\Theta\times\Xi}, \qquad t\in[0,\alpha],\ \gamma\in P_2,\ h\in C\times\Theta\times\Xi, \tag{2.3.15}$$

where $N_1:=e^{L_1N_0\alpha}$;

(ii) there exists $N_2\ge 0$ such that

$$|z_t(\cdot,\gamma,h)|_{W^{1,\infty}}\le N_2|h|_\Gamma, \qquad t\in[0,\alpha],\ \gamma\in P_2,\ h\in\Gamma. \tag{2.3.16}$$

Next we show that the linear operators $z(t,\gamma,\cdot)$ and $z_t(\cdot,\gamma,\cdot)$ are continuous in t and γ, assuming that γ belongs to P_2. First we need the following result.

Lemma 2.3.4 *Assume (A1) (i)–(iii), (A2) (i)–(v). Let $\gamma\in P_2$, $h=(h^\varphi,h^\theta,h^\xi)\in\Gamma$, $h_k=(h_k^\varphi,h_k^\theta,h_k^\xi)\in\Gamma$ $(k\in\mathbb{N})$ be a sequence such that $|h_k|_\Gamma\to 0$ as $k\to\infty$, and $\gamma+h_k\in P_2$ for $k\in\mathbb{N}$. Let $x(s):=x(s,\gamma)$, $x^k(s):=x(s,\gamma+h_k)$, $u(s):=s-\tau(s,x_s,\xi)$,* *and* $u^k(s) := s - \tau(s, x^k_s, \xi + h^\xi_k)$. *Then there exists a nonnegative sequence* $c_{0,k}$ *such that* $c_{0,k} \to 0$ *as* $k \to \infty$, *and*

$$|L(s,x^k)h - L(s,x)h| \le c_{0,k}|h|_\Gamma + L_1 L_2 |\dot{x}(u^k(s)) - \dot{x}(u(s))||h|_\Gamma \tag{2.3.17}$$

for a.e. $s \in [0,\alpha]$, $k \in \mathbb{N}$ *and* $h \in \Gamma$.

Proof We have

$$\begin{aligned}
&L(s,x^k)(h^\varphi,h^\theta,h^\xi) - L(s,x)(h^\varphi,h^\theta,h^\xi)\\
&\quad= \Big(D_2 f(s,x^k_s,x^k(u^k(s)),\theta+h^\theta_k) - D_2 f(s,x_s,x(u(s)),\theta)\Big)h^\varphi\\
&\qquad+\Big(D_3 f(s,x^k_s,x^k(u^k(s)),\theta+h^\theta_k) - D_3 f(s,x_s,x(u(s)),\theta)\Big)\\
&\qquad\quad\times\Big(-\dot{x}^k(u^k(s))\Big)\Big(D_2\tau(s,x^k_s,\xi+h^\xi_k)h^\varphi + D_3\tau(s,x^k_s,\xi+h^\xi_k)h^\xi\Big)\\
&\qquad+D_3 f(s,x_s,x(u(s)),\theta)\Big(-\dot{x}^k(u^k(s)) + \dot{x}(u^k(s)))\Big)\\
&\qquad\quad\times\Big(D_2\tau(s,x^k_s,\xi+h^\xi_k)h^\varphi + D_3\tau(s,x^k_s,\xi+h^\xi_k)h^\xi\Big)\\
&\qquad+D_3 f(s,x_s,x(u(s)),\theta)\Big(-\dot{x}(u^k(s)) + \dot{x}(u(s)))\Big)\\
&\qquad\quad\times\Big(D_2\tau(s,x^k_s,\xi+h^\xi_k)h^\varphi + D_3\tau(s,x^k_s,\xi+h^\xi_k)h^\xi\Big)\\
&\qquad+D_3 f(s,x_s,x(u(s)),\theta)\Big(-\dot{x}(u(s))\Big)\\
&\qquad\quad\times\Big[\Big(D_2\tau(s,x^k_s,\xi+h^\xi_k) - D_2\tau(s,x_s,\xi)\Big)h^\varphi\\
&\qquad\qquad+\Big(D_3\tau(s,x^k_s,\xi+h^\xi_k) - D_3\tau(s,x_s,\xi)\Big)h^\xi\Big]\\
&\qquad+\Big(D_3 f(s,x^k_s,x^k(u^k(s)),\theta+h^\theta_k) - D_3 f(s,x_s,x(u(s)),\theta)\Big)h^\varphi(-\tau(s,x^k_s,\xi+h^\xi_k))\\
&\qquad+D_3 f(s,x_s,x(u(s)),\theta)\Big(h^\varphi(-\tau(s,x^k_s,\xi+h^\xi_k)) - h^\varphi(-\tau(s,x_s,\xi))\Big)\\
&\qquad+\Big(D_4 f(s,x^k_s,x^k(u^k(s)),\theta+h^\theta_k) - D_4 f(s,x_s,x(u(s)),\theta)\Big)h^\theta, \qquad s\in[0,\alpha].
\end{aligned}$$

Relations (2.2.1), (2.2.2), (2.2.4) and the Mean Value Theorem give

$$\begin{aligned}
|x^k(u^k(s)) - x(u(s))| &\le |x^k(u^k(s)) - x(u^k(s))| + |x(u^k(s)) - x(u(s))|\\
&\le L|h_k|_\Gamma + N|u^k(s) - u(s)|\\
&\le K_2|h_k|_\Gamma,
\end{aligned}\tag{2.3.18}$$

with $K_2 := L + NK_0$,

$$|x^k_s - x_s|_C + |x^k(u^k(s)) - x(u(s))| + |h^\theta_k|_\Theta \le K_3|h_k|_\Gamma, \tag{2.3.19}$$

with $K_3 := L + K_2 + 1$, and

$$|x_s^k - x_s|_C + |h_k^\xi|_\Xi \le (L+1)|h_k|_\Gamma. \tag{2.3.20}$$

Combining the above estimates with (A1) (ii), (A2) (ii), (2.2.1), (2.2.2), (2.2.4) and the definition of Ω_f and Ω_τ we get

$$\begin{aligned}
&|L(s,x^k)(h^\varphi,h^\theta,h^\xi) - L(s,x)(h^\varphi,h^\theta,h^\xi)| \\
&\quad\le\ \Omega_f\Big(K_3|h_k|_\Gamma\Big)|h^\varphi|_C + \Omega_f\Big(K_3|h_k|_\Gamma\Big)NL_2(|h^\varphi|_C + |h^\xi|_\Xi) \\
&\qquad + L_1L|h_k|_\Gamma L_2(|h^\varphi|_C + |h^\xi|_\Xi) + L_1\Big|\dot{x}(u^k(s)) - \dot{x}(u(s))\Big|L_2(|h^\varphi|_C + |h^\xi|_\Xi) \\
&\qquad + L_1N\Omega_\tau\Big((L+1)|h_k|_\Gamma)\Big)(|h^\varphi|_C + |h^\xi|_\Xi) + \Omega_f\Big(K_3|h_k|_\Gamma\Big)|h^\varphi|_C \\
&\qquad + L_1|\dot{h}^\varphi|_{L^\infty}K_0|h_k|_\Gamma + \Omega_f\Big(K_3|h_k|_\Gamma\Big)|h^\theta|_\Theta, \qquad s\in[0,\alpha],
\end{aligned}$$

which yields (2.3.17) with $c_{0,k} := N_0\Omega_f\Big(K_3|h_k|_\Gamma\Big) + L_1L_2L|h_k|_\Gamma + L_1N\Omega_\tau\Big((L+1)|h_k|_\Gamma\Big) + L_1K_0|h_k|_\Gamma$, where N_0 is defined by (2.3.12). □

Lemma 2.3.5 *Assume (A1) (i)–(iii), (A2) (i)–(v). Let $\gamma\in P_2$, and $x(t) := x(t,\gamma)$ for $t\in[-r,\alpha]$. Let $h\in C\times\Omega\times\Xi$ and let $z(t,\gamma,h)$ be the corresponding solution of the IVP* (2.3.13)-(2.3.14) *on $[-r,\alpha]$. Then the maps*

$$\mathbb{R}\times\Gamma\supset[0,\alpha]\times P_2\to\mathcal{L}(\Gamma,\mathbb{R}^n),\quad (t,\gamma)\mapsto z(t,\gamma,\cdot)$$

and

$$\mathbb{R}\times\Gamma\supset[0,\alpha]\times P_2\to\mathcal{L}(\Gamma,C),\quad (t,\gamma)\mapsto z_t(\cdot,\gamma,\cdot)$$

are continuous.

Proof Let $\gamma\in P_2$ be fixed, and let $h_k = (h_k^\varphi,h_k^\theta,h_k^\xi)\in\Gamma$ ($k\in\mathbb{N}$) be a sequence such that $|h_k|_\Gamma\to 0$ as $k\to\infty$ and $\gamma+h_k\in P_2$ for $k\in\mathbb{N}$. For a fixed $h=(h^\varphi,h^\theta,h^\xi)\in\Gamma$ we define the short notations $x^k(t) := x(t,\gamma+h_k)$, $x(t) := x(t,\gamma)$, $u^k(t) := t-\tau(t,x_t^k,\xi+h_k^\xi)$, $u(t) := t-\tau(t,x_t,\xi)$, $z^{k,h}(t) := z(t,\gamma+h_k,h)$ and $z^h(t) := z(t,\gamma,h)$. The functions $z^{k,h}$ and z^h satisfy

$$\begin{aligned}
z^{k,h}(t) &= h^\varphi(0) + \int_0^t L(s,x^k)(z_s^{k,h},h^\theta,h^\xi)\,ds, \qquad t\in[0,\alpha],\\
z^h(t) &= h^\varphi(0) + \int_0^t L(s,x)(z_s^h,h^\theta,h^\xi)\,ds, \qquad t\in[0,\alpha],
\end{aligned}$$

and therefore for $t\in[0,\alpha]$

$$|z^{k,h}(t)-z^h(t)| \le \int_0^t\Big|\Big(L(s,x^k)-L(s,x)\Big)(z_s^h,h^\theta,h^\xi)+L(s,x^k)(z_s^{k,h}-z_s^h,0,0)\Big|\,ds. \tag{2.3.21}$$

We have by (2.3.16) and $N_2 \ge 1$

$$|(z_s^h, h^\theta, h^\xi)|_\Gamma \le N_2|h|_\Gamma + |h^\theta|_\Theta + |h^\xi|_\Xi \le (N_2+1)|h|_\Gamma. \tag{2.3.22}$$

Then (2.3.11), (2.3.17), (2.3.21) and (2.3.22) imply

$$|z^{k,h}(t) - z^h(t)| \le c_{1,k}|h|_\Gamma + \int_0^t L_1 N_0 |z_s^{k,h} - z_s^h|_C\, ds, \qquad t \in [0,\alpha], \tag{2.3.23}$$

where $c_{1,k}$ is defined by

$$c_{1,k} \;:=\; \alpha c_{0,k}(N_2+1) + L_1 L_2 (N_2+1) \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds.$$

Lemmas 1.2.11 and 2.2.3 yield that $\int_0^{\alpha^*} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds \to 0$ as $k \to \infty$. If $\alpha^* < \alpha$, then define

$$\Omega_x(\varepsilon) := \max\Big\{|\dot{x}(s) - \dot{x}(\bar{s})|\colon\ |s - \bar{s}| \le \varepsilon,\ s, \bar{s} \in [0,\alpha]\Big\}.$$

The continuity of $\dot{x}$ on $[0,\alpha]$ yields $\Omega_x(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore

$$\int_{\alpha^*}^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds \le \Omega_x(K_0|h_k|_\Gamma)\alpha \to 0, \qquad k \to \infty,$$

and so

$$\lim_{k\to\infty} \int_\alpha^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds = 0. \tag{2.3.24}$$

Hence $c_{1,k} \to 0$ as $k \to \infty$.

Lemma 1.2.1 is applicable for (2.3.23), since $|z_0^{k,h} - z_0^h|_C = 0$, and it gives

$$|z^{k,h}(t) - z^h(t)| \le |z_t^{k,h} - z_t^h|_C \le c_{1,k} N_1 |h|_\Gamma, \qquad t \in [0,\alpha], \tag{2.3.25}$$

where $N_1 := e^{L_1 N_0 \alpha}$. Therefore we get for $t \in [0,\alpha]$

$$|z(t,\gamma+h_k,\cdot) - z(t,\gamma,\cdot)|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)} \le |z_t(\cdot,\gamma+h_k,\cdot) - z_t(\cdot,\gamma,\cdot)|_{\mathcal{L}(W^{1,\infty},C)} \le c_{1,k}N_1 \tag{2.3.26}$$

for all $k \in \mathbb{N}$.

Let $t \in [0,\alpha]$ be fixed, and let ν_k be a sequence of real numbers such that $t + \nu_k \in [0,\alpha]$ for $k \in \mathbb{N}$ and $\nu_k \to 0$ as $k \to \infty$. Then (2.3.16) and the Mean Value Theorem yield

$$|z_{t+\nu_k}(\cdot,\gamma+h_k,\cdot) - z_t(\cdot,\gamma+h_k,\cdot)|_{\mathcal{L}(\Gamma,C)} \le N_2|\nu_k|, \qquad k \ge k_0.$$

Combining this relation with (2.3.26) and $c_{1,k} \to 0$ we get

$$\begin{aligned}
&|z(t+\nu_k,\gamma+h_k,\cdot) - z(t,\gamma,\cdot)|_{\mathcal{L}(\Gamma,\mathbb{R}^n)}\\
&\quad\le\ |z_{t+\nu_k}(\cdot,\gamma+h_k,\cdot) - z_t(\cdot,\gamma,\cdot)|_{\mathcal{L}(\Gamma,C)}\\
&\quad\le\ |z_{t+\nu_k}(\cdot,\gamma+h_k,\cdot) - z_t(\cdot,\gamma+h_k,\cdot)|_{\mathcal{L}(\Gamma,C)} + |z_t(\cdot,\gamma+h_k,\cdot) - z_t(\cdot,\gamma,\cdot)|_{\mathcal{L}(\Gamma,C)}\\
&\quad\le\ N_2|\nu_k| + c_{1,k}N_1\\
&\quad\to\ 0, \qquad \text{as } k \to \infty.
\end{aligned}$$

This completes the proof. $\square$

In Lemma 2.3.8 below we will show that under additional conditions, the function $\gamma \mapsto z(t,\gamma,\cdot)$ is Lipschitz continuous. To obtain this higher smoothness first consider the next lemma.

Lemma 2.3.6 *Assume (A1) (i)–(iv), (A2) (i)–(iv) and $\gamma = (\varphi,\theta,\xi) \in P$ is such that $\varphi \in W^{2,\infty}$. Then there exists $K_4 = K_4(\gamma) \geq 0$ such that the solution $x(t) = x(t,\gamma)$ of the IVP* (2.1.1)-(2.1.2) *satisfies*

$$|\dot{x}(t) - \dot{x}(\bar{t})| \leq K_4|t-\bar{t}| \qquad \text{for } t,\bar{t} \in [-r,0) \quad \text{and} \quad t,\bar{t} \in (0,\alpha]. \tag{2.3.27}$$

Moreover, if in addition $\gamma \in \mathcal{P}$, then $x \in W^{2,\infty}([-r,\alpha],\mathbb{R}^n)$, and

$$|\dot{x}(t) - \dot{x}(\bar{t})| \leq K_4|t-\bar{t}| \qquad \text{for } t,\bar{t} \in [-r,\alpha]. \tag{2.3.28}$$

Proof The Mean Value Theorem and the definition of the $W^{2,\infty}$-norm yield

$$|\dot{x}(t) - \dot{x}(\bar{t})| = |\dot{\varphi}(t) - \dot{\varphi}(\bar{t})| \leq |\varphi|_{W^{2,\infty}}|t-\bar{t}|, \qquad t,\bar{t} \in [-r,0).$$

For $t,\bar{t} \in (0,\alpha]$ it follows from (A1) (ii), (iv), (A2) (ii), (iv), (2.2.1) and (2.2.6) with $k=0$

$$\begin{aligned} |\dot{x}(t) - \dot{x}(\bar{t})| &= |f(t,x_t,x(u(t)),\theta) - f(\bar{t},x_{\bar{t}},x(u(\bar{t})),\theta)| \\ &\leq L_1\Big(|t-\bar{t}| + |x_t - x_{\bar{t}}|_C + |x(u(t)) - x(u(\bar{t}))|\Big) \\ &\leq L_1\Big(1+N+NL_2(1+N)\Big)|t-\bar{t}|. \end{aligned}$$

Hence (2.3.27) is satisfied with $K_4 := \max\{|\varphi|_{W^{2,\infty}}, L_1[1+N+NL_2(1+N)]\}$.

If $\gamma \in \mathcal{P}$, then $\dot{x}$ is continuous, and (2.3.27) yields that it is Lipschitz continuous on $[-r,\alpha]$ with the Lipschitz constant K_4, so, in particular, $x \in W^{2,\infty}([-r,\alpha],\mathbb{R}^n)$. □

We will need the following class of initial functions in the next lemma.

Definition 2.3.7 *Let $PW^{2,\infty}$ denote the set of functions $\varphi \in W^{1,\infty}$ which are piecewise $W^{2,\infty}$-functions, i.e., there exists a finite mesh $-r = t_0 < t_1 < \ldots < t_m = 0$ such that $\dot{\varphi}$ is Lipschitz continuous on the intervals (t_i,t_{i+1}) for $i = 0,\ldots,m-1$, and has continuous one-sided derivatives at t_i for $i = 0,\ldots,m$. We define a norm on $PW^{2,\infty}$ by*

$$|\varphi|_{PW^{2,\infty}} := \max\{|\varphi|_C, |\dot{\varphi}|_{L^\infty}, |\ddot{\varphi}|_{L^\infty}\}.$$

Note that any function $\varphi \in PW^{2,\infty}$ is almost everywhere differentiable and twice differentiable, but both $\dot{\varphi}$ and $\ddot{\varphi}$ may have discontinuity at the mesh points. A typical example of a $PW^{2,\infty}$-function is a spline function defined on $[-r,0]$.

The next lemma gives sufficient conditions under the solutions of the IVP (2.3.13)-(2.3.14) depend Lipschitz continuously on the parameters. This result will be essential to prove the convergence of the quasilinearization sequence in Chapter 3.

Lemma 2.3.8 *Assume (A1) (i)–(v), (A2) (i)–(vi), and $\gamma^* = (\varphi^*, \theta^*, \xi^*) \in P_1$. Then there exists $\delta^* > 0$ such that for every $m \in \mathbb{N}$ and $K \geq 0$ there exists a nonnegative constant $N_3 = N_3(\gamma^*, \delta^*, m, K)$ such that for every $\gamma = (\varphi, \theta, \xi) \in \mathcal{B}_\Gamma(\gamma^*; \delta^*)$ satisfying $\varphi \in PW^{2,\infty}$ with $|\varphi|_{PW^{2,\infty}} \leq K$, and the number of points of discontinuity of $\dot\varphi$ and $\ddot\varphi$ in $(-r, 0)$ is less or equal to m, there exists $\delta > 0$ such that for every sequence $h_k \in \Gamma$ with $|h_k|_\Gamma \leq \delta$ for $k \in \mathbb{N}$ and all $h \in \Gamma$ the functions $z^{k,h}(t) := z(t, \gamma + h_k, h)$ and $z^h(t) := z(t, \gamma, h)$ satisfy*

$$|z^{k,h}(t) - z^h(t)| \leq |z^{k,h}_t - z^h_t|_C \leq N_3 |h_k|_\Gamma |h|_\Gamma, \qquad t \in [0, \alpha], \quad h \in \Gamma. \tag{2.3.29}$$

Proof Since P_1 is an open subset of P (see [58] and [50]), there exists a $\delta_0 > 0$ such that $\mathcal{B}_\Gamma(\gamma^*; \delta_0) \subset P_1$. For a fixed $\gamma \in \mathcal{B}_\Gamma(\gamma^*; \delta_0)$ we define $x(t) := x(t, \gamma)$, $x^*(t) := x(t, \gamma^*)$, $u(t) := t - \tau(t, x_t, \xi)$ and $u^*(t) := t - \tau(t, x^*_t, \xi^*)$. Introduce

$$M^* := \min\Big\{ \operatorname*{ess\,inf}_{s \in [0, \alpha^*]} \dot u^*(s),\ 1 \Big\}.$$

Then $\gamma^* \in P_1$ yields $M^* > 0$, and u^* is strictly monotone increasing on $[0, \alpha^*]$. Let $0 < M < M^*$ be fixed. It follows from Lemma 2.2.3 that there exists $0 < \delta^* \leq \delta_0$ such that if $\gamma \in \mathcal{B}_\Gamma(\gamma^*; \delta^*)$, then $\dot u(s) \geq M$ for a.e. $s \in [0, \alpha^*]$, and, in particular, u is also strictly monotone increasing on $[0, \alpha^*]$.

Fix $m \in \mathbb{N}$ and $K \geq 0$, and $\gamma = (\varphi, \theta, \xi) \in \mathcal{B}_\Gamma(\gamma^*; \delta^*)$ be fixed such that $\varphi \in PW^{2,\infty}$, $|\varphi|_{PW^{2,\infty}} \leq K$, and the points of discontinuity of φ in $(-r, 0)$ is less or equal to m. Let $\delta_1 \geq 0$ be such that $\mathcal{B}_\Gamma(\gamma; \delta_1) \subset \mathcal{B}_\Gamma(\gamma^*; \delta^*)$, and let $h_k \in \Gamma$ ($k \in \mathbb{N}$) be a sequence satisfying $|h_k|_\Gamma \leq \delta_1$ for $k \in \mathbb{N}$. Let $x^k(t) := x(t, \gamma + h_k)$ and $u^k(t) := t - \tau(t, x^k_t, \xi + h^\xi_k)$. Let $-r < t_1 < \cdots < t_\ell < 0$ be the points of discontinuity of φ (from Definition 2.3.7), and define $t_0 := -r$ and $t_{\ell+1} := 0$. Then by the assumption on γ we have $\ell \leq m$.

It follows easily from the proof of Lemma 2.3.6 that $K^*_4 := \max\{K, L_1[1 + N + NL_2(1 + N)]\}$ satisfies

$$|\dot x(t) - \dot x(\bar t)| \leq K^*_4 |t - \bar t| \qquad \text{for } t, \bar t \in (t_i, t_{i+1}), \quad i = 0, \ldots, \ell, \quad t, \bar t \in (0, \alpha) \tag{2.3.30}$$

and for all $\gamma \in \mathcal{B}_\Gamma(\gamma^*; \delta^*) \cap \mathcal{B}_{PW^{2,\infty}}(0; K)$.

Let $\varepsilon_0 := \min\{t_{i+1} - t_i \colon i = 0, \ldots, \ell\}$. Let $\delta_2 := \min\Big\{\delta_1, \frac{M\varepsilon_0}{K_0}\Big\}$. Then if $|h_k|_\Gamma < \delta_2$ for all $k \in \mathbb{N}$, then by (2.2.4) we have

$$|u^k(s) - u(s)| \leq K_0 |h_k|_\Gamma \leq M\varepsilon_0 \leq \varepsilon_0, \qquad k \in \mathbb{N}, \quad s \in [0, \alpha^*]. \tag{2.3.31}$$

Since $u(0) \leq 0$, there exist $s_i \in [0, \alpha^*]$ and $j \in \{0, 1, \ldots, \ell + 1\}$ such that $u(s_i) = t_i$ for $i = j, \ldots, \ell + 1$. By the strict monotonicity of u we have $0 \leq s_j < \cdots < s_{\ell+1} \leq \alpha^*$. Similarly, let $s_{k,i}$ and j_k be such that $u^k(s_{k,i}) = t_i$ for $i = j_k, \ldots, \ell + 1$, $k \in \mathbb{N}$. We again have $0 \leq s_{k,j_k} < \cdots < s_{k,\ell+1} \leq \alpha^*$.

Next we show that if $|h_k|_\Gamma < \delta_2$ for $k \in \mathbb{N}$, then

$$|s_{k,i} - s_i| \le \frac{K_0}{M}|h_k|_\Gamma \le \varepsilon_0, \qquad i = \max(j, j_k), \dots, \ell + 1, \quad k \in \mathbb{N}. \tag{2.3.32}$$

First consider the case when $s_{k,i} \ge s_i$ for some $i \in \{\max(j, j_k), \dots, \ell+1\}$ and $k \in \mathbb{N}$. The definitions of M, δ^*, δ_1, δ_2, s_i and $s_{k,i}$ and (2.3.31) imply

$$M(s_{k,i} - s_i) \le u(s_{k,i}) - u(s_i) = u(s_{k,i}) - u^k(s_{k,i}) \le K_0|h_k|_\Gamma \le M\varepsilon_0, \qquad k \in \mathbb{N}$$

for all $i = \max(j, j_k), \dots, \ell+1$. We have then $0 \le s_{k,i} - s_i \le \varepsilon_0$. In the opposite case when $s_{k,i} < s_i$ we get the same way that $0 \le s_i - s_{k,i} \le \frac{K_0}{M}|h_k|_\Gamma \le \varepsilon_0$, which yields (2.3.32).

We distinguish 3 cases. Case (1): If $j = 0$, then $s_j = 0$, moreover, $j_k = 0$ and $s_{k,j_k} = 0$ for $u^k(0) = 0$, and $j_k = 1$ and $s_{k,j_k} > 0$ for $u^k(0) > -r$. Case (2): If $s_j = 0$ and $j > 0$, then $u(0) = t_j$, moreover, $j_k = j+1$ and $s_{k,j+1} > 0$ for $u^k(0) > u(0)$, and $j_k = j$ and $s_{k,j} \ge 0$ for $u^k(0) \le u(0)$. Case (3): Consider the case when $s_j > 0$ and $j > 0$. Then $t_{j-1} < u(0) < t_j$, and let $\Delta := \min(u(0) - t_{j-1}, t_j - u(0))$ and $\delta_3 := \min\{\delta_2, \frac{\Delta}{K_0}\}$. Then if $|h_k|_\Gamma < \delta_3$ for all $k \in \mathbb{N}$, then $|u^k(s) - u(s)| \le K_0|h_k|_\Gamma < \Delta$ for s close to 0, and hence $j_k = j$, and $u^k(s), u(s) \in (t_{j-1}, t_j)$ for $0 \le s < \min(s_j, s_{k,j})$, and $t_{j-1} < u^k(s) < t_j < u(s)$ for $s \in (\min(s_j, s_{k,j}), \max(s_j, s_{k,j}))$.

Now we consider Case (3) above. Suppose $|h_k|_\Gamma < \delta_3$ for all $k \in \mathbb{N}$. Define $a_{k,i} := \min(s_i, s_{k,i})$ and $b_{k,i} := \max(s_i, s_{k,i})$ for $i = j, \dots, \ell+1$. Then for $i = j, \dots, \ell$ and $k \in \mathbb{N}$ we have

$$b_{k,i} - a_{k,i} = |s_i - s_{k,i}| \le \frac{K_0}{M}|h_k|_\Gamma, \tag{2.3.33}$$

$b_{k,i} < a_{k,i+1}$, and $u(s), u^k(s) \in (t_i, t_{i+1})$ for $s \in (b_{k,i}, a_{k,i+1})$. For definiteness suppose $(a_{k,i}, b_{k,i}) = (s_i, s_{k,i})$ (the opposite case is similar). Then for $s \in (a_{k,i}, b_{k,i})$ we have $u(s) \in (t_i, t_{i+1})$ and $u^k(s) \in (t_{i-1}, t_i)$. Therefore (2.3.30) and (2.2.4) imply

$$\begin{aligned}
|\dot{x}(u(s)) - \dot{x}(u^k(s))| &\le |\dot{x}(u(s)) - \dot{x}(t_i+)| + |\dot{x}(t_i+) - \dot{x}(t_i-)| + |\dot{x}(t_i-) - \dot{x}(u^k(s))| \\
&\le K_4^*(u(s) - t_i) + |\dot{x}(t_i+) - \dot{x}(t_i-)| + K_4^*(t_i - u^k(s)) \\
&\le K_4^*|u(s) - u^k(s)| + |\dot{x}(t_i+) - \dot{x}(t_i-)| \\
&\le K_4^* K_0 |h_k|_\Gamma + |\dot{x}(t_i+) - \dot{x}(t_i-)|.
\end{aligned} \tag{2.3.34}$$

Then (A1) (ii), (2.2.2) and (2.3.18) give for $t \in [0, \alpha]$

$$\begin{aligned}
|\dot{x}(t)| &\le |f(t, x_t, x(u(t)), \theta) - f(t, x_t^*, x^*(u^*(t)), \theta^*)| + |f(t, x_t^*, x^*(u^*(t)), \theta^*)| \\
&\le L_1(|x_t - x_t^*|_C + |x(u(t)) - x^*(u^*(t))| + |\theta - \theta^*|_\Theta) + \max_{t\in[0,\alpha]} |f(t, x_t^*, x^*(u^*(t)), \theta^*)| \\
&\le L_1(L + K_2 + 1)|\gamma - \gamma^*|_\Gamma + \max_{t\in[0,\alpha]} |f(t, x_t^*, x^*(u^*(t)), \theta^*)| \\
&\le \widehat{K},
\end{aligned}$$

where $\widehat{K} := L_1(L + K_2 + 1)\delta^* + \max_{t\in[0,\alpha]} |f(t, x_t^*, x^*u^*(t)), \theta^*)|$. Then, in particular, $|\dot{x}(0+)| \le \widehat{K}$ for all $\gamma \in \mathcal{B}_\Gamma(\gamma^*; \delta^*)$, and so (2.3.34) yields for all $i = j, \ldots, \ell$ and $k \in \mathbb{N}$

$$|\dot{x}(u(s)) - \dot{x}(u^k(s))| \le K_4^* K_0 |h_k|_\Gamma + 2K^*, \qquad s \in (a_{k,i}, b_{k,i}), \tag{2.3.35}$$

where $K^* := \max\{K, \widehat{K}\}$. Note that it is easy to check that (2.3.35) holds for the case $(a_{k,i}, b_{k,i}) = (s_{k,i}, s_i)$, too.

Therefore by (2.2.4), (2.3.30), (2.3.33), (2.3.35) and $\ell \le m$ we have

$$\begin{aligned}
&\int_0^{\alpha^*} |\dot{x}(u(s)) - \dot{x}(u^k(s))|\,ds \\
&\quad = \int_0^{a_{k,j}} |\dot{x}(u(s)) - \dot{x}(u^k(s))|\,ds + \sum_{i=j}^{\ell} \int_{a_{k,i}}^{b_{k,i}} |\dot{x}(u(s)) - \dot{x}(u^k(s))|\,ds \\
&\qquad + \sum_{i=j}^{\ell} \int_{b_{k,i}}^{a_{k,i+1}} |\dot{x}(u(s)) - \dot{x}(u^k(s))|\,ds + \int_{b_{k,\ell+1}}^{\alpha^*} |\dot{x}(u(s)) - \dot{x}(u^k(s))|\,ds \\
&\quad \le a_{k,j} K_4^* K_0 |h_k|_\Gamma + \sum_{i=j}^{\ell} (b_{k,i} - a_{k,i}) K_4^* K_0 |h_k|_\Gamma + \sum_{i=j}^{\ell} (b_{k,i} - a_{k,i}) 2K^* \\
&\qquad + \sum_{i=j}^{\ell} (a_{k,i+1} - b_{k,i}) K_4^* K_0 |h_k|_\Gamma + (\alpha^* - b_{k,\ell+1}) K_4^* K_0 |h_k|_\Gamma \\
&\quad \le \Big(\alpha^* K_4^* K_0 + m \frac{K_0}{M} 2K^*\Big) |h_k|_\Gamma.
\end{aligned} \tag{2.3.36}$$

Ineqauality (2.3.36) can be obtained similarly for the Cases (1) and (2).

Assumptions (A1) (v) and (A2) (vi) imply that $\Omega_f(\varepsilon) \le L_3\varepsilon$ and $\Omega_\tau(\varepsilon) \le L_5\varepsilon$ for $\varepsilon \ge 0$ with $L_3 = L_3(\alpha, M_1, M_2, M_3)$ and $L_5 = L_5(\alpha, M_1, M_4)$. Therefore the definition of $c_{0,k}$, $c_{1,k}$ and (2.3.36) yield the existence of an $L^* \ge 0$ such that $c_{1,k} \le L^* |h_k|_\Gamma$ for all h_k satisfying $|h_k|_\Gamma < \delta$ for some $\delta > 0$. Then (2.3.29) follows from (2.3.25) with $N_3 := L^* N_1$. $\square$

Now we are ready to prove the Fréchet-differentiability of the function $x(t, \gamma)$ wrt γ. We will denote this derivative by $D_2 x(t, \gamma)$.

Theorem 2.3.9 *Assume (A1) (i)–(iii), (A2) (i)–(v), and let P_2 be defined by* (2.3.9). *Then the functions*

$$\mathbb{R} \times \Gamma \supset [0, \alpha] \times P \to \mathbb{R}^n, \qquad (t, \gamma) \mapsto x(t, \gamma)$$

and

$$\mathbb{R} \times \Gamma \supset [0, \alpha] \times P \to C, \qquad (t, \gamma) \mapsto x_t(\cdot, \gamma)$$

are both differentiable wrt γ *for every* $\gamma \in P_2$*, and*

$$D_2 x(t,\gamma)h = z(t,\gamma,h), \qquad h \in \Gamma,\ t \in [0,\alpha],\ \gamma \in P_2, \tag{2.3.37}$$

and

$$D_2 x_t(\cdot,\gamma)h = z_t(\cdot,\gamma,h), \qquad h \in \Gamma,\ t \in [0,\alpha],\ \gamma \in P_2, \tag{2.3.38}$$

where $z(t,\gamma,h)$ *is the solution of the IVP* (2.3.13)-(2.3.14) *for* $t \in [0,\alpha]$, $\gamma \in P_2$ *and* $h \in \Gamma$. *Moreover, the functions*

$$\mathbb{R}\times\Gamma \supset [0,\alpha]\times P_2 \to \mathcal{L}(\Gamma,\mathbb{R}^n), \qquad (t,\gamma) \mapsto D_2 x(t,\gamma)$$

and

$$\mathbb{R}\times\Gamma \supset [0,\alpha]\times P_2 \to \mathcal{L}(\Gamma,C), \qquad (t,\gamma) \mapsto D_2 x_t(\cdot,\gamma)$$

are continuous.

Proof Let $\gamma = (\varphi,\theta,\xi) \in P_2$ be fixed, and let $h_k = (h_k^\varphi, h_k^\theta, h_k^\xi) \in \Gamma$ ($k \in \mathbb{N}$) be a sequence with $|h_k|_\Gamma \to 0$ as $k \to \infty$ and $\gamma + h_k \in P$ for $k \in \mathbb{N}$. To simplify notation, let $x^k(t) := x(t,\gamma+h_k)$, $x(t) := x(t,\gamma)$, $u(s) := s - \tau(s,x_s,\xi)$, $u^k(s) := s - \tau(s,x_s^k,\xi+h_k^\xi)$ and $z^{h_k}(t) := z(t,\gamma,h_k)$. Then

$$\begin{aligned} x^k(t) &= \varphi(0) + h_k^\varphi(0) + \int_0^t f(s,x_s^k,x^k(u^k(s)),\theta+h_k^\theta)\,ds, \qquad t \in [0,\alpha],\\ x(t) &= \varphi(0) + \int_0^t f(s,x_s,x(u(s)),\theta)\,ds, \qquad t \in [0,\alpha], \end{aligned}$$

and

$$z^{h_k}(t) = h_k^\varphi(0) + \int_0^t L(s,x)(z_s^{h_k},h_k^\theta,h_k^\xi)\,ds, \qquad t \in [0,\alpha].$$

We have

$$\begin{aligned} x^k(t) - x(t) - z^{h_k}(t) &= \int_0^t \Big(f(s,x_s^k,x^k(u^k(s)),\theta+h_k^\theta) - f(s,x_s,x(u(s)),\theta)\\ &\qquad - L(s,x)(z_s^{h_k},h_k^\theta,h_k^\xi)\Big)ds. \end{aligned} \tag{2.3.39}$$

The definitions of ω_f and $L(s,x)$ (see (2.3.3) and (2.3.10), respectively) yield for $s \in [0,\alpha]$

$$\begin{aligned} &f(s,x_s^k,x^k(u^k(s)),\theta+h_k^\theta) - f(s,x_s,x(u(s)),\theta) - L(s,x)(z_s^{h_k},h_k^\theta,h_k^\xi)\\ &\quad= D_2 f(s,x_s,x(u(s)),\theta)(x_s^k - x_s - z_s^{h_k}) + D_3 f(s,x_s,x(u(s)),\theta)\Big(x^k(u^k(s)) - x(u(s))\Big)\\ &\qquad + D_3 f(s,x_s,x(u(s)),\theta)\Big(\dot{x}(u(s))\Big(D_2\tau(s,x_s,\xi)z_s^{h_k} + D_3\tau(s,x_s,\xi)h_k^\xi\Big) - z^{h_k}(u(s))\Big)\\ &\qquad + \omega_f(s,x_s,x(u(s),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta). \end{aligned} \tag{2.3.40}$$

Relation (2.3.4) and simple manipulations give

$$\begin{aligned}
& x^k(u^k(s)) - x(u(s)) + \dot{x}(u(s))\Big(D_2\tau(s,x_s,\xi)z_s^{h_k} + D_3\tau(s,x_s,\xi)h_k^\xi\Big) - z^{h_k}(u(s)) \\
&\quad = x^k(u^k(s)) - x(u^k(s)) - z^{h_k}(u^k(s)) + x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s)) \\
&\qquad -\dot{x}(u(s))\omega_\tau(s,x_s,\xi,x_s^k,\xi+h_k^\xi) - \dot{x}(u(s))D_2\tau(s,x_s,\xi)(x_s^k - x_s - z_s^{h_k}) \\
&\qquad +z^{h_k}(u^k(s)) - z^{h_k}(u(s)).
\end{aligned} \tag{2.3.41}$$

Relation (2.2.4) and (2.3.16) imply

$$|z^{h_k}(u^k(s)) - z^{h_k}(u(s))| \le N_2|h_k|_\Gamma|u^k(s) - u(s)| \le N_2K_0|h_k|_\Gamma^2. \tag{2.3.42}$$

Using (2.2.1), (A1) (ii), (A2) (ii), and combining (2.3.39), (2.3.40), (2.3.41) and (2.3.42) we get

$$\begin{aligned}
& |x^k(t) - x(t) - z^{h_k}(t)| \\
&\quad \le \int_0^t \Big[L_1\Big(|x_s^k - x_s - z_s^{h_k}|_C + |x^k(u^k(s)) - x(u^k(s)) - z^{h_k}(u^k(s))| \\
&\qquad\quad + |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))| \\
&\qquad\quad + N|\omega_\tau(s,x_s,\xi,x_s^k,\xi+h_k^\xi)| + NL_2|x_s^k - x_s - z_s^{h_k}|_C + N_2K_0|h_k|_\Gamma^2\Big) \\
&\qquad\quad + |\omega_f(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta)|\Big]\,ds, \qquad t\in[0,\alpha].
\end{aligned} \tag{2.3.43}$$

Let N_0 be defined by (2.3.12). Then

$$|x^k(t)-x(t)-z^{h_k}(t)| \le a_k+b_k+c_k+d_k+L_1N_0\int_0^t |x_s^k-x_s-z_s^{h_k}|_C\,ds, \qquad t\in[0,\alpha], \tag{2.3.44}$$

where

$$a_k := \int_0^\alpha |\omega_f(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta)|\,ds, \tag{2.3.45}$$

$$b_k := L_1N\int_0^\alpha |\omega_\tau(s,x_s,\xi,s,x_s^k,\xi+h_k^\xi)|\,ds, \tag{2.3.46}$$

$$c_k := L_1\int_0^\alpha |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))|\,ds, \tag{2.3.47}$$

and

$$d_k := \alpha N_2K_0|h_k|_\Gamma^2. \tag{2.3.48}$$

Since $|x_0^k - x_0 - z_0|_C = 0$, Lemma 1.2.1 is applicable for (2.3.44), and it yields

$$|x^k(t) - x(t) - z^{h_k}(t)| \le |x_t^k - x_t - z_t|_C \le (a_k+b_k+c_k+d_k)N_1, \qquad t\in[0,\alpha], \tag{2.3.49}$$

where $N_1 := e^{L_1 N_0 \alpha}$, and hence

$$\frac{|x^k(t) - x(t) - z^{h_k}(t)|}{|h_k|_\Gamma} \le \frac{|x^k_t - x_t - z^{h_k}_t|_C}{|h_k|_\Gamma} \le \frac{a_k + b_k + c_k + d_k}{|h_k|_\Gamma} N_1, \quad t \in [0, \alpha], \tag{2.3.50}$$

which proves both (2.3.37) and (2.3.38), since Lemmas 2.3.1, 2.3.2 and (2.3.48) show that

$$\lim_{k\to\infty} \frac{a_k + b_k + c_k + d_k}{|h_k|_\Gamma} = 0. \tag{2.3.51}$$

The continuity of $D_2 x(t, \gamma)$ follows from Lemma 2.3.5. □

Remark 2.3.10 We comment that for $\gamma \in P_1$ the statements of Theorem 2.3.9 are valid without assumptions (A2) (iv) and (v), since they are needed only to prove (2.2.5), which is the key assumption of Lemma 1.2.11. If $\gamma \in P_1$, then both u and u^k are monotone increasing (for large enough k), so Lemma 1.2.6 can be used instead of Lemma 1.2.11. Also, continuous differentiability of x wrt the parameters holds in a neighborhood of γ, since P_1 is open in P. See Theorem 4.7 in [50] for a related result.

2.4 Second-order differentiability wrt the parameters

To obtain second-order differentiability wrt the parameters we need more smoothness of the initial functions. Therefore we introduce the parameter set

$$\Gamma_2 := W^{2,\infty} \times \Theta \times \Xi$$

equipped with the norm $|h|_{\Gamma_2} := |h^\varphi|_{W^{2,\infty}} + |h^\theta|_\Theta + |h^\xi|_\Xi$. We will show in Theorem 2.4.16 below that the parameter map

$$\Gamma_2 \supset (P_2 \cap \Gamma_2) \to \mathbb{R}^n, \qquad \gamma \to x(t,\gamma)$$

is twice differentiable at every point $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. The proof will be based on a sequence of Lemmas.

We assume throughout this section

(H) $\gamma = (\varphi,\theta,\xi) \in P_2 \cap \Gamma_2$, $h = (h^\varphi, h^\theta, h^\xi) \in \Gamma$, $h_k = (h_k^\varphi, h_k^\theta, h_k^\xi) \in \Gamma$ $(k \in \mathbb{N})$ are so that $|h_k|_\Gamma \to 0$ as $k \to \infty$, $\gamma + h_k \in P_2$ for $k \in \mathbb{N}$, and $|h_k|_\Gamma \neq 0$ for $k \in \mathbb{N}$. Let $x^k(t) := x(t, \gamma + h_k)$ and $x(t) := x(t,\gamma)$ be the solutions of the IVP (2.1.1)-(2.1.2), $z^{k,h}(t) := D_2 x(t, \gamma + h_k)h$ and $z^h(t) := D_2 x(t,\gamma)h$ be the solutions of the IVP (2.3.13)-(2.3.14).

The simplifying notations for $t \in [0,\alpha]$ and $k \in \mathbb{N}$

$$\begin{aligned}
u(t) &:= t - \tau(t, x_t, \xi),\\
u^k(t) &:= t - \tau(t, x_t^k, \xi + h_k^\xi),\\
\mathbf{v}(t) &:= (t, x_t, x(u(t)), \theta),\\
\mathbf{v}^k(t) &:= (t, x_t^k, x^k(u^k(t)), \theta),\\
A(t, h^\varphi, h^\xi) &:= D_2\tau(t, x_t, \xi)h^\varphi + D_3\tau(t, x_t, \xi)h^\xi,\\
A^k(t, h^\varphi, h^\xi) &:= D_2\tau(t, x_t^k, \xi + h_k^\xi)h^\varphi + D_3\tau(t, x_t^k, \xi + h_k^\xi)h^\xi,\\
E(t, h^\varphi, h^\xi) &:= -\dot{x}(u(t))A(t, h^\varphi, h^\xi) + h^\varphi(-\tau(t, x_t, \xi)), \quad \text{a.e. } t \in [0,\alpha],\\
E^k(t, h^\varphi, h^\xi) &:= -\dot{x}^k(u^k(t))A^k(t, h^\varphi, h^\xi) + h^\varphi(-\tau(t, x_t^k, \xi + h_k^\xi)), \quad \text{a.e. } t \in [0,\alpha],\\
F(t, h^\varphi, h^\xi) &:= -\ddot{x}(u(t))A(t, h^\varphi, h^\xi) + \dot{h}^\varphi(-\tau(t, x_t, \xi)), \quad \text{a.e. } t \in [0,\alpha],\\
F^k(t, h^\varphi, h^\xi) &:= -\ddot{x}^k(u^k(t))A^k(t, h^\varphi, h^\xi) + \dot{h}^\varphi(-\tau(t, x_t^k, \xi + h_k^\xi)), \quad \text{a.e. } t \in [0,\alpha]
\end{aligned}$$

will be used throughout this section. For simplicity of the notation we define $h_0 := 0 = (0,0,0) \in \Gamma$, and accordingly, $x^0 := x$, $u^0 := u$, $z^{0,h} := z^h$, $A^0 := A$, $E^0 := E$. Note that in all the above abbreviations the dependence on γ is omitted from the notation but it should be kept in mind. With these notations the operator $L(t,x)$ defined by (2.3.10) can be written shortly as

$$L(t,x)h = D_2 f(\mathbf{v}(t))h^\varphi + D_3 f(\mathbf{v}(t))E(t,h^\varphi,h^\xi) + D_4 f(\mathbf{v}(t))h^\theta.$$

Lemma 2.4.1 *Assume (A1) (i)–(iii), (A2) (i)–(v), and (H). Then*

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |\dot{x}^k(s) - \dot{x}(s) - \dot{z}^{h_k}(s)|\, ds = 0, \tag{2.4.1}$$

and

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |\dot{x}^k(u^k(s)) - \dot{x}(u^k(s)) - \dot{z}^{h_k}(u^k(s))|\, ds = 0. \tag{2.4.2}$$

Proof Using (2.3.39), (2.3.43), (2.3.44) and (2.3.49) we get

$$\begin{aligned}
&\int_0^\alpha |\dot{x}^k(s) - \dot{x}(s) - \dot{z}^{h_k}(s)|\, ds \\
&\le \int_0^\alpha \Big[L_1 \Big(|x_s^k - x_s - z_s^{h_k}|_C + |x^k(u^k(s)) - x(u^k(s)) - z^{h_k}(u^k(s))| \\
&\qquad + |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))| \\
&\qquad + N|\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi)| + NL_2 |x_s^k - x_s - z_s^{h_k}|_C + N_2 K_0 |h_k|_\Gamma^2 \Big) \\
&\qquad + |\omega_f(s, x_s, x(u(s)), \theta, x_s^k, x^k(u^k(s)), \theta + h_k^\theta)| \Big] ds \\
&\le a_k + b_k + c_k + d_k + L_1 N_0 \int_0^\alpha |x_s^k - x_s - z_s^{h_k}|_C\, ds \\
&\le (a_k + b_k + c_k + d_k)(1 + L_1 N_0 N_1 \alpha),
\end{aligned}$$

where a_k, b_k, c_k and d_k are defined by (2.3.45)–(2.3.48), respectively. Then (2.4.1) is obtained from (2.3.51).

Relation (2.4.2) follows from (2.4.1), $x^k(s) - x(s) - z^{h_k}(s) = 0$ for $s \in [-r, 0]$, $|\dot{x}^k(s) - \dot{x}(s) - \dot{z}^{h_k}(s)| \le (L + N_2)|h_k|_\Gamma$ for $s \in [-r, 0]$, and Lemmas 1.2.12 and 2.2.3. □

Lemma 2.4.2 *Assume (A1) (i)–(v), (A2) (i)–(vi), (H) and $\gamma \in \mathcal{P}$. Then there exists $N_4 = N_4(\gamma) \ge 0$ such that*

$$|\dot{z}^h(s) - \dot{z}^h(\bar{s})| \le N_4 |h|_{\Gamma_2} |s - \bar{s}|, \quad \text{for} \quad s, \bar{s} \in [-r, 0) \quad \text{and} \quad s, \bar{s} \in (0, \alpha], \quad h \in \Gamma_2. \tag{2.4.3}$$

Proof For $h \in \Gamma_2$, i.e., $h^\varphi \in W^{2,\infty}$, the function $\dot{h}^\varphi$ is continuous, and for $s, \bar{s} \in [-r, 0)$

$$|\dot{z}^h(s) - \dot{z}^h(\bar{s})| = |\dot{h}^\varphi(s) - \dot{h}^\varphi(\bar{s})| \le |h^\varphi|_{W^{2,\infty}} |s - \bar{s}| \le |h|_{\Gamma_2} |s - \bar{s}|.$$

Since $\gamma \in \mathcal{P}$, $L(s,x)$ is defined and continuous for all $s \in [0,\alpha]$, so $\dot{z}^h$ is continuous on $(0,\alpha]$. For $s, \bar{s} \in (0,\alpha]$ (2.3.11) and (2.3.13) imply

$$\begin{aligned} |\dot{z}^h(s) - \dot{z}^h(\bar{s})| &= |L(s,x)(z^h_s, h^\theta, h^\xi) - L(\bar{s},x)(z^h_{\bar{s}}, h^\theta, h^\xi)| \\ &\le |[L(s,x) - L(\bar{s},x)](z^h_s, h^\theta, h^\xi)| + |L(\bar{s},x)(z^h_s - z^h_{\bar{s}}, 0, 0)| \\ &\le |[D_2 f(\mathbf{v}(s)) - D_2 f(\mathbf{v}(\bar{s}))] z^h_s| + |[D_3 f(\mathbf{v}(s)) - D_3 f(\mathbf{v}(\bar{s}))] E(s, z^h_s, h^\xi)| \\ &\quad + |D_3 f(\mathbf{v}(\bar{s}))[E(s, z^h_s, h^\xi) - E(\bar{s}, z^h_{\bar{s}}, h^\xi)]| \\ &\quad + |[D_4 f(\mathbf{v}(s)) - D_4 f(\mathbf{v}(\bar{s}))] h^\theta| + L_1 N_0 |z^h_s - z^h_{\bar{s}}|_C. \end{aligned} \tag{2.4.4}$$

We have by (2.2.1) and (2.2.6) with $k = 0$ for $s, \bar{s} \in [0,\alpha]$

$$|\mathbf{v}(s) - \mathbf{v}(\bar{s})| \le |s - \bar{s}| + |x_s - x_{\bar{s}}|_C + |x(u(s)) - x(u(\bar{s}))| \le K_5 |s - \bar{s}| \tag{2.4.5}$$

and

$$|(s, x_s, \xi) - (\bar{s}, x_{\bar{s}}, \xi)| \le (1 + N)|s - \bar{s}| \tag{2.4.6}$$

with $K_5 := (1 + N + NL_2(1 + N))$ and $(1 + N) := 1 + N$. Let $L_3 := L_3(\alpha, M_1, M_2, M_3)$ and $L_5 := L_5(\alpha, M_1, M_2, M_3)$ be defined by (A1) (v) and (A2) (vi), respectively.

The definition of A, (A2) (ii) and (2.3.15) give

$$|A(s, z^h_s, h^\xi)| \le |D_2 \tau(s, x_s, \xi) z^h_s| + |D_3 \tau(s, x_s, \xi) h^\xi| \le K_6 |h|_\Gamma, \quad s \in [0,\alpha],\ h \in \Gamma,\ \gamma \in P_2 \tag{2.4.7}$$

with $K_6 := L_2(N_1 + 1)$, and by using (A2) (ii), (vi), (2.3.15), (2.3.16), (2.4.6)

$$\begin{aligned} |A(s, z^h_s, h^\xi) - A(\bar{s}, z^h_{\bar{s}}, h^\xi)| &\le |[D_2 \tau(s, x_s, \xi) - D_2 \tau(\bar{s}, x_{\bar{s}}, \xi)] z^h_s| + |D_2 \tau(\bar{s}, x_{\bar{s}}, \xi)[z^h_s - z^h_{\bar{s}}]| \\ &\quad + |[D_3 \tau(s, x_s, \xi) - D_3 \tau(\bar{s}, x_{\bar{s}}, \xi)] h^\xi| \\ &\le K_7 |s - \bar{s}| |h|_\Gamma, \qquad s, \bar{s} \in [0,\alpha] \end{aligned} \tag{2.4.8}$$

with $K_7 := L_5(1 + N)N_1 + L_2 N_2 + L_5(1 + N)$. Relations (2.2.1), (2.3.15) and (2.4.7) yield

$$\begin{aligned} |E(s, z^h_s, h^\xi)| &\le |\dot{x}(u(s))| |A(s, z^h_s, h^\xi)| + |z^h(u(s))| \\ &\le K_8 |h|_\Gamma, \qquad s \in [0,\alpha],\ h \in \Gamma,\ \gamma \in P_2 \end{aligned} \tag{2.4.9}$$

with $K_8 := NK_6 + N_1$, and using (2.2.1), (2.2.6) with $k = 0$, (2.3.16), (2.3.28), (2.4.7) and (2.4.8)

$$\begin{aligned} &|E(s, z^h_s, h^\xi) - E(\bar{s}, z^h_{\bar{s}}, h^\xi)| \\ &\quad \le |[\dot{x}(u(s)) - \dot{x}(u(\bar{s}))] A(s, z^h_s, h^\xi)| + |\dot{x}(u(\bar{s}))[A(s, z^h_s, h^\xi) - A(\bar{s}, z^h_{\bar{s}}, h^\xi)]| \\ &\qquad + |z^h(u(s)) - z^h(u(\bar{s}))| \\ &\quad \le K_9 |s - \bar{s}| |h|_\Gamma, \qquad s, \bar{s} \in [0,\alpha] \end{aligned} \tag{2.4.10}$$

with $K_9 = K_9(\gamma) := K_4L_2(1+N)K_6 + NK_7 + N_2L_2(1+N)$. Then combining (2.4.4) with (2.4.5), (2.4.9) and (2.4.10) yields

$$|\dot{z}^h(s) - \dot{z}^h(\bar{s})| \le (L_3K_5N_1 + L_3K_5K_8 + L_1K_9 + L_3K_5 + L_1N_0N_2)|s-\bar{s}||h|_\Gamma$$

for $s, \bar{s} \in [0,\alpha]$ and $h \in \Gamma$. Hence $N_4 := \max\{1, L_3K_5N_1 + L_3K_5K_8 + L_1K_9 + L_3K_5 + L_1N_0N_2\}$ satisfies (2.4.3). □

Lemma 2.4.3 *Assume (A1) (i)–(v), (A2) (i)–(vi), (H) and $\gamma \in \mathcal{P}$. Then*

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma_2}} \frac{1}{|h|_{\Gamma_2}} \int_0^\alpha |\dot{z}^h(u^k(s)) - \dot{z}^h(u(s))|\,ds = 0. \tag{2.4.11}$$

Proof Since $\gamma \in P_2$ and $u(0) \le 0$, it follows that u has finitely many zeros on $[0,\alpha]$. Let $0 \le s_1 < s_2 < \cdots < s_\ell \le \alpha$ be the mesh points where $u(s_i) = 0$, $0 < \varepsilon < \min\{s_{i+1} - s_i : i = 1,\dots,\ell-1\}/2$ be fixed, and introduce $s_i' := \min\{s_i + \varepsilon, \alpha\}$ and $s_i'' := \max\{s_i - \varepsilon, 0\}$ for $i = 1,\dots,\ell$, $s_0' := 0$, $s_{\ell+1}'' := \alpha$, and let

$$M := \min_{i=1,\dots,\ell-1} \min_{s\in[s_i', s_{i+1}'']} |u(s)|.$$

We have $M > 0$. Relation (2.2.4) yields that there exist $k_0 > 0$ such that $|u^k - u|_{C([0,\alpha],\mathbb{R})} < \frac{M}{2}$ for $k \ge k_0$. Then for $k \ge k_0$ it follows $|u^k(s)| \ge \frac{M}{2}$ for $s \in [s_i', s_{i+1}'']$ and $i = 0,\dots,\ell$. Note that $h \in \Gamma_2$ and $\gamma \in \mathcal{P}$ yield $\dot{z}^h$ is continuous on $[-r, 0)$ and $(0,\alpha]$, and (2.3.16) implies $|\dot{z}^h(s)| \le N_2|h|_\Gamma \le N_2|h|_{\Gamma_2}$ for $s \neq 0$. Therefore $|\dot{z}^h(u^k(s))| \le N_2|h|_{\Gamma_2}$ for a.e. $s \in [0,\alpha]$, since, by assumption (H), $\gamma + h_k \in P_2$, hence $u^k \in \mathcal{PM}([0,\alpha],[-r,\alpha])$. Then (2.2.4), (2.3.16) and (2.4.3) yield

$$\begin{aligned}
&\int_0^\alpha |\dot{z}^h(u^k(s)) - \dot{z}^h(u(s))|\,ds\\
&\quad\le \sum_{i=1}^{\ell} \int_{s_i''}^{s_i'} [|\dot{z}^h(u^k(s))| + |\dot{z}^h(u(s))|]\,ds + \sum_{i=0}^{\ell} \int_{s_i'}^{s_{i+1}''} |\dot{z}^h(u^k(s)) - \dot{z}^h(u(s))|\,ds\\
&\quad\le 4\ell\varepsilon N_2|h|_{\Gamma_2} + (\ell+1)\alpha N_4 K_0 |h|_{\Gamma_2}|h_k|_\Gamma.
\end{aligned}$$

This concludes the proof of (2.4.11), since $\varepsilon > 0$ can be arbitrary close to 0. □

Lemma 2.4.4 *Assume (A1) (i)–(v), (A2) (i)–(vi), (H) and $\gamma \in \mathcal{P}$. Then*

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma_2}} \frac{1}{|h|_{\Gamma_2}|h_k|_\Gamma} \int_0^\alpha |z^h(u^k(s)) - z^h(u(s)) - \dot{z}^h(u(s))(u^k(s) - u(s))|\,ds = 0. \tag{2.4.12}$$

Proof Let s_i, s_i', s_i'', ℓ, ε, M and k_0 be defined as in the proof of Lemma 2.4.3. Then $|u(s)+\nu(u^k(s)-u(s))| > \frac{M}{2}$, and $u(s)$ and $u(s)+\nu(u^k(s)-u(s))$ are both either positive or negative for $s \in [s_i', s_{i+1}'']$, $\nu \in [0,1]$ and $i = 0, \ldots, \ell$. Therefore (2.2.4) and (2.4.3) yield

$$|\dot{z}^h(u(s)+\nu(u^k(s)-u(s))) - \dot{z}^h(u(s))| \le N_4 |h|_{\Gamma_2} |u^k(s)-u(s)| \le N_4 K_0 |h|_{\Gamma_2} |h_k|_\Gamma.$$

Hence, using Fubini's Theorem, (2.2.4) and (2.3.16) we have

$$\begin{aligned}
&\int_0^\alpha |z^h(u^k(s)) - z^h(u(s)) - \dot{z}^h(u(s))(u^k(s)-u(s))|\, ds \\
&\quad\le \sum_{i=1}^{\ell} \int_{s_i''}^{s_i'} \Big(|z^h(u^k(s)) - z^h(u(s))| + |\dot{z}^h(u(s))||u^k(s)-u(s))| \Big) ds \\
&\qquad + \sum_{i=0}^{\ell} \int_{s_i'}^{s_{i+1}''} |z^h(u^k(s)) - z^h(u(s)) - \dot{z}^h(u(s))(u^k(s)-u(s))|\, ds \\
&\quad\le 4\varepsilon \ell N_2 K_0 |h|_\Gamma |h_k|_\Gamma \\
&\qquad + \sum_{i=0}^{\ell} \int_{s_i'}^{s_{i+1}''} \left| \int_0^1 [\dot{z}^h(u(s)+\nu(u^k(s)-u(s))) - \dot{z}^h(u(s))][u^k(s)-u(s)]\, d\nu \right| ds \\
&\quad\le 4\varepsilon \ell N_2 K_0 |h|_\Gamma |h_k|_\Gamma \\
&\qquad + K_0 |h_k|_\Gamma \sum_{i=0}^{\ell} \int_0^1 \int_{s_i'}^{s_{i+1}''} |\dot{z}^h(u(s)+\nu(u^k(s)-u(s))) - \dot{z}^h(u(s))| ds\, d\nu \\
&\quad\le 4\varepsilon \ell N_2 K_0 |h|_{\Gamma_2} |h_k|_\Gamma + K_0^2 (\ell+1)\alpha N_4 |h|_{\Gamma_2} |h_k|_\Gamma^2.
\end{aligned}$$

This completes the proof of (2.4.12), since $\varepsilon > 0$ is arbitrary close to 0. □

Lemma 2.4.5 *Assume (A1) (i)–(iii), (A2) (i)–(v), (H). Then*

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma} \int_0^\alpha |\dot{z}^{k,h}(s) - \dot{z}^h(s)|\, ds = 0, \tag{2.4.13}$$

and

$$\lim_{k\to\infty} \sup_{|h|_\Gamma \neq 0} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))]|\, ds = 0. \tag{2.4.14}$$

Proof For $s \in [0,\alpha]$ combining (2.3.11), (2.3.13), (2.3.17), (2.3.22) and (2.3.25) we get

$$\begin{aligned}
&|\dot{z}^{k,h}(s) - \dot{z}^h(s)| \\
&\quad\le |L(s,x^k)(z_s^{k,h} - z_s^h, 0, 0)| + |(L(s,x^k) - L(s,x))(z_s^h, h^\theta, h^\xi)| \\
&\quad\le L_1 N_0 c_{1,k} N_1 |h|_\Gamma + c_{0,k}(N_2+1)|h|_\Gamma + L_1 L_2 (N_2+1)|\dot{x}(u^k(s)) - \dot{x}(u(s))||h|_\Gamma.
\end{aligned}$$

Hence Lemmas 1.2.11 and 2.2.3 yield (2.4.13).

Define the functions
$$f^{k,h}(s) := \frac{|\dot z^{k,h}(s) - \dot z^h(s)|}{|h|_\Gamma},$$
and the set $H := \{h \in \Gamma : h \neq 0\}$. Note that (2.3.11), (2.3.13) and (2.3.15) yield $|\dot z^{k,h}(s)| = |L(s, x^k) z_s^{k,h}| \le L_1 N_0 N_1 |h|_\Gamma$ for $k \in \mathbb{N}_0$ and $s \in [0, \alpha]$, so $|f^{k,h}(s)| \le 2L_1 N_0 N_1$ for a.e. $s \in [-r, \alpha]$, $k \in \mathbb{N}$ and $h \in H$. Then it follows from (2.4.13), $z^{k,h}(s) - z^h(s) = 0$ for $s \in [-r, 0]$, and Lemmas 1.2.12 and 2.2.3 that for any fixed $\nu \in [0, 1]$
$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma} \int_0^\alpha \Big| \dot z^{k,h}\Big(u(s) + \nu(u^k(s) - u(s))\Big) - \dot z^h\Big(u(s) + \nu(u^k(s) - u(s))\Big)\Big|\, ds = 0. \tag{2.4.15}$$

(2.2.4) and Fubini's Theorem yield
$$\begin{aligned}
&\int_0^\alpha |z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))]|\, ds \\
&= \int_0^\alpha \Big| \int_0^1 \Big[\dot z^{k,h}\Big(u(s) + \nu(u^k(s) - u(s))\Big) - \dot z^h\Big(u(s) + \nu(u^k(s) - u(s))\Big)\Big] \\
&\qquad \times [u^k(s) - u(s)]\, d\nu\Big|\, ds \\
&\le K_0 |h_k|_\Gamma \int_0^1 \int_0^\alpha \Big| \dot z^{k,h}\Big(u(s) + \nu(u^k(s) - u(s))\Big) - \dot z^h\Big(u(s) + \nu(u^k(s) - u(s))\Big)\Big|\, ds\, d\nu.
\end{aligned}$$
Therefore (2.4.15) and the Dominated Convergence Theorem imply (2.4.14). □

Introduce the notation
$$p^k(t) := x^k(t) - x(t) - z^{h_k}(t).$$
Then, under the assumptions of Theorem 2.3.9, (2.3.50) and (2.3.51) give
$$\lim_{k\to\infty} \max_{s\in[-r,\alpha]} \frac{|p^k(s)|}{|h_k|_\Gamma} = 0. \tag{2.4.16}$$
To linearize equation (2.3.13) around a fixed solution z we will need the following results.

Lemma 2.4.6 *Assume (A1) (i)–(v), (A2) (i)–(vi), (H) and $\gamma \in \mathcal{P}$. Then*

(i)
$$u^k(s) - u(s) + A(s, z_s^{h_k}, h_k^\xi) = g_0^k(s), \qquad s \in [0, \alpha], \tag{2.4.17}$$

where

$$g_0^k(s) := -\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi) - D_2\tau(s, x_s, \xi)p_s^k$$

satisfies

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |g_0^k(s)|\, ds = 0; \tag{2.4.18}$$

(ii)

$$x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi) = g_1^k(s), \qquad s \in [0, \alpha], \tag{2.4.19}$$

where

$$\begin{aligned} g_1^k(s) \quad := \quad & p^k(u^k(s)) + x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s)) + \dot{x}(u(s))g_0^k(s) \\ & + z^{h_k}(u^k(s)) - z^{h_k}(u(s)) \end{aligned}$$

satisfies

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |g_1^k(s)|\, ds = 0; \tag{2.4.20}$$

and

(iii) if $h_k \in \Gamma_2$ for $k \in \mathbb{N}$, then

$$\dot{x}^k(u^k(s)) - \dot{x}(u(s)) - F(s, z_s^{h_k}, h_k^\xi) = g_2^k(s), \qquad s \in [0, \alpha], \tag{2.4.21}$$

where

$$\begin{aligned} g_2^k(s) \quad := \quad & \dot{x}^k(u^k(s)) - \dot{x}(u^k(s)) - \dot{z}^{h_k}(u^k(s)) + \dot{z}^{h_k}(u^k(s)) - \dot{z}^{h_k}(u(s)) \\ & + \dot{x}(u^k(s)) - \dot{x}(u(s)) - \ddot{x}(u(s))(u^k(s) - u(s)) \\ & - \ddot{x}(u(s))\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi) - \ddot{x}(u(s))D_2\tau(s, x_s, \xi)p_s^k \end{aligned}$$

satisfies

$$\lim_{k\to\infty} \frac{1}{|h_k|_{\Gamma_2}} \int_0^\alpha |g_2^k(s)|\, ds = 0. \tag{2.4.22}$$

Proof The definition of ω_τ and A imply

$$\begin{aligned} & u^k(s) - u(s) + A(s, z_s^{h_k}, h_k^\xi) \\ & \qquad = \quad -[\tau(s, x_s^k, \xi + h_k^\xi) - \tau(s, x_s, \xi) - D_2\tau(s, x_s, \xi)(x_s^k - x_s) - D_2\tau(s, x_s, \xi)h_k^\xi] \\ & \qquad\qquad -D_2\tau(s, x_s, \xi)(x_s^k - x_s - z_s^{h_k}), \qquad s \in [0, \alpha], \end{aligned}$$

which shows (2.4.17). (2.4.18) follows from $|D_2\tau(s, x_s, \xi)|_{\mathcal{L}(C,\mathbb{R})} \le L_2$ for $s \in [0, \alpha]$, (2.3.8) and (2.4.16).

Relation (2.3.41) and the definition of g_1^k yield (2.4.19). We have by (2.2.1) and (2.3.42)

$$\begin{aligned}\int_0^\alpha |g_1^k(s)|\,ds \;\le\; & \alpha \max_{s\in[-r,\alpha]} |p^k(s)| + \int_0^\alpha |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s)-u(s))|\,ds \\ & + N\int_0^\alpha |g_0^k(s)|\,ds + \alpha N_2 K_0 |h_k|_\Gamma^2.\end{aligned}$$

Therefore (2.4.16), (2.4.18), and Lemmas 2.3.1 and 2.2.3 yield (2.4.20).

Simple computation and the definition of g_2^k imply (2.4.21) immediately. Note that $\gamma \in \mathcal{P}$ yields that $\dot{x}$ is continuous on $[-r,\alpha]$, and $\varphi \in W^{2,\infty}$ and Lemma 2.3.6 imply that $x \in W^{2,\infty}([-r,\alpha],\mathbb{R}^n)$. Then (2.2.5) and Lemma 2.3.1 with $y = \dot{x}$ yield

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s)) - \ddot{x}(u(s))(u^k(s)-u(s))|\,ds = 0. \tag{2.4.23}$$

We have by (2.3.27) and Lemma 1.2.10 that $|\ddot{x}(u(s))| \le K_4$ for a.e. $s \in [0,\alpha]$, therefore

$$\begin{aligned}\int_0^\alpha |g_2^k(s)|\,ds \;\le\; & \int_0^\alpha |\dot{x}^k(u^k(s)) - \dot{x}(u^k(s)) - \dot{z}^{h_k}(u^k(s))|\,ds \\ & + \int_0^\alpha |\dot{z}^{h_k}(u^k(s)) - \dot{z}^{h_k}(u(s))|\,ds \\ & + \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s)) - \ddot{x}(u(s))(u^k(s)-u(s))|\,ds \\ & + K_4 \int_0^\alpha |\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi)|\,ds + \alpha K_4 L_2 \max_{s\in[0,\alpha]} |p_s^k|_C.\end{aligned}$$

Hence (2.3.8), (2.4.2), (2.4.11), (2.4.16) and (2.4.23) imply (2.4.22). □

We define the notations

$$\begin{aligned}&\omega_{D_2\tau}(s,\bar{\varphi},\bar{\xi},\varphi,\xi,\psi) \\ &\quad := D_2\tau(s,\varphi,\xi)\psi - D_2\tau(s,\bar{\varphi},\bar{\xi})\psi - D_{22}\tau(s,\bar{\varphi},\bar{\xi})\langle\psi,\varphi-\bar{\varphi}\rangle - D_{23}\tau(s,\bar{\varphi},\bar{\xi})\langle\psi,\xi-\bar{\xi}\rangle \\ &\omega_{D_3\tau}(s,\bar{\varphi},\bar{\xi},\varphi,\xi,\chi) \\ &\quad := D_3\tau(s,\varphi,\xi)\chi - D_3\tau(s,\bar{\varphi},\bar{\xi})\chi - D_{32}\tau(s,\bar{\varphi},\bar{\xi})\langle\chi,\varphi-\bar{\varphi}\rangle - D_{33}\tau(s,\bar{\varphi},\bar{\xi})\langle\chi,\xi-\bar{\xi}\rangle\end{aligned}$$

for $s \in [0,\alpha]$, $\bar{\varphi},\varphi \in \Omega_1$, $\bar{\xi},\xi \in \Omega_4$, $\psi \in C$ and $\chi \in \Xi$.

Lemma 2.4.7 *Assume (A2) (i)–(vii) and (H). Then*

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})|\,ds = 0, \tag{2.4.24}$$

and

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\omega_{D_3\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, h^\xi)|\,ds = 0. \tag{2.4.25}$$

Proof Let $L_5 = L_5(\alpha, M_1, M_3)$ be defined by (A2) (vi). Then (A2) (vi), (2.2.2), (2.3.15) and (2.3.20) yield for $s \in [0,\alpha]$

$$\begin{aligned}
|D_2\tau(s, x_s^k, \xi + h_k^\xi) z_s^{k,h} - D_2\tau(s, x_s, \xi) z_s^{k,h}| &\leq L_5(L+1)N_1 |h_k|_\Gamma |h|_\Gamma, \\
|D_{22}\tau(s, x_s, \xi)\langle z_s^{k,h}, x_s^k - x_s\rangle &\leq L_5 N_1 L |h|_\Gamma |h_k|_\Gamma, \\
|D_{23}\tau(s, x_s, \xi)\langle z_s^{k,h}, h_k^\xi\rangle &\leq L_5 N_1 |h|_\Gamma |h_k|_\Gamma,
\end{aligned}$$

and hence,

$$|\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})| \leq 2L_5(L+1)N_1 |h_k|_\Gamma |h|_\Gamma, \qquad s \in [0,\alpha].$$

On the other hand, for $s \in [0,\alpha]$, $k \in \mathbb{N}$ and $0 \neq h \in \Gamma$ such that $|x_s^k - x_s|_C + |h_k^\xi|_\Gamma \neq 0$ and $|z_s^{k,h}|_C \neq 0$, assumption (A2) (vii), (2.2.2) and (2.3.15) yield

$$\begin{aligned}
&\sup_{|h|_\Gamma \neq 0} \frac{|\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})|}{|h|_\Gamma |h_k|_\Gamma} \\
&\quad = \sup_{|h|_\Gamma \neq 0} \frac{|\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})|}{(|x_s^k - x_s|_C + |h_k^\xi|_\Gamma)|z_s^{k,h}|_C} \cdot \frac{(|x_s^k - x_s|_C + |h_k^\xi|_\Gamma)|z_s^{k,h}|_C}{|h|_\Gamma |h_k|_\Gamma} \\
&\quad \leq (L+1)N_1 \sup_{|h|_\Gamma \neq 0} \frac{|\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})|}{(|x_s^k - x_s|_C + |h_k^\xi|_\Gamma)|z_s^{k,h}|_C} \\
&\quad \to 0, \qquad k \to \infty.
\end{aligned}$$

Note that for s, k and h such that $|x_s^k - x_s|_C + |h_k^\xi|_\Gamma = 0$ or $|z_s^{k,h}|_C = 0$, $|\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})| = 0$. Therefore the Dominated Convergence Theorem implies (2.4.24).

The proof of (2.4.25) is similar. □

For a.e. $s \in [0,\alpha]$, $h, y \in \Gamma$ we introduce the bilinear operators by

$$\begin{aligned}
G(s)\langle(h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle &:= D_{22}\tau(s, x_s, \xi)\langle h^\varphi, y^\varphi\rangle + D_{23}\tau(s, x_s, \xi)\langle h^\varphi, y^\xi\rangle \\
&\quad + D_{32}\tau(s, x_s, \xi)\langle h^\xi, y^\varphi\rangle + D_{33}\tau(s, x_s, \xi)\langle h^\xi, y^\xi\rangle, \\
H(s)\langle(h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle &:= -A(s, h^\varphi, h^\xi)F(s, y^\varphi, y^\xi) - \dot{x}(u(s))G(s)\langle(h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle \\
&\quad - \dot{h}^\varphi(-\tau(s, x_s, \xi))A(s, y^\varphi, y^\xi),
\end{aligned}$$

and

$$\begin{aligned}
B(s)\langle h, y\rangle := \; & D_{22}f(\mathbf{v}(s))\langle h^\varphi, y^\varphi\rangle + D_{23}f(\mathbf{v}(s))\langle h^\varphi, E(s, y^\varphi, y^\xi)\rangle + D_{24}f(\mathbf{v}(s))\langle h^\varphi, y^\theta\rangle \\
& + D_{32}f(\mathbf{v}(s))\langle E(s, h^\varphi, h^\xi), y^\varphi\rangle + D_{33}f(\mathbf{v}(s))\langle E(s, h^\varphi, h^\xi), E(s, y^\varphi, y^\xi)\rangle \\
& + D_{34}f(\mathbf{v}(s))\langle E(s, h^\varphi, h^\xi), y^\theta\rangle + D_{42}f(\mathbf{v}(s))\langle h^\theta, y^\varphi\rangle \\
& + D_{43}f(\mathbf{v}(s))\langle h^\theta, E(s, y^\varphi, y^\xi)\rangle + D_{44}f(\mathbf{v}(s))\langle h^\theta, y^\theta\rangle \\
& + D_3f(\mathbf{v}(s))H(s)\langle(h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle.
\end{aligned}$$

Note that G, H and B correspond to γ, but this dependence is omitted for simplicity in the notation.

For $\gamma \in P_2$ consider the corresponding solution x of the IVP (2.1.1)-(2.1.2), and let z^h and z^y be the solutions of the IVP (2.3.13)-(2.3.14) corresponding to a fixed $h, y \in \Gamma$. We consider the IVP

$$\begin{aligned} \dot{w}(t) &= L(t,x)(w_t,0,0) + B(t)\langle (z_t^h, h^\theta, h^\xi), (z_t^y, y^\theta, y^\xi)\rangle, \quad \text{a.e. } t \in [0,\alpha], &(2.4.26)\\ w(t) &= 0, \qquad t \in [-r,0]. &(2.4.27)\end{aligned}$$

The IVP (2.4.26)-(2.4.27) is a Carathéodory type inhomogeneous linear delay system with time-dependent but state-independent delays. It is easy to see that under assumptions (A1) (i)–(vi), (A2) (i)–(vii) the IVP (2.4.26)-(2.4.27) has a unique solution on $[-r,\alpha]$, which will be denoted by $w^{h,y}(t) := w(t,\gamma,h,y)$. It is easy to see that $\Gamma\times\Gamma \to \mathbb{R}^n$, $(h,y) \mapsto w(t,\gamma,h,y)$ is a bilinear map for a fixed $t \in [0,\alpha]$ and $\gamma \in P_2$. In Lemma 2.4.12 below we will show that this bilinear map is bounded.

We need the further notation

$$q^{k,h}(s) \;:=\; z^{k,h}(s) - z^h(s) - w^{h,h_k}(s), \quad s \in [-r,\alpha].$$

Lemma 2.4.8 *Assume (A2) (i)–(vi) and (H). Then there exists $K_{10} \geq 0$ such that*

$$|A^k(s, z_s^{j,h}, h^\xi) - A(s, z_s^{j,h}, h^\xi)| \leq K_{10}|h|_\Gamma |h_k|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N},\ j \in \mathbb{N}_0, \quad (2.4.28)$$

and there exists a sequence $c_{2,k} \geq 0$ satisfying $c_{2,k} \to 0$ as $k \to \infty$ such that

$$|A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)| \leq c_{2,k}|h|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N}. \quad (2.4.29)$$

Proof Let $L_5 = L_5(\alpha, M_1, M_3)$ be defined by (A2) (vi). To show (2.4.29) we use (2.2.2), (2.3.15), (2.3.20) and (A2) (vi) to get

$$\begin{aligned} &|A^k(s, z_s^{j,h}, h^\xi) - A(s, z_s^{j,h}, h^\xi)| \\ &\quad\leq |D_2\tau(s, x_s^k, \xi + h_k^\xi) z_s^{j,h} - D_2\tau(s, x_s, \xi) z_s^{j,h}| + |D_3\tau(s, x_s^k, \xi + h_k^\xi) h^\xi - D_3\tau(s, x_s, \xi) h^\xi| \\ &\quad\leq L_5(L+1)|h_k|_\Gamma N_1 |h|_\Gamma + L_5(L+1)|h_k|_\Gamma |h|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N},\ j \in \mathbb{N}_0,\end{aligned}$$

which yields (2.4.28). Using (2.3.25), (2.4.29) and (A2) (ii) we get

$$\begin{aligned} &|A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)| \\ &\qquad\leq |A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)| + |A(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)| \\ &\qquad\leq K_{10}|h|_\Gamma |h_k|_\Gamma + |D_2\tau(s, x_s, \xi)(z_s^{k,h} - z_s^h)| \\ &\qquad\leq K_{10}|h_k|_\Gamma |h|_\Gamma + L_2 c_{1,k} N_1 |h|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N},\end{aligned}$$

therefore (2.4.29) holds. $\square$

Lemma 2.4.9 *Assume (A1) (i)–(v), (A2) (i)–(vii), (H) and $\gamma \in \mathcal{P}$. Then*

$$\begin{aligned} A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - A(s, w_s^{h,h_k}, 0) \\ = \ A(s, q_s^{k,h}, 0) + g_3^{k,h}(s), \qquad s \in [0,\alpha],\ h \in \Gamma,\ k \in \mathbb{N}, \end{aligned} \tag{2.4.30}$$

where

$$\begin{aligned} g_3^{k,h}(s) \ :=\ & D_{22}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, x_s^k - x_s\rangle + D_{22}\tau(s, x_s, \xi)\langle z_s^h, p_s^k\rangle \\ & + D_{23}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, h_k^\xi\rangle + D_{32}\tau(s, x_s, \xi)\langle h^\xi, p_s^k\rangle \\ & + \omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h}) + \omega_{D_3\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, h^\xi) \end{aligned}$$

satisfies

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |g_3^{k,h}(s)|\, ds = 0; \tag{2.4.31}$$

and if $h_k \in \Gamma_2$ for $k \in \mathbb{N}$, then

$$\begin{aligned} E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - H(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - E(s, w_s^{h,h_k}, 0) \\ = \ E(s, q_s^{k,h}, 0) + g_4^{k,h}(s), \qquad a.e.\ s \in [0,\alpha],\ h \in \Gamma,\ k \in \mathbb{N} \end{aligned} \tag{2.4.32}$$

with

$$\begin{aligned} g_4^{k,h}(s) \ :=\ & -[\dot{x}^k(u^k(s)) - \dot{x}(u(s))][A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)] - g_2^k(s) A(s, z_s^{k,h}, h^\xi) \\ & - \dot{x}(u(s)) g_3^{k,h}(s) + z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))] \\ & + z^h(u^k(s)) - z^h(u(s)) - \dot{z}^h(u(s))(u^k(s) - u(s)) \\ & + \dot{z}^h(u(s))\Big(u^k(s) - u(s) + A(s, z_s^{h_k}, h_k^\xi)\Big) \end{aligned}$$

satisfying

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma_2}} \frac{1}{|h|_{\Gamma_2} |h_k|_{\Gamma_2}} \int_0^\alpha |g_4^{k,h}(s)|\, ds = 0. \tag{2.4.33}$$

Proof The definitions of $A^k, A, G, g_3^{k,h}, \omega_{D_2\tau}, \omega_{D_3\tau}$ and relation

$$A(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi) - A(s, w_s^{h,h_k}, 0) = A(s, z_s^{k,h} - z_s^h - w_s^{h,h_k}, 0)$$

yield

$$\begin{aligned} & A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - A(s, w_s^{h,h_k}, 0) \\ & \quad = \ A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle + A(s, q_s^{k,h}, 0) \\ & \quad = \ D_2\tau(s, x_s^k, \xi + h_k^\xi) z_s^{k,h} - D_2\tau(s, x_s, \xi) z_s^{k,h} - D_{22}\tau(s, x_s, \xi)\langle z_s^{k,h}, x_s^k - x_s\rangle \\ & \qquad - D_{23}\tau(s, x_s, \xi)\langle z_s^{k,h}, h_k^\xi\rangle + D_{22}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, x_s^k - x_s\rangle \\ & \qquad + D_{22}\tau(s, x_s, \xi)\langle z_s^h, p_s^k\rangle + D_{23}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, h_k^\xi\rangle \\ & \qquad + D_3\tau(t, x_s^k, \xi + h_k^\xi) h^\xi - D_3\tau(s, x_s, \xi) h^\xi - D_{32}\tau(s, x_s, \xi)\langle h^\xi, x_s^k - x_s\rangle \\ & \qquad - D_{33}\tau(s, x_s, \xi)\langle h^\xi, h_k^\xi\rangle + D_{32}\tau(s, x_s, \xi)\langle h^\xi, p_s^k\rangle + A(s, q_s^{k,h}, 0) \\ & \quad = \ A(s, q_s^{k,h}, 0) + g_3^{k,h}(s). \end{aligned}$$

Let $L_5 = L_5(\alpha, M_1, M_3)$ be defined by (A2) (vi). Then we have by (2.2.2), (2.3.15) and (2.3.25)

$$\begin{aligned}\int_0^\alpha |g_3^{k,h}(s)|\,ds \quad \leq \quad & \alpha L_5 c_{1,k} N_1 |h|_\Gamma L |h_k|_\Gamma + \alpha L_5 N_1 |h|_\Gamma \max_{s\in[0,\alpha]} |p_s^k|_C + \alpha L_5 c_{1,k} N_1 |h|_\Gamma |h_k|_\Gamma \\ & + \alpha L_5 |h|_\Gamma \max_{s\in[0,\alpha]} |p_s^k|_C + \int_0^\alpha |\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})|\,ds \\ & + \int_0^\alpha |\omega_{D_3\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, h^\xi)|\,ds.\end{aligned}$$

Hence $c_{1,k} \to 0$ as $k \to \infty$, (2.4.16), (2.4.24) and (2.4.25) imply (2.4.31).

Relation

$$E(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - E(s, w_s^{h,h_k}, 0) = E(s, z_s^{k,h} - z_s^h - w_s^{h,h_k}, 0)$$

and the definition of E, E^k and H give

$$\begin{aligned} & E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - H(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - E(s, w_s^{h,h_k}, 0) \\ & \quad = \quad E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^{k,h}, h^\xi) - H(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle + E(s, q_s^{k,h}, 0) \\ & \quad = \quad -\dot{x}^k(u^k(s)) A^k(s, z_s^{k,h}, h^\xi) + \dot{x}(u(s)) A(s, z_s^{k,h}, h^\xi) + z^{k,h}(u^k(s)) - z^{k,h}(u(s)) \\ & \qquad\quad + A(s, z_s^h, h^\xi) F(s, z_s^{h_k}, h_k^\xi) + \dot{x}(u(s)) G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle \\ & \qquad\quad + \dot{z}^h(u(s)) A(s, z_s^{h_k}, h_k^\xi) - E(s, q_s^{k,h}, 0) \\ & \quad = \quad -[\dot{x}^k(u^k(s)) - \dot{x}(u(s))][A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)] \\ & \qquad\quad -[\dot{x}^k(u^k(s)) - \dot{x}(u(s)) - F(s, z_s^{h_k}, h_k^\xi)] A(s, z_s^{k,h}, h^\xi) \\ & \qquad\quad -\dot{x}(u(s)) \Big[A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle\Big] \\ & \qquad\quad + z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))] \\ & \qquad\quad + z^h(u^k(s)) - z^h(u(s)) - \dot{z}^h(u(s))(u^k(s) - u(s)) \\ & \qquad\quad + \dot{z}^h(u(s))\Big(u^k(s) - u(s) + A(s, z_s^{h_k}, h_k^\xi)\Big) + E(s, q_s^{k,h}, 0),\end{aligned}$$

which, together with (2.4.21) and (2.4.30), yields (2.4.32).

To prove (2.4.33) first note that by (2.2.2), (2.2.4) and (2.3.28)

$$\begin{aligned}|\dot{x}^k(u^k(s)) - \dot{x}(u(s))| \quad \leq \quad & |\dot{x}^k(u^k(s)) - \dot{x}(u^k(s))| + |\dot{x}(u^k(s)) - \dot{x}(u(s))| \\ \leq \quad & L|h_k|_\Gamma + K_4 K_0 |h_k|_\Gamma. \end{aligned} \tag{2.4.34}$$

Hence (2.4.28) and (2.4.34) give

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\dot{x}^k(u^k(s)) - \dot{x}(u(s))||A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)|\,ds = 0.$$

Relations(2.2.1), (2.4.7), (2.4.22) and (2.4.31) imply for $h_k \in \Gamma_2$ for $k \in \mathbb{N}$

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_{\Gamma_2}} \int_0^\alpha |g_2^k(s) A(s, z_s^{k,h}, h^\xi)|\, ds \le \lim_{k\to\infty} \frac{K_6}{|h_k|_{\Gamma_2}} \int_0^\alpha |g_2^k(s)|\, ds = 0$$

and

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\dot{x}(u(s)) g_3^{k,h}(s)|\, ds \le \lim_{k\to\infty} \frac{N}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |g_3^{k,h}(s)|\, ds = 0.$$

The above limits and (2.4.12), (2.4.14), $|\dot{z}^h(u(s))| \le N_2 |h|_{\Gamma_2}$ and (2.4.18) yield (2.4.33). □

Lemma 2.4.10 *Assume (A2) (i)–(vii), (H) and $\gamma \in \mathcal{P}$. Then there exist $K_{11} = K_{11}(\gamma) \ge 0$ and a nonnegative sequence $c_{3,k} = c_{3,k}(\gamma)$ satisfying $c_{3,k} \to 0$ as $k \to \infty$ such that*

$$\begin{aligned} |F(s, z_s^h, h^\xi)| &\le K_{11}|h|_\Gamma, && \text{a.e. } s \in [0,\alpha],\ h \in \Gamma, && (2.4.35)\\ |E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi)| &\le c_{3,k}|h|_\Gamma, && \text{a.e. } s \in [0,\alpha],\ k \in \mathbb{N}, && (2.4.36) \end{aligned}$$

and, if in addition, (A2) (viii) holds, there exists a nonnegative sequence $c_{4,k} = c_{4,k}(\gamma)$ satisfying $c_{4,k} \to 0$ as $k \to \infty$ such that

$$\int_0^\alpha |F^k(s, z_s^{k,h}, h^\xi) - F(s, z_s^h, h^\xi)|\, ds \le c_{4,k}|h|_{\Gamma_2}, \qquad \text{a.e. } s \in [0,\alpha],\ k \in \mathbb{N},\ h \in \Gamma_2. \tag{2.4.37}$$

Proof The definition of F, (2.3.27) and (2.4.7) imply immediately (2.4.35) with $K_{11} := K_4 K_6 + 1$.

Relations (2.2.1), (2.2.2), (2.2.4), (2.3.15), (2.3.16), (2.3.25), (2.4.7), (2.4.29), (2.4.34) and (H2) (ii) yield for a.e. $s \in [0,\alpha]$

$$\begin{aligned} &|E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi)|\\ &\quad\le |\dot{x}^k(u^k(s)) - \dot{x}(u(s))||A^k(s, z_s^{k,h}, h^\xi)|\\ &\qquad + |\dot{x}(u(s))|\Big|A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)\Big| + |z^{k,h}(u^k(s)) - z^h(u^k(s))|\\ &\qquad + |z^h(u^k(s)) - z^h(u(s))|\\ &\quad\le (L + K_4K_0)|h_k|_\Gamma K_6 |h|_\Gamma + N c_{2,k}|h|_\Gamma + c_{1,k}N_1|h|_\Gamma + N_2|h|_\Gamma K_0 |h_k|_\Gamma, \end{aligned}$$

which proves (2.4.36).

$$\begin{aligned} &|F^k(s, z_s^h, h^\xi) - F(s, z_s^h, h^\xi)|\\ &\quad\le \Big(|\ddot{x}^k(u^k(s)) - \ddot{x}(u^k(s))| + |\ddot{x}(u^k(s)) - \ddot{x}(u(s))|\Big)|A^k(s, z_s^h, h^\xi)|\\ &\qquad + |\ddot{x}(u(s))|\Big|A^k(s, z_s^h, h^\xi) - A^k(s, z_s^h, h^\xi)\Big| + |\dot{z}^h(u^k(s)) - \dot{z}^h(u(s))|. \end{aligned}$$

For $t \in (0, \alpha]$ we have by (A2) (viii) that

$$\begin{aligned}
|\ddot{x}^k(t) - \ddot{x}(t)| &= \Big| \frac{d}{dt} f(t, x_t^k, x^k(u^k(t)), \theta + h_k^\theta) - \frac{d}{dt} f(t, x_t, x(u(t)), \theta) \Big| \\
&\le L_6(|x_t^k - x_t|_C + |h_k^\theta|_\Theta + |h_k^\xi|_\Xi) \\
&\le L_6(L+1)|h_k|_\Gamma.
\end{aligned}$$

For $t \in [-r, 0)$ and $h \in \Gamma_2$ we get

$$|\ddot{x}^k(t) - \ddot{x}(t)| = |\ddot{h}_k^\varphi(t)| \le |h_k|_{\Gamma_2}.$$

Using that $\ddot{x} \in L^\infty([-r, \alpha], \mathbb{R}^n)$, similarly to (2.3.24) we can argue that

$$\lim_{k\to\infty} \int_0^\alpha |\ddot{x}(u^k(s)) - \ddot{x}(u(s))|\, ds = 0.$$

Then the above relations, $|\ddot{x}(u(s))| \le K_4$ for a.e. $s \in [0, \alpha]$, (2.4.7), (2.4.11) and (2.4.28) yield (2.4.37). □

For a.e. $s \in [0, \alpha]$, $h, y \in \Gamma$ and $k \in \mathbb{N}$ we introduce the bilinear operators by

$$\begin{aligned}
G^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle &:= D_{22}\tau(s, x_s^k, \xi + h_k^\xi)\langle h^\varphi, y^\varphi\rangle + D_{23}\tau(s, x_s^k, \xi + h_k^\xi)\langle h^\varphi, y^\xi\rangle \\
&\quad + D_{32}\tau(s, x_s^k, \xi + h_k^\xi)\langle h^\xi, y^\varphi\rangle + D_{33}\tau(s, x_s^k, \xi + h_k^\xi)\langle h^\xi, y^\xi\rangle, \\
H^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle &:= -A^k(s, h^\varphi, h^\xi) F^k(s, y^\varphi, y^\xi) \\
&\quad - \dot{x}^k(u^k(s)) G^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle \\
&\quad - \dot{h}^\varphi(-\tau(s, x_s^k, \xi + h_k^\xi)) A^k(s, y^\varphi, y^\xi),
\end{aligned}$$

and

$$\begin{aligned}
B^k(s)\langle h, y\rangle := \; & D_{22} f(\mathbf{v}^k(s))\langle h^\varphi, y^\varphi\rangle + D_{23} f(\mathbf{v}^k(s))\langle h^\varphi, E^k(s, y^\varphi, y^\xi)\rangle \\
& + D_{24} f(\mathbf{v}^k(s))\langle h^\varphi, y^\theta\rangle + D_{32} f(\mathbf{v}^k(s))\langle E^k(s, h^\varphi, h^\xi), y^\varphi\rangle \\
& + D_{33} f(\mathbf{v}^k(s))\langle E^k(s, h^\varphi, h^\xi), E^k(s, y^\varphi, y^\xi)\rangle \\
& + D_{34} f(\mathbf{v}^k(s))\langle E^k(s, h^\varphi, h^\xi), y^\theta\rangle + D_{42} f(\mathbf{v}^k(s))\langle h^\theta, y^\varphi\rangle \\
& + D_{43} f(\mathbf{v}^k(s))\langle h^\theta, E^k(s, y^\varphi, y^\xi)\rangle + D_{44} f(\mathbf{v}^k(s))\langle h^\theta, y^\theta\rangle \\
& + D_3 f(\mathbf{v}^k(s)) H^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle.
\end{aligned}$$

Lemma 2.4.11 *Assume (A1) (i)–(vi), (A2) (i)–(vii). Then for every $\gamma \in P_2$ there exists $K_{12} = K_{12}(\gamma) \ge 0$ such that*

$$|B(s)\langle (z_s^h, h^\theta, h^\xi), (z_s^y, y^\theta, y^\xi)\rangle| \le K_{12}|h|_\Gamma |y|_\Gamma, \quad \text{a.e. } s \in [-r, \alpha],\; h, y \in \Gamma,\; \gamma \in P_2. \tag{2.4.38}$$

If in addition (A2) (viii) holds, then for every $\gamma \in P_2 \cap \mathcal{P}$ there exists a nonnegative sequence $c_{5,k} = c_{5,k}(\gamma)$ such that $c_{5,k} \to 0$ as $k \to \infty$, and

$$\int_0^\alpha \Big| B^k(s)\langle (z_s^h, h^\theta, h^\xi), (z_s^y, y^\theta, y^\xi)\rangle - B(s)\langle (z_s^h, h^\theta, h^\xi), (z_s^y, y^\theta, y^\xi)\rangle \Big| \, ds \le c_{5,k} |h|_{\Gamma_2} |y|_{\Gamma_2}, \tag{2.4.39}$$

for $h, y \in \Gamma_2$.

Proof Let $L_3 = L_3(\alpha, M_1, M_2, M_3)$ and $L_5 = L_5(\alpha, M_1, M_4)$ be the Lipschitz constants from (A1) (v) and (A2) (vi), respectively. Then the definition of G, (A2) (vi) and (2.3.15) yield

$$|G(s)\langle (z_s^h, h^\xi), (z_s^y, y^\xi)\rangle| \le 4L_5 N_1^2 |h|_\Gamma |y|_\Gamma, \qquad h, y \in \Gamma, \quad s \in [0, \alpha]. \tag{2.4.40}$$

Then definition of H, (2.2.1), (2.3.15), (2.3.27), (2.4.7), (2.4.35) and (2.4.40) imply

$$|H(s)\langle (z_s^h, h^\xi), (z_s^y, y^\xi)\rangle| \le K_{13} |h|_\Gamma |y|_\Gamma, \quad h, y \in \Gamma, \quad \text{a.e. } s \in [0, \alpha] \tag{2.4.41}$$

with $K_{13} = K_{13}(\gamma) := K_6(K_4K_6 + 1) + N4L_5N_1^2 + K_6$. Therefore we have by the definition of B, (2.4.9) and (2.4.41)

$$|B(s)\langle h, y\rangle| \le L_3(4 + 4K_8 + K_8^2 + K_{13}) |h|_\Gamma |y|_\Gamma, \qquad \text{a.e. } s \in [0, \alpha],$$

which, together with (2.3.22), yields (2.4.38).

Define the set $M_4^* := \{\xi\} \cup \{h_k^\xi : \ k \in \mathbb{N}\}$. It is easy to show that $M_4^* \subset M_4$ is a compact subset of Ξ. Define

$$\begin{aligned}\Omega_{2,\tau}(\varepsilon) \;:=\; \max_{i,j=2,3} \sup\Big\{ &|D_{ij}\tau(s, \psi, \eta) - D_{ij}\tau(s, \bar\psi, \bar\eta)|_{\mathcal{L}^2(X_i \times X_j, \mathbb{R})} : \\ &s \in [0, \alpha],\ \psi, \bar\psi \in M_1, \eta, \bar\eta \in M_4^*,\ |\psi - \bar\psi|_C + |\eta - \bar\eta|_\Xi \le \varepsilon \Big\},\end{aligned}$$

where $X_2 := C$ and $X_3 := \Xi$. Assumption (A2) (vii) and the compactness of $[0, \alpha] \times M_1 \times M_4^*$ yields that $\Omega_{2,\tau}(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Then (2.3.15) and (2.3.20) give

$$\begin{aligned} |[G^k(s) - G(s)]\langle (z_s^h, h^\xi), (z_s^y, y^\xi)\rangle| \;\le\; & |[D_{22}\tau(s, x_s^k, \xi + h_k^\xi) - D_{22}\tau(s, x_s, \xi)]\langle z_s^h, z_s^y\rangle| \\ &+ |[D_{23}\tau(s, x_s^k, \xi + h_k^\xi) - D_{23}\tau(s, x_s^k, \xi + h_k^\xi)]\langle z_s^h, y^\xi\rangle| \\ &+ |[D_{32}\tau(s, x_s^k, \xi + h_k^\xi) - D_{32}\tau(s, x_s^k, \xi + h_k^\xi)]\langle h^\xi, z_s^y\rangle| \\ &+ |[D_{33}\tau(s, x_s^k, \xi + h_k^\xi) - D_{33}\tau(s, x_s^k, \xi + h_k^\xi)]\langle h^\xi, y^\xi\rangle| \\ \le\; & \Omega_{2,\tau}\Big((L+1)|h_k|_\Gamma\Big)(N_1 + 1)^2 |h|_\Gamma |y|_\Gamma, \qquad s \in [0, \alpha]. \end{aligned} \tag{2.4.42}$$

Relations (2.2.1), (2.2.2), (2.2.4), (2.3.15), (2.3.16), (2.4.7), (2.4.28), (2.4.34), (2.4.35), (2.4.37), (2.4.40) and (2.4.42) imply

$$
\begin{aligned}
&\int_0^\alpha |[H^k(s)-H(s)]\langle (z_s^h,h^\xi),(z_s^y,y^\xi)\rangle|\,ds\\
&\quad\le \int_0^\alpha \Big(|[A^k(s,z_s^h,h^\xi)-A(s,z_s^h,h^\xi)]F(s,z_s^y,y^\xi)|\\
&\qquad\qquad +|A^k(s,z_s^h,h^\xi)[F^k(s,z_s^y,y^\xi)-F(s,z_s^y,y^\xi)]|\\
&\qquad\qquad +|[\dot x^k(u^k(s))-\dot x(u(s))]G^k(s)\langle (z_s^h,h^\xi),(z_s^y,y^\xi)\rangle|\\
&\qquad\qquad +|\dot x(u(s))[G^k(s)-G(s)]\langle (z_s^h,h^\xi),(z_s^y,y^\xi)\rangle|\\
&\qquad\qquad +|[\dot z^h(u^k(s))-\dot z^h(u(s))]A^k(s,z_s^y,y^\xi)|\\
&\qquad\qquad +|\dot z^h(u(s))[A^k(s,z_s^y,y^\xi)-A(s,z_s^y,y^\xi)]|\Big)\,ds\\
&\quad\le \alpha K_{10}|h|_\Gamma|h_k|_\Gamma K_{11}|y|_\Gamma+K_6|h|_\Gamma c_{4,k}|y|_{\Gamma_2}+(L+K_4K_0)|h_k|_\Gamma 4L_5N_1^2|h|_\Gamma|y|_\Gamma\\
&\qquad +N\Omega_{2,\tau}\Big((L+1)|h_k|_\Gamma\Big)(N_1+1)^2|h|_\Gamma|y|_\Gamma\\
&\qquad +\int_0^\alpha |\dot z^h(u^k(s))-\dot z^h(u(s))|\,ds\,K_6|y|_\Gamma+\alpha N_2|h|_{\Gamma_2}K_{10}|h|_\Gamma|y|_\Gamma\\
&\quad\le c_{6,k}|h|_{\Gamma_2}|y|_{\Gamma_2}
\end{aligned}
\tag{2.4.43}
$$

with some appropriate sequence $c_{6,k}=c_{6,k}(\gamma)$ satisfying $c_{6,k}\to 0$ as $k\to\infty$, where in the last estimate we used (2.4.11).

Simple manipulations give

$$
\begin{aligned}
&|[B^k(s)-B(s)]\langle (z_s^h,h^\theta,h^\xi),(z_s^y,y^\theta,y^\xi)\rangle|\\
&\quad\le |[D_{22}f(\mathbf{v}^k(s))-D_{22}f(\mathbf{v}(s))]\langle z_s^h,z_s^y\rangle|\\
&\qquad +|[D_{23}f(\mathbf{v}^k(s))-D_{23}f(\mathbf{v}(s))]\langle z_s^h,E^k(s,z_s^y,y^\xi)\rangle|\\
&\qquad +|D_{23}f(\mathbf{v}(s))\langle z_s^h,E^k(s,z_s^y,y^\xi)-E(s,z_s^y,y^\xi)\rangle|\\
&\qquad +|[D_{24}f(\mathbf{v}^k(s))-D_{24}f(\mathbf{v}(s))]\langle z_s^h,y^\theta\rangle|\\
&\qquad +|[D_{32}f(\mathbf{v}^k(s))-D_{32}f(\mathbf{v}(s))]\langle E^k(s,z_s^h,h^\xi),z_s^y\rangle|\\
&\qquad +|D_{32}f(\mathbf{v}(s))\langle E^k(s,z_s^h,h^\xi)-E(s,z_s^h,h^\xi),z_s^y\rangle|\\
&\qquad +|[D_{33}f(\mathbf{v}^k(s))-D_{33}f(\mathbf{v}(s))]\langle E^k(s,z_s^h,h^\xi),E^k(s,z_s^y,y^\xi)\rangle|\\
&\qquad +|D_{33}f(\mathbf{v}(s))\langle E^k(s,z_s^h,h^\xi)-E(s,z_s^h,h^\xi),E^k(s,z_s^y,y^\xi)\rangle|\\
&\qquad +|D_{33}f(\mathbf{v}(s))\langle E(s,z_s^h,h^\xi),E^k(s,z_s^y,y^\xi)-E(s,z_s^y,y^\xi)\rangle|\\
&\qquad +|[D_{34}f(\mathbf{v}^k(s))-D_{34}f(\mathbf{v}(s))]\langle E^k(s,z_s^h,h^\xi),y^\theta\rangle|\\
&\qquad +|D_{34}f(\mathbf{v}(s))\langle E^k(s,z_s^h,h^\xi)-E(s,z_s^h,h^\xi),y^\theta\rangle|\\
&\qquad +|[D_{42}f(\mathbf{v}^k(s))-D_{42}f(\mathbf{v}(s))]\langle h^\theta,z_s^y\rangle|
\end{aligned}
$$

$$
\begin{aligned}
&+|[D_{43}f(\mathbf{v}^k(s)) - D_{43}f(\mathbf{v}(s))]\langle h^\theta, E^k(s, z_s^y, y^\xi)\rangle \\
&+|D_{43}f(\mathbf{v}(s))\langle h^\theta, E^k(s, z_s^y, y^\xi) - E(s, z_s^y, y^\xi)\rangle \\
&+|[D_{44}f(\mathbf{v}^k(s)) - D_{44}f(\mathbf{v}(s))]\langle h^\theta, y^\theta\rangle| \\
&+|[D_3f(\mathbf{v}^k(s)) - D_3f(\mathbf{v}(s))]H^k(s)\langle (z_s^h, h^\xi), (z_s^y, y^\xi)\rangle| \\
&+|D_3f(\mathbf{v}(s))[H^k(s) - H(s)]\langle (z_s^h, h^\xi), (z_s^y, y^\xi)\rangle|.
\end{aligned}
\tag{2.4.44}
$$

Define the set $M_3^* := \{\theta\} \cup \{h_k^\theta \colon k \in \mathbb{N}\}$. Clearly, $M_3^* \subset M_3$ is a compact subset of Θ.

Define

$$
\begin{aligned}
\Omega_{2,f}(\varepsilon) \quad := \quad \max_{i,j=2,3,4} \sup\Big\{ &|D_{ij}f(s,\psi,v,\eta) - D_{ij}f(s,\bar\psi,\bar v,\bar\eta)|_{\mathcal{L}^2(Y_i\times Y_j,\mathbb{R})} \colon \\
& s \in [0,\alpha],\ \psi,\bar\psi \in M_1, v,\bar v \in M_2,\ \eta,\bar\eta \in M_3^*, \\
& |\psi - \bar\psi|_C + |v - \bar v| + |\eta - \bar\eta|_\Theta \le \varepsilon \Big\},
\end{aligned}
$$

where $Y_2 := C$, $Y_3 := \mathbb{R}^n$ and $Y_4 := \Theta$. Assumption (A1) (vi) and the compactness of $[0,\alpha] \times M_1 \times M_2 \times M_3^*$ yields that $\Omega_{2,f}(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Then combining (2.4.44) with (2.3.19), $|D_{ij}f(\mathbf{v}^k(s)) - D_{ij}f(\mathbf{v}(s))|_{\mathcal{L}^2(Y_i\times Y_j,\mathbb{R}^n)} \le \Omega_{2,f}\Big(K_3|h_k|_\Gamma\Big)$ for $i,j = 2,3,4$, $|D_if(\mathbf{v}^k(s))|_{\mathcal{L}(Y_i,\mathbb{R}^n)} \le L_1$ for $i = 2,3,4$, $s \in [0,\alpha]$ and $k \in \mathbb{N}_0$, (2.3.15), (2.4.9), (2.4.36), (2.4.41), (2.4.43) and (2.4.44). yields (2.4.39)

□

Lemma 2.4.12 *Assume (A1) (i)–(vi), (A2) (i)–(vii), $\gamma \in P_2$. Then there exists $N_5 = N_5(\gamma) \ge 0$ such that the solution of the IVP* (2.4.26)-(2.4.27) *satisfies*

$$
|w^{h,y}(t)| \le N_5|h|_\Gamma|y|_\Gamma, \qquad t \in [-r,\alpha], \quad h,y \in \Gamma. \tag{2.4.45}
$$

Proof It follows from (2.4.26) and (2.4.27) that

$$
w^{h,y}(t) = \int_0^t B(s)\langle (z_s^h, h^\theta, h^\xi), (z_s^y, y^\theta, y^\xi)\rangle\, ds + \int_0^t L(s,x)(w_s^{h,y}, 0, 0)\, ds, \qquad t \in [0,\alpha].
$$

Therefore (2.3.11) and (2.4.38) yield

$$
|w^{h,y}(t)| \le K_{12}|h|_\Gamma|y|_\Gamma + L_1N_0 \int_0^t |w_s^{h,y}|_C\, ds, \qquad t \in [0,\alpha].
$$

Since $w^{h,y}(t) = 0$ for $t \in [-r,0]$, Lemma 1.2.1 gives (2.4.45) with $N_5 := K_{12}e^{L_1N_0\alpha}$. □

Lemma 2.4.13 *Assume (A1) (i)–(vi), (A2) (i)–(viii), (H). For $h, y \in \Gamma_2$ and $k \in \mathbb{N}$ let $w^{h,y}(t) := w(t,\gamma,h,y)$ and $w^{k,h,y}(t) := w(t,\gamma+h_k,h,y)$ be the solutions of the IVP* (2.4.26)-(2.4.27)*. Then there exists a nonnegative sequence $c_{7,k} = c_{7,k}(\gamma)$ such that*

$$|w_t^{k,h,y} - w_t^{h,y}|_C \le c_{7,k}|h|_{\Gamma_2}|y|_{\Gamma_2}, \qquad t \in [0,\alpha], \quad h, y \in \Gamma_2. \tag{2.4.46}$$

Proof It follows from (2.3.11), (2.3.17), (2.3.26), (2.4.26), (2.4.38), (2.4.34) and (2.4.45)

$$\begin{aligned}
&|w^{k,h,y}(t) - w^{h,y}(t)| \\
&\le \int_0^t \Big(|[L(s,x^k) - L(s,x)](w_s^{k,h,y},0,0)| + |L(s,x)(w_s^{k,h,y} - w_s^{h,y},0,0)| \Big) ds \\
&\quad + \int_0^t \Big(|B^k(s)\langle (z_s^{k,h},h^\theta,h^\xi),(z_s^{k,y} - z_s^y,0,0)\rangle| + |B^k(s)\langle (z_s^{k,h} - z_s^h,0,0),(z_s^y,y^\theta,y^\xi)\rangle| \\
&\qquad + |B^k(s)\langle (z_s^h,h^\theta,h^\xi),(z_s^y,y^\theta,y^\xi)\rangle - B(s)\langle (z_s^h,h^\theta,h^\xi),(z_s^y,y^\theta,y^\xi)\rangle| \Big) ds \\
&\le \alpha c_{0,k} N_5 |h|_\Gamma |y|_\Gamma + L_1 L_2 \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds N_5 |h|_\Gamma |y|_\Gamma \\
&\quad + L_1 N_0 \int_0^t |w_s^{k,h,y} - w_s^{h,y}|_C\, ds + 2\alpha K_{12} c_{1,k} N_1^2 |h|_\Gamma |y|_\Gamma + \alpha c_{5,k} |h|_\Gamma |y|_\Gamma \\
&\le c_{8,k} |h|_\Gamma |y|_\Gamma + L_1 N_0 \int_0^t |w_s^{k,h,y} - w_s^{h,y}|_C\, ds,
\end{aligned}$$

where $c_{8,k} = c_{8,k}(\gamma) := \alpha c_{0,k} N_5 + L_1 L_2 (L + K_4 K_0) N_5 |h_k|_\Gamma + 2\alpha K_{12} c_{1,k} N_1^2 + \alpha c_{5,k}$. Then Lemma 1.2.1 is applicable, since $|w_0^{k,h,y} - w_0^{h,y}|_C = 0$, and it yields (2.4.46) with $c_{7,k} := c_{8,k} e^{L_1 N_0 \alpha}$. □

We define

$$\begin{aligned}
\omega_{D_2 f}(\mathbf{v}(s),\mathbf{v}^k(s),\psi) &:= D_2 f(\mathbf{v}^k(s))\psi - D_2 f(\mathbf{v}(s))\psi - D_{22} f(\mathbf{v}(s))\langle \psi, x_s^k - x_s\rangle \\
&\quad - D_{23} f(\mathbf{v}(s))\langle \psi, x^k(u^k(s)) - x(u(s))\rangle - D_{24} f(\mathbf{v}(s))\langle \psi, h_k^\theta\rangle, \\
\omega_{D_3 f}(\mathbf{v}(s),\mathbf{v}^k(s),v) &:= D_3 f(\mathbf{v}^k(s))v - D_3 f(\mathbf{v}(s))v - D_{32} f(\mathbf{v}(s))\langle v, x_s^k - x_s\rangle \\
&\quad - D_{33} f(\mathbf{v}(t))\langle v, x^k(u^k(s)) - x(u(s))\rangle - D_{34} f(\mathbf{v}(s))\langle v, h_k^\theta\rangle, \\
\omega_{D_4 f}(\mathbf{v}(s),\mathbf{v}^k(s),\eta) &:= D_4 f(\mathbf{v}^k(s))\eta - D_4 f(\mathbf{v}(s))\eta - D_{42} f(\mathbf{v}(s))\langle \eta, x_s^k - x_s\rangle \\
&\quad - D_{43} f(\mathbf{v}(s))\langle \eta, x^k(u^k(s)) - x(u(s))\rangle - D_{44} f(\mathbf{v}(s))\langle \eta, h_k^\theta\rangle
\end{aligned}$$

for $s \in [0,\alpha]$, $\psi \in C$, $v \in \mathbb{R}^n$ and $\eta \in \Theta$.

The proof of the following lemma is similar to that of Lemma 2.4.7.

Lemma 2.4.14 *Assume (A1) (i)–(vi) and (H). Then*

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\omega_{D_2 f}(s, x_s, x(u(s)), \theta, x_s^k, x^k(u^k(s)), \theta + h_k^\theta, z_s^{k,h})|\, ds = 0, \tag{2.4.47}$$

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma}}\frac{1}{|h|_\Gamma|h_k|_\Gamma}\int_0^\alpha|\omega_{D_3f}(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta,E^k(s,z_s^{k,h},h^\xi))|\,ds=0,\tag{2.4.48}$$

and

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma}}\frac{1}{|h|_\Gamma|h_k|_\Gamma}\int_0^\alpha|\omega_{D_4f}(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta,h_k^\theta)|\,ds=0.\tag{2.4.49}$$

Lemma 2.4.15 *Assume (A1) (i)–(vi), (A2) (i)–(vii), (H), $\gamma\in\mathcal{P}$ and $h_k\in\Gamma_2$ for $k\in\mathbb{N}$. Then*

$$\begin{aligned}&L(s,x^k)(z_s^{k,h},h^\theta,h^\xi)-L(s,x)(z_s^h+w_s^{h,h_k},h^\theta,h^\xi)-B(s)\Big\langle(z_s^h,h^\theta,h^\xi),(z_s^{h_k},h_k^\theta,h_k^\xi)\Big\rangle\\ &\quad= L(s,x)(q_s^{k,h},0,0)+g_5^{k,h}(s),\qquad a.e.\ s\in[0,\alpha],\end{aligned}\tag{2.4.50}$$

where

$$\begin{aligned}g_5^{k,h}(s)\;:=\;&D_{22}f(\mathbf{v}(s))\langle z_s^{k,h}-z_s^h,x_s^k-x_s\rangle+D_{22}f(\mathbf{v}(s))\langle z_s^h,p_s^k\rangle\\ &+D_{23}f(\mathbf{v}(s))\langle z_s^{k,h}-z_s^h,x^k(u^k(s))-x(u(s))\rangle-D_{23}f(\mathbf{v}(s))\langle z_s^h,g_1^k(s)\rangle\\ &+D_{24}f(\mathbf{v}(s))\langle z_s^{k,h}-z_s^h,h_k^\theta\rangle+D_{32}f(\mathbf{v}(s))\langle E(s,z_s^h,h^\xi),p_s^k\rangle\\ &+D_{32}f(\mathbf{v}(s))\langle E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi),x_s^k-x_s\rangle\\ &+D_{33}f(\mathbf{v}(s))\langle E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi),x^k(u^k(s))-x(u(s))\rangle\\ &+D_{33}f(\mathbf{v}(s))\langle E(s,z_s^h,h^\xi),g_1^k(s)\rangle+D_3f(\mathbf{v}(s))g_4^{k,h}(s)\\ &+D_{34}f(\mathbf{v}(s))\langle E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi),h_k^\theta\rangle\\ &+D_{42}f(\mathbf{v}(s))\langle h^\theta,p_s^k\rangle+D_{43}f(\mathbf{v}(s))\langle h^\theta,g_1^k(s)\rangle+\omega_{D_2f}(\mathbf{v}(s),\mathbf{v}^k(s),z_s^{k,h})\\ &+\omega_{D_3f}(\mathbf{v}(s),\mathbf{v}^k(s),E^k(s,z_s^{k,h},h^\xi))+\omega_{D_4f}(\mathbf{v}(s),\mathbf{v}^k(s),h_k^\theta)\end{aligned}$$

satisfies

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma_2}}\frac{1}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}}\int_0^\alpha|g_5^{k,h}(s)|\,ds=0.\tag{2.4.51}$$

Proof Straightforward manipulations yield for a.e. $s\in[0,\alpha]$

$$\begin{aligned}&L(s,x^k)(z_s^{k,h},h^\theta,h^\xi)-L(s,x)(z_s^h+w_s^{h,h_k},h^\theta,h^\xi)-B(s)\Big\langle(z_s^h,h^\theta,h^\xi),(z_s^{h_k},h_k^\theta,h_k^\xi)\Big\rangle\\ &=\;D_2f(\mathbf{v}^k(s))z_s^{k,h}-D_2f(\mathbf{v}(s))z_s^{k,h}+D_2f(\mathbf{v}(s))(z_s^{k,h}-z_s^h-w_s^{h,h_k})\\ &\quad+D_3f(\mathbf{v}^k(s))E^k(s,z_s^{k,h},h^\xi)-D_3f(\mathbf{v}(s))E^k(s,z_s^{k,h},h^\xi)\\ &\quad+D_3f(\mathbf{v}(s))\Big(E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi)\Big)+D_4f(\mathbf{v}^k(s))h^\theta-D_4f(\mathbf{v}(s))h^\theta\\ &\quad-D_3f(\mathbf{v}(s))E(s,w_s^{h,h_k},0)-B(s)\Big\langle(z_s^h,h^\theta,h^\xi),(z_s^{h_k},h_k^\theta,h_k^\xi)\Big\rangle\end{aligned}$$

$$\begin{aligned}
= \; & D_2 f(\mathbf{v}^k(s)) z_s^{k,h} - D_2 f(\mathbf{v}(s)) z_s^{k,h} - D_{22} f(\mathbf{v}(s)) \langle z_s^{k,h}, x_s^k - x_s \rangle \\
& - D_{23} f(\mathbf{v}(s)) \langle z_s^{k,h}, x^k(u^k(s)) - x(u(s)) \rangle - D_{24} f(\mathbf{v}(s)) \langle z_s^{k,h}, h_k^\theta \rangle \\
& + D_2 f(\mathbf{v}(s)) q_s^{k,h} + D_{22} f(\mathbf{v}(s)) \langle z_s^{k,h} - z_s^h, x_s^k - x_s \rangle + D_{22} f(\mathbf{v}(s)) \langle z_s^h, p_s^k \rangle \\
& + D_{23} f(\mathbf{v}(s)) \langle z_s^{k,h} - z_s^h, x^k(u^k(s)) - x(u(s)) \rangle \\
& + D_{23} f(\mathbf{v}(s)) \langle z_s^h, x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi) \rangle + D_{24} f(\mathbf{v}(s)) \langle z_s^{k,h} - z_s^h, h_k^\theta \rangle \\
& + D_3 f(\mathbf{v}^k(s)) E^k(s, z_s^{k,h}, h^\xi) - D_3 f(\mathbf{v}(s)) E^k(s, z_s^{k,h}, h^\xi) \\
& - D_{32} f(\mathbf{v}(s)) \langle E^k(s, z_s^{k,h}, h^\xi), x_s^k - x_s \rangle \\
& - D_{33} f(\mathbf{v}(s)) \langle E^k(s, z_s^{k,h}, h^\xi), x^k(u^k(s)) - x(u(s)) \rangle - D_{34} f(\mathbf{v}(s)) \langle E^k(s, z_s^{k,h}, h^\xi), h_k^\theta \rangle \\
& + D_{32} f(\mathbf{v}(s)) \langle E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi), x_s^k - x_s \rangle + D_{32} f(\mathbf{v}(s)) \langle E(s, z_s^h, h^\xi), p_s^k \rangle \\
& + D_{33} f(\mathbf{v}(s)) \langle E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi), x^k(u^k(s)) - x(u(s)) \rangle \\
& + D_{33} f(\mathbf{v}(s)) \langle E(s, z_s^h, h^\xi), x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi) \rangle \\
& + D_{34} f(\mathbf{v}(s)) \langle E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi), h_k^\theta \rangle \\
& + D_3 f(\mathbf{v}(s)) [E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - H(s) \langle (z_s^h, h^\xi), (z_s^{k,h}, h_k^\xi) \rangle - E(s, w_s^{h,h_k}, 0)] \\
& + D_4(\mathbf{v}^k(s)) h^\theta - D_4(\mathbf{v}(s)) h^\theta - D_{42} f(\mathbf{v}(s)) \langle h^\theta, x_s^k - x_s \rangle \\
& - D_{43} f(\mathbf{v}(s)) \langle h^\theta, x^k(u^k(s)) - x(u(s)) \rangle - D_{44} f(\mathbf{v}(s)) \langle h^\theta, h_k^\theta \rangle \\
& + D_{42} f(\mathbf{v}(s)) \langle h^\theta, p_s^k \rangle + D_{43} f(\mathbf{v}(s)) \langle h^\theta, x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi) \rangle,
\end{aligned}$$

which implies (2.4.50), using (2.4.19) and (2.4.32). Let $L_3 = L_3(\alpha, M_1, M_2, M_3)$ be defined by (A1) (iv). Then (A1) (iv), (2.2.2), (2.3.16), (2.3.18), (2.3.25), (2.4.9) and (2.4.36) yield

$$\begin{aligned}
& \int_0^\alpha |g_5^{k,h}(s)| \, ds \\
\leq \; & \alpha L_3 c_{1,k} N_1 |h|_\Gamma L |h_k|_\Gamma + \alpha L_3 N_1 |h|_\Gamma \max_{s \in [0,\alpha]} |p_s^k|_C + \alpha L_3 c_{1,k} N_1 |h|_\Gamma K_2 |h_k|_\Gamma \\
& + L_3 N_1 |h|_\Gamma \int_0^\alpha |g_{1,k}(s)| \, ds + \alpha L_3 c_{1,k} N_1 |h|_\Gamma |h_k|_\Gamma + \alpha L_3 K_8 |h|_\Gamma \max_{s \in [0,\alpha]} |p_s^k|_C \\
& + \alpha L_3 c_{3,k} |h|_\Gamma L |h_k|_\Gamma + \alpha L_3 c_{3,k} |h|_\Gamma K_2 |h_k|_\Gamma \\
& + L_3 K_8 |h|_\Gamma \int_0^\alpha |g_{1,k}(s)| \, ds + L_1 \int_0^\alpha |g_{3,k}^h(s)| \, ds + \alpha L_3 c_{3,k} |h|_\Gamma |h_k|_\Gamma \\
& + L_3 |h|_\Gamma \max_{s \in [0,\alpha]} |p_s^k|_C + L_3 |h|_\Gamma \int_0^\alpha |g_{1,k}(s)| \, ds + \int_0^\alpha |\omega_{D_2 f}(\mathbf{v}(s), \mathbf{v}^k(s), z_s^{k,h})| \, ds \\
& + \int_0^\alpha |\omega_{D_3 f}(\mathbf{v}(s), \mathbf{v}^k(s), E^k(s, z_s^{k,h}, h^\xi))| \, ds + \int_0^\alpha |\omega_{D_4 f}(\mathbf{v}(s), \mathbf{v}^k(s), h_k^\theta)| \, ds.
\end{aligned}$$

Hence $c_{1,k} \to 0$, $c_{3,k} \to 0$ as $k \to \infty$, (2.4.16), (2.4.20), (2.4.31), (2.4.47), (2.4.48) and (2.4.49) imply (2.4.51).

$\square$

Now we are ready to prove the main result of this section.

Theorem 2.4.16 *Assume (A1) (i)–(vi), (A2) (i)–(vii). Then for* $t \in [0, \alpha]$ *the maps*

$$\Gamma_2 \supset (P_2 \cap \Gamma_2) \to \mathbb{R}^n, \quad \gamma \mapsto x(t, \gamma)$$

and

$$\Gamma_2 \supset (P_2 \cap \Gamma_2) \to C, \quad \gamma \mapsto x_t(\cdot, \gamma)$$

are twice differentiable wrt γ *for every* $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$, *and*

$$D_{22}x(t, \gamma)\langle h, y\rangle = w^{h,y}(t), \qquad h, y \in \Gamma_2,$$

and

$$D_{22}x_t(\cdot, \gamma)\langle h, y\rangle = w_t^{h,y}, \qquad h, y \in \Gamma_2,$$

where $w^{h,y}$ *is the solution of the IVP* (2.4.26)-(2.4.27). *Moreover, if in addition, (A2) (viii) holds, then the maps*

$$\mathbb{R} \times \Gamma_2 \supset \Big([0, \alpha] \times (P_2 \cap \Gamma_2 \cap \mathcal{P})\Big) \to \mathcal{L}^2(\Gamma_2 \times \Gamma_2, \mathbb{R}^n), \quad (t, \gamma) \mapsto D_{22}x(t, \gamma)$$

and

$$\mathbb{R} \times \Gamma_2 \supset \Big([0, \alpha] \times (P_2 \cap \Gamma_2 \cap \mathcal{P})\Big) \to \mathcal{L}^2(\Gamma_2 \times \Gamma_2, C), \quad (t, \gamma) \mapsto D_{22}x_t(\cdot, \gamma)$$

are continuous.

Proof It follows from Theorem 2.3.9 that $D_2x(t, \gamma) \in \mathcal{L}(\Gamma, \mathbb{R}^n)$ exists for all $\gamma \in P_2$ and $t \in [0, \alpha]$. Since $|h|_\Gamma \le |h|_{\Gamma_2}$ for all $h \in \Gamma_2$, it follows that $D_2x(t, \gamma)\Big|_{\Gamma_2} \in \mathcal{L}(\Gamma_2, \mathbb{R}^n)$, and $D_2x(t, \gamma)\Big|_{\Gamma_2}$ is the derivtive of the map $\Gamma_2 \supset (P_2 \cap \Gamma_2) \to \mathbb{R}^n$, $\gamma \to x(t, \gamma)$. For simplicity, the restiction of $D_2x(t, \gamma)$ to Γ_2 will be denoted by $D_2x(t, \gamma)$, as well. Theorem 2.3.9 yields that $D_2x(t, \gamma)h = z(t, \gamma, h)$, where $z(t, \gamma, h)$ is the solution of the IVP (2.3.13)-(2.3.14) for $h \in \Gamma_2$.

Let $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$ be fixed, $h_k = (h_k^\varphi, h_k^\theta, h_k^\xi) \in \Gamma_2$ ($k \in \mathbb{N}$) be a sequence such that $\gamma + h_k \in P_2$ for $k \in \mathbb{N}$, $0 \ne h = (h^\varphi, h^\theta, h^\xi) \in \Gamma_2$. Let $x(t) := x(t, \gamma)$ and $x^k(t) := x(t, \gamma + h_k)$ be the solutions of the IVP (2.1.1)-(2.1.2), $z^h(t) := D_2x(t, \gamma)h$ and $z^{k,h}(t) := D_2x(t, \gamma + h_k)h$ be the solution of the IVP (2.3.13)-(2.3.14), and $w^{h,h_k}(t)$ be the solution of the IVP (2.4.26)-(2.4.27) corresponding to parameters h and h_k. Then we have for $t \in [0, \alpha]$

$$\begin{aligned}
z^{k,h}(t) &= h^\varphi(0) + \int_0^t L(s, x^k)(z_s^{k,h}, h^\theta, h^\xi)\, ds, \\
z^h(t) &= h^\varphi(0) + \int_0^t L(s, x)(z_s^h, h^\theta, h^\xi)\, ds, \\
w^{h,h_k}(t) &= \int_0^t \Big(L(s, x)(w_s^{h,h_k}, 0, 0) + B(s)\Big\langle (z_s^h, h^\theta, h^\xi), (z_s^{h_k}, h_k^\theta, h_k^\xi)\Big\rangle\Big) ds.
\end{aligned}$$

Hence Lemma 2.4.15 and the definition of $q^{k,h}$ give

$$\begin{aligned} q^{k,h}(t) &= \int_0^t \Big(L(s,x^k)(z_s^{k,h},h^\theta,h^\xi) - L(s,x)(z_s^h + w_s^{h,h_k},h^\theta,h^\xi) \\ &\qquad -B(s)\Big\langle (z_s^h,h^\theta,h^\xi),(z_s^{h_k},h_k^\theta,h_k^\xi)\Big\rangle\Big)\,ds \\ &= \int_0^t g_5^{k,h}(s)\,ds + \int_0^t L(s,x)(q_s^{k,h},0,0)\,ds, \qquad t\in[0,\alpha], \end{aligned}$$

so (2.3.11) yields

$$|q^{k,h}(t)| \le \int_0^t |g_5^{k,h}(s)|\,ds + \int_0^t |L(s,x)(q_s^{k,h},0,0)|\,ds \le \int_0^\alpha |g_5^{k,h}(s)|\,ds + L_1N_0\int_0^t |q_s^{k,h}|_C\,ds,$$

for $t\in[0,\alpha]$. Using that $q^{k,h}(t)=0$ for $t\in[-r,0]$, Lemma 1.2.1 implies

$$|q^{k,h}(t)| \le |q_t^{k,h}|_C \le N_1\int_0^\alpha |g_5^{k,h}(s)|\,ds, \qquad t\in[0,\alpha],$$

where $N_1 := e^{L_1N_0\alpha}$. Therefore (2.4.51) yields for $t\in[0,\alpha]$

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma_2}} \frac{|q^{k,h}(t)|}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}} \le \lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma_2}} \frac{|q_t^{k,h}|_C}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}} \le \lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma_2}} \frac{N_1}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}}\int_0^\alpha |g_5^{k,h}(s)|\,ds = 0,$$

which completes the proof of the second-order differentiability wrt parameters. The continuity of $D_{22}x(t,\gamma)$ follows from Lemma 2.4.13. □

We note that the method used in this section to prove the existence of the second order derivative $D_{22}x(t,\gamma)$ can not be used to prove the existence of the third order derivative, since some parts of the proof relied on the assumption that the parameter γ satisfies the compatibility condition $\gamma\in\mathcal{P}$. The key step to show the existence of higher order derivatives is to get rid of this assumption in the proof of Theorem 2.4.16.

Chapter 3

Parameter estimation by quasilinearization

3.1 Introduction

Estimation of unknown parameters in various classes of differential equations, and in particular in FDEs, has been investigated by many authors (see, e.g., [6, 7, 14, 15, 17, 51, 52, 54, 55, 59, 79]).

In this chapter we consider again the nonlinear SD-DDE (2.1.1)

$$\dot{x}(t) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \theta\Big), \qquad t \in [0, T] \tag{3.1.1}$$

with the associated initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{3.1.2}$$

For simplicity we assume throughout this chapter that (3.1.1) is a scalar equation, which is defined on the whole space, i.e., we suppose

(B1) $n = 1$, $\Omega_1 = C$, $\Omega_2 = \mathbb{R}$, $\Omega_3 = \Theta$, and $\Omega_4 = \Xi$.

By Theorem 2.2.1, (A1) (i)–(ii), (A2) (i)-(ii) and (B1) imply that the IVP (3.1.1)-(3.1.2) has a unique solution $x(t, \gamma)$ on an interval $[-r, \alpha]$ and $\gamma \in P$, where P is a neighborhood of a fixed parameter $\widehat{\gamma} \in \Gamma$, and the parameter map $\Gamma \to \mathbb{R},\ \gamma \mapsto x(t, \gamma)$ is differentiable for every $\gamma \in P_1$.

We assume that the parameter $\gamma = (\varphi, \xi, \theta) \in \Gamma$ is unknown, but there are measurements $X_0, X_1, \ldots, X_l$ of the solution at the points $t_0, t_1, \ldots, t_l \in [0, \alpha]$. Our goal is to find a parameter value which minimizes the least square cost function

$$J(\gamma) := \sum_{i=0}^{l} (x(t_i, \gamma) - X_i)^2 \tag{3.1.3}$$

over the parameter space Γ. Denote this infinite dimensional minimization problem by $\mathcal{P}$.

The method of quasilinearization for parameter estimation was introduced for ODEs in [8] and was applied to identify finite dimensional parameters in FDEs in [14] and [15]. The method uses the derivative of the solution wrt the parameters. This problem was studied, e.g., in [13], [42], [43], [63] for several classes of state-independent FDEs, and see Section 2.1 for SD-DDEs.

Next we briefly show the derivation of the quasilinearization method following the procedure suggested in [62]. Let Γ^N be an N-dimensional subspace of the parameter space Γ, and let $\gamma_k = (\varphi^k, \theta_k, \xi_k) \in \Gamma^N$ be fixed, and consider the corresponding solution of the IVP (2.1.1)-(2.1.2), $x(t, \gamma_k)$. For a fixed $i \in \{0, 1, \ldots, \ell\}$ take first order Taylor-approximation of $x(t_i, \gamma)$ around the parameter γ_k:

$$x(t_i, \gamma) \approx x(t_i, \gamma_k) + D_2 x(t_i, \gamma_k)(\gamma - \gamma_k),$$

and consider the approximate cost function restricted to the subspace Γ^N defined by

$$J^{k,N}(\gamma) := \sum_{i=0}^{l} \Big(x(t_i, \gamma_k) + D_2 x(t_i, \gamma_k)(\gamma - \gamma_k) - X_i \Big)^2, \qquad \gamma \in \Gamma^N.$$

We solve the minimization problem $\mathcal{P}^{k,N}$:

$$\min_{\gamma \in \Gamma^N} J^{k,N}(\gamma).$$

Fix a basis $\{\chi_1^N, \ldots, \chi_N^N\}$ for the finite dimensional subspace Γ^N, and let

$$\gamma_k := \sum_{j=1}^{N} c_j^k \chi_j^N \qquad \text{and} \qquad \gamma := \sum_{j=1}^{N} c_j \chi_j^N.$$

We introduce the vectors $\mathbf{c}^k = (c_1^k, \ldots, c_N^k)^T \in \mathbb{R}^N$ and $\mathbf{c} = (c_1, \ldots, c_N)^T \in \mathbb{R}^N$. Then we can identify the finite dimensional parameters γ_k and $\gamma \in \Gamma^N$ with the vectors $\mathbf{c}^k$ and $\mathbf{c} \in \mathbb{R}^N$, so we simply write $x(t_i, \mathbf{c}^k)$ and $J^{k,N}(\mathbf{c})$ instead of $x(t_i, \gamma_k)$ and $J^{k,N}(\gamma)$. Then we have

$$\begin{aligned} J^{k,N}(\mathbf{c}) &= \sum_{i=0}^{l} \Big(x(t_i, \mathbf{c}^k) + D_2 x(t_i, \mathbf{c}^k) \sum_{j=1}^{N} (c_j - c_j^k) \chi_j^N - X_i \Big)^2 \\ &= \sum_{i=0}^{l} \Big(x(t_i, \mathbf{c}^k) - X_i + \sum_{j=1}^{N} (c_j - c_j^k) D_2 x(t_i, \mathbf{c}^k) \chi_j^N \Big)^2. \end{aligned}$$

To find the minimizer of $J^{k,N}(\mathbf{c})$ first consider

$$\frac{\partial}{\partial c_p} J^{k,N}(\mathbf{c}) = 2 \sum_{i=0}^{l} \Big(x(t_i, \mathbf{c}^k) - X_i + \sum_{j=1}^{N} (c_j - c_j^k) D_2 x(t_i, \mathbf{c}^k) \chi_j^N \Big) D_2 x(t_i, \mathbf{c}^k) \chi_p^N.$$

We introduce the N-dimensional vectors

$$\mathbf{m}(t_i, \mathbf{c}^k) := \Big(D_2 x(t_i, \mathbf{c}^k)\chi_1^N, \ldots, D_2 x(t_i, \mathbf{c}^k)\chi_N^N\Big)^T, \tag{3.1.4}$$

$$\mathbf{b}(\mathbf{c}^k) := \sum_{i=0}^{l} \mathbf{m}(t_i, \mathbf{c}^k)(x(t_i, \mathbf{c}^k) - X_i) \tag{3.1.5}$$

and the $N \times N$ matrix

$$\mathbf{D}(\mathbf{c}^k) := \sum_{i=0}^{l} \mathbf{m}(t_i, \mathbf{c}^k)\mathbf{m}^T(t_i, \mathbf{c}^k). \tag{3.1.6}$$

Then $\frac{\partial}{\partial c_p} J^{k,N}(\mathbf{c}) = 0$ for $p = 1, \ldots, N$, if and only if

$$\mathbf{D}(\mathbf{c}^k)(\mathbf{c} - \mathbf{c}^k) = -\mathbf{b}(\mathbf{c}^k). \tag{3.1.7}$$

We note that the Hessian of $J^{k,N}(\mathbf{c})$ is $2\mathbf{D}(\mathbf{c}^k)$.

Lemma 3.1.1 $\mathbf{D}(\mathbf{c}^k)$ *is a positive semi-definite $N \times N$ matrix, and it is positive definite, if and only if there is no $\mathbf{u} \in \mathbb{R}^N$ such that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \perp \mathbf{m}(t_i, \mathbf{c}^k)$ for $i = 0, \ldots, N$.*

Proof Let $\mathbf{u} \in \mathbb{R}^N$ and consider

$$\mathbf{u}^T\mathbf{D}(\mathbf{c}^k)\mathbf{u} = \sum_{i=0}^{l} \mathbf{u}^T\mathbf{m}(t_i, \mathbf{c}^k)\mathbf{m}^T(t_i, \mathbf{c}^k)\mathbf{u} = \sum_{i=0}^{l} \Big(\mathbf{m}^T(t_i, \mathbf{c}^k)\mathbf{u}\Big)^T \mathbf{m}^T(t_i, \mathbf{c}^k)\mathbf{u} \geq 0,$$

which yields the statement of the lemma. □

Assuming that $\mathbf{D}(\mathbf{c}^k)$ is invertible for all $k = 0, 1, \ldots$, we define the quasilinearization method by the iteration

$$\mathbf{c}^{k+1} = \mathbf{c}^k - \mathbf{D}^{-1}(\mathbf{c}^k)\mathbf{b}(\mathbf{c}^k), \qquad k = 0, 1, \ldots. \tag{3.1.8}$$

Lemma 3.1.1 and the previous calculation imply that $\mathbf{c}^{k+1}$ is the unique minimizer of $J^{k,N}(\mathbf{c})$.

This is the same scheme that was used in [14] and [15] except that there the parameter space was finite dimensional, and the set $\{\chi_1^N, \ldots, \chi_N^N\}$ was the canonical basis of $\mathbb{R}^N$. In our examples the parameter space will be the space of Lipschitz continuous functions, and therefore $D_2 x(t_i, \mathbf{c}^k)$ is a linear functional defined on the space of $W^{1,\infty}$-functions, and $D_2 x(t_i, \mathbf{c}^k)\chi_j^N$ denotes the value of the linear functional applied to the function χ_j^N. For the derivation of this method for ODEs with finite dimensional parameters we refer to [8].

3.2 Convergence results

In this section we show the local convergence of the scheme (3.1.8) supposing the existence of an exact fit solution of the parameter estimation problem $\mathcal{P}$. We assume

(B2) $\Gamma^N \subset \Gamma$ is a finite dimensional subspace for all $N \in \mathbb{N}$;

(B3) there exists $\gamma^* \in \Gamma$, for which $J(\gamma^*) = 0$.

The next theorem studies the convergence of the quasilinearization scheme (3.1.8) in the case when $\gamma^* \in \Gamma^N$ for some $N \in \mathbb{N}$.

Definition 3.2.1 *We say that the sequence* $\mathbf{c}^k \in \mathbb{R}^N$ *converges to* $\mathbf{c}^* \in \mathbb{R}^N$ *superlinearly, if there exists a sequence* $\varepsilon_k \geq 0$ *such that* $\varepsilon_k \to 0$ *as* $k \to \infty$*, and*

$$|\mathbf{c}^{k+1} - \mathbf{c}^*| \leq \varepsilon_k |\mathbf{c}^k - \mathbf{c}^*|, \qquad k \in \mathbb{N}.$$

As in Section 2.3, we define the parameter set $P_1 := \{\gamma \in \Gamma\colon\ x(\cdot,\gamma) \in X(\alpha,\xi)\}$, where

$$X(\alpha,\xi) := \Big\{x \in W^{1,\infty}([-r,\alpha],\mathbb{R})\colon\ \operatorname{ess\,inf}\{\frac{d}{dt}(t - \tau(t,x_t,\xi))\colon\ \text{a.e. } t \in [0,\alpha^*]\} > 0\Big\}.$$

We know (see [48] and [58]) that P_1 is an open subset of Γ, and it follows from Theorem 2.3.9 and Remark 2.3.10 that for every $\hat{\gamma} \in P_1$ there exists a $\delta > 0$ such that $D_2 x(t,\gamma) \in \mathcal{L}(\Gamma,\mathbb{R})$ exists and it is continuous for $t \in [0,\alpha]$ and $\gamma \in \mathcal{B}_\Gamma(\hat{\gamma};\, \delta)$.

Theorem 3.2.2 *Assume (A1) (i)–(iii), (A2) (i)–(iii) and (B1)–(B3). Suppose* $\gamma^* \in P_1$*, and suppose* $\gamma^* = \sum_{j=1}^N c_j^* \chi_j^N \in \Gamma^N$ *for some* $N \in \mathbb{N}$*, and* $\mathbf{D}(\mathbf{c}^*)$ *is invertible where* $\mathbf{c}^* := (c_1^*, \ldots, c_N^*)^T$*. Then for this* N *the quasilinearization sequence* (3.1.8) *is locally superlinearly convergent to* $\mathbf{c}^*$*.*

Proof It follows from Theorem 2.3.9 and Remark 2.3.10 that there exists $\delta_1 > 0$ such that $D_2 x(t,\gamma) \in \mathcal{L}(\Gamma,\mathbb{R})$ exists and it is continuous for $t \in [0,\alpha]$ and $\gamma \in \mathcal{B}_\Gamma(\gamma^*;\, \delta_1)$. Then there exists $\delta_2 > 0$ such that for $|\mathbf{c} - \mathbf{c}^*| < \delta_2$ it follows that the corresponding parameter $\gamma = \sum_{j=1}^N c_j \chi_j^N \in \mathcal{B}_\Gamma(\gamma^*;\, \delta_1)$. Hence $\mathbf{D}(\mathbf{c})$ is well-defined and continuous on $\mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\, \delta_2)$. Since $\mathbf{D}(\mathbf{c})$ is invertible at $\mathbf{c}^*$ and continuous, there exist $0 < \delta_3 \leq \delta_2$ and $d > 0$ such that $\mathbf{D}(\mathbf{c})$ is invertible and satisfies

$$\Big|\mathbf{D}^{-1}(\mathbf{c})\Big| \leq d, \qquad \text{for } \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\, \delta_3).$$

Then the function

$$\mathbf{g}\colon\ \mathbb{R}^N \supset \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\, \delta_3) \to \mathbb{R}^N, \qquad \mathbf{g}(\mathbf{c}) := \mathbf{c} - \mathbf{D}^{-1}(\mathbf{c})\mathbf{b}(\mathbf{c})$$

is well-defined. Consider

$$\begin{aligned} g(\mathbf{c}) - \mathbf{c}^* &= \mathbf{c} - \mathbf{c}^* - \mathbf{D}^{-1}(\mathbf{c})\mathbf{b}(\mathbf{c}) \\ &= \mathbf{D}^{-1}(\mathbf{c})\Big(\mathbf{D}(\mathbf{c})(\mathbf{c} - \mathbf{c}^*) - \mathbf{b}(\mathbf{c})\Big) \\ &= \mathbf{D}^{-1}(\mathbf{c})\sum_{i=0}^{l}\mathbf{m}(t_i,\mathbf{c})\Big(\mathbf{m}^T(t_i,\mathbf{c})(\mathbf{c} - \mathbf{c}^*) - (x(t_i,\mathbf{c}) - X_i)\Big). \end{aligned} \tag{3.2.1}$$

Now using the exact fit-to-data assumption, $\mathbf{c}^*$ satisfies $x(t_i, \mathbf{c}^*) = X_i$ for $i = 1, \dots, N$, hence (3.2.1) yields

$$g(\mathbf{c}) - \mathbf{c}^* = -\mathbf{D}^{-1}(\mathbf{c})\sum_{i=0}^{l}\mathbf{m}(t_i,\mathbf{c})\Big(x(t_i,\mathbf{c}) - x(t_i,\mathbf{c}^*) - \mathbf{m}^T(t_i,\mathbf{c})(\mathbf{c} - \mathbf{c}^*)\Big). \tag{3.2.2}$$

It follows from (2.3.15) that

$$|D_2 x(t_i,\mathbf{c})\chi_j^N| \le N_1|\chi_j^N|_\Gamma \qquad \text{for } i = 0,\dots,\ell,\ \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\, \delta_3),\ \text{and } j = 1,\dots,N.$$

Then there exists $m_0 > 0$ such that

$$|\mathbf{m}(t_i,\mathbf{c})| \le m_0, \qquad i = 0,\dots,\ell, \quad \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\, \delta_3). \tag{3.2.3}$$

Hence (3.2.2) implies

$$|g(\mathbf{c}) - \mathbf{c}^*| \le d m_0 \sum_{i=0}^{l}\Big|x(t_i,\mathbf{c}) - x(t_i,\mathbf{c}^*) - \mathbf{m}^T(t_i,\mathbf{c})(\mathbf{c} - \mathbf{c}^*)\Big|, \qquad \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\, \delta_3).$$

We have

$$\mathbf{m}^T(t_i,\mathbf{c})(\mathbf{c} - \mathbf{c}^*) = D_2 x(t_i,\gamma)(\gamma - \gamma^*),$$

where $\gamma := \sum_{j=1}^N c_j\chi_j^N$ and $\gamma^* := \sum_{j=1}^N c_j^*\chi_j^N$. Therefore

$$\begin{aligned} &x(t_i,\mathbf{c}) - x(t_i,\mathbf{c}^*) - \mathbf{m}^T(t_i,\mathbf{c})(\mathbf{c} - \mathbf{c}^*) \\ &\quad = \ D_2 x(t_i,\gamma^*)(\gamma - \gamma^*) - D_2 x(t_i,\gamma)(\gamma - \gamma^*) + \omega(t_i,\gamma^*,\gamma), \end{aligned} \tag{3.2.4}$$

where

$$\omega(t_i,\gamma^*,\gamma) := x(t_i,\gamma) - x(t_i,\gamma^*) - D_2 x(t_i,\gamma^*)(\gamma - \gamma^*) \tag{3.2.5}$$

satisfies

$$\lim_{\gamma\to\gamma^*}\frac{|\omega(t_i,\gamma^*,\gamma)|}{|\gamma - \gamma^*|_\Gamma} = 0, \qquad i = 0,\dots,\ell.$$

Define the vector norm on $\mathbb{R}^N$ by

$$\|\mathbf{c}\| := \Big|\sum_{j=1}^N c_j\chi_j^N\Big|_\Gamma = |\gamma|_\Gamma, \qquad \mathbf{c} \in \mathbb{R}^N.$$

Since all vector norms on $\mathbb{R}^N$ are equivalent, there exist positive constants C_1 and C_1^* such that $C_1^*|\mathbf{c}| \leq \|\mathbf{c}\| = |\gamma|_\Gamma \leq C_1|\mathbf{c}|$ for all $\mathbf{c} \in \mathbb{R}^N$. Then we have

$$\lim_{\mathbf{c}\to\mathbf{c}^*} \frac{|\omega(t_i,\gamma^*,\gamma)|}{|\mathbf{c}-\mathbf{c}^*|} = \lim_{\mathbf{c}\to\mathbf{c}^*} \frac{|\omega(t_i,\gamma^*,\gamma)|}{|\gamma-\gamma^*|_\Gamma}\frac{\|\mathbf{c}-\mathbf{c}^*\|}{|\mathbf{c}-\mathbf{c}^*|} \leq C_1 \lim_{\gamma\to\gamma^*} \frac{|\omega(t_i,\gamma^*,\gamma)|}{|\gamma-\gamma^*|_\Gamma} = 0.$$

Hence (3.2.4) yields

$$\begin{aligned} |g(\mathbf{c}) - \mathbf{c}^*| &\leq dm_0 \sum_{i=0}^{l} \Big| x(t_i,\mathbf{c}) - x(t_i,\mathbf{c}^*) - \mathbf{m}^T(t_i,\mathbf{c})(\mathbf{c}-\mathbf{c}^*) \Big| \\ &\leq w(\mathbf{c}^*,\mathbf{c})|\mathbf{c}-\mathbf{c}^*|, \qquad \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\,\delta_3), \end{aligned} \tag{3.2.6}$$

where

$$w(\mathbf{c}^*,\mathbf{c}) := dm_0 \sum_{i=0}^{l} \Big(C_1 |D_2x(t_i,\gamma^*) - D_2x(t_i,\gamma)|_{\mathcal{L}(\Gamma,\mathbb{R})} + \frac{|\omega(t_i,\gamma^*,\gamma)|}{|\mathbf{c}-\mathbf{c}^*|} \Big) \tag{3.2.7}$$

satisfies

$$\lim_{\mathbf{c}\to\mathbf{c}^*} w(\mathbf{c}^*,\mathbf{c}) = 0. \tag{3.2.8}$$

Hence for every $0 < \nu < 1$ there exists $0 < \delta_4 \leq \delta_3$ such that $|w(\mathbf{c}^*,\mathbf{c})| \leq \nu$ for $\mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\,\delta_4)$. Then the convergence of the sequence (3.1.8) follows from (3.2.6) for all $\mathbf{c}^0 \in \mathcal{B}_{\mathbb{R}^N}(\mathbf{c}^*;\,\delta_4)$, and the superlinear speed of the convergence follows from (3.2.6) and (3.2.8). □

Next we study the case when γ^* does not belong to Γ^N for any N, but we assume that for each N the cost function J restricted to the finite dimensional parameter set Γ^N has a local infimum at $\overline{\gamma}_N \in \Gamma^N$. Then

$$J'(\overline{\gamma}_N)\chi_j^N = 2\sum_{i=0}^{\ell}(x(t_i,\overline{\gamma}_N) - X_i)D_2x(t_i,\overline{\gamma}_N)\chi_j^N = 0, \qquad j = 1,\ldots,N. \tag{3.2.9}$$

We assume also that

(B4) for each $N \in \mathbb{N}$ the basis functions $\chi_j^N := (\chi_j^{\varphi,N}, \chi_j^{\theta,N}, \chi_j^{\xi,N})$ satisfy $\chi_j^{\varphi,N} \in PW^{2,\infty}$ for $j = 1,\ldots,N$, and there exist mesh points $-r < t_1 < \cdots < t_m < 0$, where $m = m(N)$, such that $\dot{\chi}_j^{\varphi,N}$ and $\ddot{\chi}_j^{\varphi,N}$ have points of discontinuity only at t_i for all $j = 1,\ldots,N$;

(B5) for each $N \in \mathbb{N}$ the fixed basis functions in Γ^N satisfy $\sum_{j=1}^N |\chi_j^N|_\Gamma \leq 1$;

(B6) for each $N \in \mathbb{N}$ the cost function J restricted to the finite dimensional parameter set Γ^N has a local infimum at $\overline{\gamma}_N \in \Gamma^N$.

For the rest of this section, for simplicity, we use the 1-norm on $\mathbb{R}^n$, i.e., $|\mathbf{c}|_1 := \sum_{j=1}^N |c_j|$. The corresponding induced matrix norm on $\mathbb{R}^{N\times N}$ is denoted also by $|\cdot|_1$.

Theorem 3.2.3 *Assume (A1) (i)–(v), (A2) (i)–(vi), and (B1)–(B7). Suppose γ^* in (B3) satisfies $\gamma^* \in P_1$. Let $\delta^* > 0$ be defined by Lemma 2.3.8, for a fixed $N \in \mathbb{N}$ let $\overline{\gamma}_N := \sum_{j=1}^N \overline{c}_j \chi_j^N$ be defined by (B6), $\overline{\mathbf{c}}^N := (\overline{c}_1, \ldots, \overline{c}_N)^T$, $m = m(N)$ and $\chi_j^{\varphi,N}$ ($j = 1, \ldots, N$) be defined by (B4), let*

$$K := \max\Big\{|\overline{\mathbf{c}}^N|_1 + \delta^*,\ (|\overline{\mathbf{c}}^N|_1 + \delta^*) \max_{j=1,\ldots,N} |\ddot{\chi}_j^{\varphi,N}|_{L^\infty}\Big\},$$

and let $N_3 = N_3(\gamma^*, \delta^*, m, K)$ be defined by Lemma 2.3.8. Then if $\overline{\gamma}_N \in \mathcal{B}_\Gamma(\gamma^*;\, \delta^*)$, the matrix $\mathbf{D}(\overline{\mathbf{c}}^N)$ exists, it is invertible and satisfies

$$|\mathbf{D}^{-1}(\overline{\mathbf{c}}^N)|_1 N_3 \sum_{i=0}^{\ell} |x(t_i, \overline{\mathbf{c}}^N) - X_i| < 1,$$

then for this fixed N the quasilinearization sequence (3.1.8) *is locally convergent to $\overline{\mathbf{c}}^N$.*

Proof Througout this proof we associate to the vectors $\mathbf{c} := (c_1, \ldots, c_N)^T \in \mathbb{R}^N$ and $\overline{\mathbf{c}}^N := (\overline{c}_1, \ldots, \overline{c}_N)^T \in \mathbb{R}^N$ the parameters $\gamma_{\mathbf{c}} := \sum_{j=1}^N c_j \chi_j^N \in \Gamma^N$ and $\overline{\gamma}_N := \sum_{j=1}^N \overline{c}_j \chi_j^N \in \Gamma^N$, respectively.

We have by (B5) that $|\chi_j^N|_\Gamma \le 1$ for all $j = 1, \ldots, N$, hence

$$|\gamma_{\mathbf{c}}|_\Gamma \le \sum_{j=1}^N |c_i||\chi_j^N|_\Gamma \le |\mathbf{c}|_1, \qquad \mathbf{c} \in \mathbb{R}^N. \tag{3.2.10}$$

As in the proof of Theorem 3.2.2, let δ_1 be such that $D_2 x(t, \gamma) \in \mathcal{L}(\Gamma, \mathbb{R})$ exists and it is continuous for $t \in [0, \alpha]$ and $\gamma \in \mathcal{B}_\Gamma(\gamma^*;\, \delta_1)$. Let $\delta^* > 0$ be defined by Lemma 2.3.8, and suppose that N is such that $\overline{\gamma}_N := \sum_{j=1}^N \overline{c}_j \chi_j^N \in \mathcal{B}_\Gamma(\gamma^*;\, \delta^*)$. Let $\delta_2 > 0$ be such that $\mathcal{B}_\Gamma(\overline{\gamma}_N;\, \delta_2) \subset \mathcal{B}_\Gamma(\gamma^*;\, \delta^*)$. Then (3.2.10) implies that $\gamma_{\mathbf{c}} \in \mathcal{B}_\Gamma(\overline{\gamma}_N;\, \delta_2)$ for $\mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_2\big)$.

We use the notation $\gamma_{\mathbf{c}} = (\varphi_{\mathbf{c}}, \theta_{\mathbf{c}}, \xi_{\mathbf{c}}) \in \Gamma^N$. Then

$$|\varphi_{\mathbf{c}}|_{W^{1,\infty}} \le |\gamma_{\mathbf{c}}|_\Gamma \le |\mathbf{c}|_1 \le |\overline{\mathbf{c}}^N|_1 + \delta_2, \qquad \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_2\big).$$

It follows from assumption (B4) that $\chi_j^{\varphi,N} \in PW^{2,\infty}$, so

$$|\ddot{\varphi}_{\mathbf{c}}|_{L^\infty} \le \sum_{j=1}^N |c_i||\ddot{\chi}_j^{\varphi,N}|_{L^\infty} \le |\mathbf{c}|_1 \max_{j=1,\ldots,N} |\ddot{\chi}_j^{\varphi,N}|_{L^\infty},$$

and therefore $|\varphi_{\mathbf{c}}|_{PW^{2,\infty}} \le K$ for $\mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_2\big)$.

Let $\overline{\delta} > 0$ corresponding to $\overline{\gamma}_N \in \mathcal{B}_\Gamma(\gamma^*;\, \delta^*)$ be defined by Lemma 2.3.8. Then $\mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \overline{\delta}\big)$ implies $\gamma_{\mathbf{c}} \in \mathcal{B}_\Gamma\big(\overline{\gamma}_N;\, \overline{\delta}\big)$ using (3.2.10). For every d satisfying

$$|\mathbf{D}(\overline{\mathbf{c}}^N)|_1 N_3 \sum_{i=0}^{\ell} |x(t_i, \overline{\mathbf{c}}^N) - X_i| < d N_3 \sum_{i=0}^{\ell} |x(t_i, \overline{\mathbf{c}}^N) - X_i| < 1 \tag{3.2.11}$$

there exists $0 < \delta_3 \le \overline{\delta}$ such that $\mathbf{D}(\mathbf{c})$ exists and is invertible for $\mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_3\big)$, and $|\mathbf{D}^{-1}(\mathbf{c})| \le d$ for $\mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_3\big)$.

Then the function $\mathbf{g}(c) := \mathbf{c} - \mathbf{D}^{-1}(\mathbf{c})\mathbf{b}(\mathbf{c})$ is well-defined on $\mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_3\big)$, and similarly to (3.2.1) it satisfies

$$g(\mathbf{c}) - \overline{\mathbf{c}}^N = \Big(\mathbf{D}(\mathbf{c})\Big)^{-1} \sum_{i=0}^{l} \mathbf{m}(t_i, \mathbf{c}) \Big(\mathbf{m}^T(t_i, \mathbf{c})(\mathbf{c} - \overline{\mathbf{c}}^N) - (x(t_i, \mathbf{c}) - X_i)\Big). \tag{3.2.12}$$

It follows from (3.2.9) that

$$\sum_{i=0}^{\ell} (x(t_i, \overline{\mathbf{c}}^N) - X_i)\mathbf{m}(t_i, \overline{\mathbf{c}}^N) = \mathbf{0},$$

hence combining it with (3.2.12) gives

$$\begin{aligned} g(\mathbf{c}) - \overline{\mathbf{c}}^N \;&=\; \Big(\mathbf{D}(\mathbf{c})\Big)^{-1} \sum_{i=0}^{l} \mathbf{m}(t_i, \mathbf{c}) \Big(\mathbf{m}^T(t_i, \mathbf{c})(\mathbf{c} - \overline{\mathbf{c}}^N) - (x(t_i, \mathbf{c}) - x(t_i, \overline{\mathbf{c}}^N)\Big) \\ &\quad - \Big(\mathbf{D}(\mathbf{c})\Big)^{-1} \sum_{i=0}^{l} \Big(\mathbf{m}(t_i, \mathbf{c}) - \mathbf{m}(t_i, \overline{\mathbf{c}}^N)\Big)(x(t_i, \overline{\mathbf{c}}^N) - X_i). \end{aligned} \tag{3.2.13}$$

Then using (2.3.29) and (B5) we get

$$\begin{aligned} |\mathbf{m}(t_i, \mathbf{c}) - \mathbf{m}(t_i, \overline{\mathbf{c}}^N)|_1 \;&=\; \sum_{j=1}^{N} |D_2 x(t_i, \gamma_{\mathbf{c}})\chi_j^N - D_2 x(t_i, \overline{\gamma}^N)\chi_j^N| \\ &\le\; N_3 |\gamma_{\mathbf{c}} - \overline{\gamma}^N|_\Gamma \sum_{j=1}^{N} |\chi_j^N|_\Gamma \\ &\le\; N_3 |\mathbf{c} - \overline{\mathbf{c}}^N|_1, \qquad i = 0, \dots, \ell, \quad \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_3\big). \end{aligned} \tag{3.2.14}$$

Let m_0, ω and w be defined by (3.2.3), (3.2.5) and (3.2.7), respectively. Then (3.2.6), (3.2.13) and (3.2.14) yield

$$\begin{aligned} |g(\mathbf{c}) - \overline{\mathbf{c}}^N|_1 \;&\le\; d m_0 \sum_{i=0}^{l} \Big|\mathbf{m}^T(t_i, \mathbf{c})(\mathbf{c} - \overline{\mathbf{c}}^N) - (x(t_i, \mathbf{c}) - x(t_i, \overline{\mathbf{c}}^N))\Big|_1 \\ &\quad + d \sum_{i=0}^{l} |\mathbf{m}(t_i, \mathbf{c}) - \mathbf{m}(t_i, \overline{\mathbf{c}}^N)|_1 |x(t_i, \overline{\mathbf{c}}^N) - X_i| \\ &\le\; (w(\overline{\mathbf{c}}^N, \mathbf{c}) + A_N)|\mathbf{c} - \overline{\mathbf{c}}^N|_1, \qquad \mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_3\big), \end{aligned} \tag{3.2.15}$$

where by (3.2.11)

$$A_N := dN_3 \sum_{i=0}^{l} |x(t_i, \overline{\mathbf{c}}^N) - X_i| < 1.$$

Let ν be such that $A_N < \nu < 1$. Then (3.2.8) yields that there exists $0 < \delta_4 \leq \delta_3$ such that $0 \leq w(\overline{\mathbf{c}}^N, \mathbf{c}) < \nu - A_N$ for $\mathbf{c} \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_4\big)$. Therefore (3.2.15) gives

$$|\mathbf{c}^{k+1} - \overline{\mathbf{c}}^N|_1 \leq \nu |\mathbf{c}^k - \overline{\mathbf{c}}^N|_1, \qquad \mathbf{c}^0 \in \mathcal{B}_{\mathbb{R}^N}\big(\overline{\mathbf{c}}^N;\, \delta_4\big),$$

which proves the local convergence of (3.1.8) to $\overline{\mathbf{c}}^N$. $\square$

3.3 Numerical examples

In all of the numerical examples we present below only one component of the parameter vector (φ, θ, ξ) is considered to be unknown, the other two components will be given. So the parameter set Γ will be identified with either $W^{1,\infty}$, Θ or Ξ. Also, θ and ξ below will be coefficient functions in the equations, so we will use $W^{1,\infty}([0,\alpha],\mathbb{R})$ as the parameter set Θ or Ξ. In all this three cases we approximate the functions of $W^{1,\infty}$ or $W^{1,\infty}([0,\alpha],\mathbb{R})$ by linear splines. Hence in the examples we define Γ^N as the space of linear spline functions with equally distant node points $\nu_1, \nu_2, \ldots, \nu_N$ of the domain $[-r, 0]$ or $[0, \alpha]$. Let $\{\lambda_1^N, \ldots, \lambda_N^N\}$ be the usual "hat" functions corresponding to the mesh $\{\nu_1, \ldots, \nu_N\}$ satisfying $\lambda_i^N(\nu_j) = 0$ if $i \neq j$, and $\lambda_i^N(\nu_i) = 1$. Then the basis of Γ^N will be the scaled "hat" functions $\{\chi_1^N, \ldots, \chi_N^N\}$ defined by $\chi_i^N(t) := \frac{1}{N|\lambda_i^N|_{W^{1,\infty}}}\lambda_i^N$ for $i = 1, \ldots, N$. Then Γ^N and $\{\chi_1^N, \ldots, \chi_N^N\}$ satisfy assumptions (B2), (B4) and (B5).

Example 3.3.1 Consider the scalar delay equation

$$\dot{x}(t) = \theta(t)x\Big(t - \xi^2(t)x^2(t) - 1\Big), \qquad t \in [0,3], \tag{3.3.1}$$

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{3.3.2}$$

If we take

$$\xi(t) := \frac{20}{(t+4)^2}, \qquad \theta(t) := \frac{2t+8}{(t+2)^2} \quad \text{and} \quad \varphi(t) := \frac{1}{20}(t+4)^2 \tag{3.3.3}$$

as the parameters in (3.3.1)-(3.3.2), then the solution of the corresponding IVP (3.3.1)–(3.3.2) is

$$x(t) = \frac{1}{20}(t+4)^2. \tag{3.3.4}$$

Note that along with the “true“ solution (3.3.4), the time lag function is $t - x^2(t)\xi^2(t) - 1 = t - 2$, so $r \geq 2$ is needed in (3.3.2) to generate solution (3.3.4).

We used the function (3.3.4) to generate measurements at the points $t_i = 0.2i, \quad i = 0, 1, \ldots, 15$. In this example let ξ and φ be defined by (3.3.3), and consider θ as an unknown parameter in the equation. The derivative of the solution $x(t, \theta)$ of the IVP (3.3.1)–(3.3.2) with respect to θ applied to a fixed function $h \in W^{1,\infty}([0, 3], \mathbb{R})$ is denoted by $z(t) := z(t, \theta, h) = D_2 x(t, \theta)h$, and it satisfies the variational equation

$$\begin{aligned} \dot{z}(t) &= \theta(t)\Big[-\dot{x}\Big(t - \xi^2(t)x^2(t) - 1\Big)\xi^2(t)2x(t)z(t) + z\Big(t - \xi^2(t)x^2(t) - 1\Big)\Big] \\ &\quad + h(t)x\Big(t - \xi^2(t)x^2(t) - 1\Big), \qquad t \in [0, 3], \qquad (3.3.5) \\ z(t) &= 0, \qquad t \in [-2, 0]. \qquad (3.3.6) \end{aligned}$$

This IVP and also the IVP (3.3.1)-(3.3.2) are solved numerically by the approximation technique introduced in [41] to obtain the derivative values used in (3.1.4). In all the numerical runnings below step-size 0.05 was used in the numerical simulation.

First we computed iteration (3.1.8) starting from the constant 0 initial parameter value. The numerical results can be seen in Figures 1 and 2 using $N = 3$ and $N = 8$ dimensional linear spline approximations of the coefficient function θ. In the figures the solid curve represents the ”true“ parameter function θ, and the dotted curves are the spline approximations obtained by the quasilinearization sequence (3.1.8). We observe good approximation of the ”true” parameter θ in two steps. In Tables 1 and 2 the value of the least square cost function $J(\theta^{(k)})$ at the kth iteration, and the the error of the spline iteration function at the node points $\Delta_i^{(k)} = |\theta^{(k)}(\nu_i) - \theta(\nu_i)|$ are presented.

Let $P^N f$ denote the projection of the function f to the space of N-dimensional linear spline functions (with equi-distant node points). In Figures 3 and 4 and Tables 3 and 4 the numerical results of the iteration (3.1.8) can be seen starting from the initial parameter guess $\theta^{(0)}(t) = P^3(4 \sin 5t)$ and $\theta^{(0)}(t) = P^8(4 \sin 5t)$, respectively. As in the previous running, a quick convergence is observed.

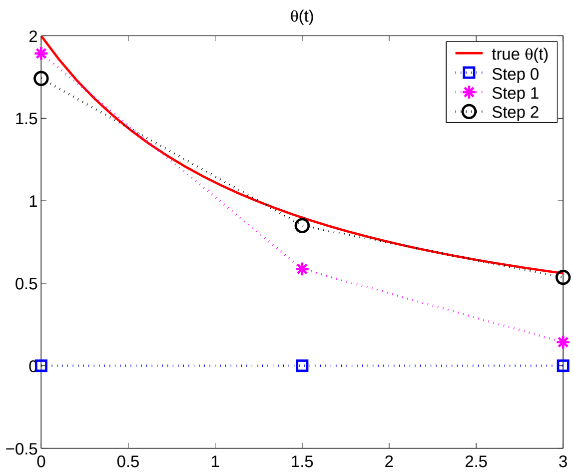

Figure 1: $\theta^{(0)}(t) = 0, \ N = 3$

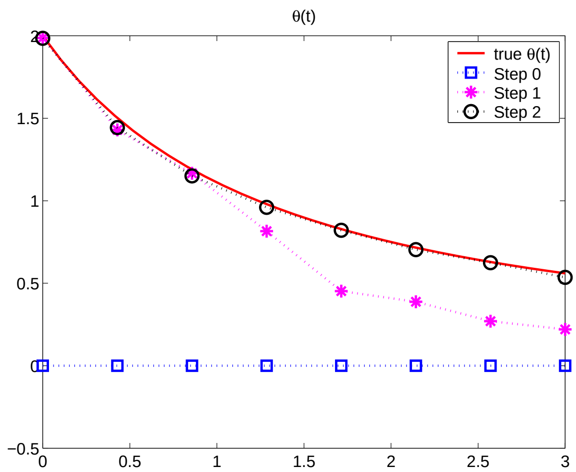

Figure 2: $\theta^{(0)}(t) = 0, \ N = 8$

Table 1: $\theta^{(0)}(t) = 0,\ N = 3$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$
0:	13.257248	2.00000	0.89796	0.56000
1:	0.583975	0.10736	0.31157	0.41742
2:	0.000202	0.25890	0.04866	0.02411

Table 2: $\theta^{(0)}(t) = 0,\ N = 8$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$	$\Delta_4^{(k)}$	$\Delta_5^{(k)}$	$\Delta_6^{(k)}$	$\Delta_7^{(k)}$	$\Delta_8^{(k)}$
0:	13.257248	2.00000	1.50173	1.19000	0.97921	0.82840	0.71581	0.62891	0.56000
1:	0.577428	0.01275	0.07210	0.02331	0.16346	0.37610	0.32800	0.35868	0.33955
2:	0.000007	0.01554	0.05837	0.03913	0.01889	0.00730	0.01190	0.00464	0.02400

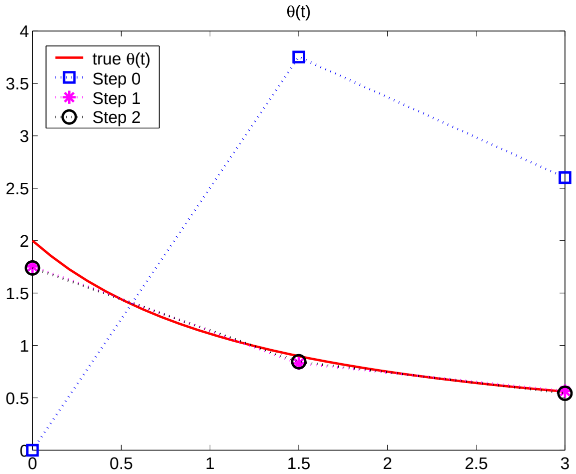

Figure 3: $\theta^{(0)}(t) = P^3(4\sin 5t),\ N = 3$

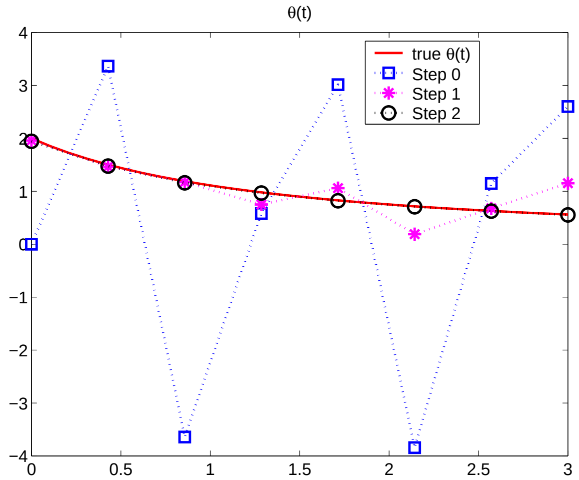

Figure 4: $\theta^{(0)}(t) = P^8(4\sin 5t),\ N = 8$

Table 3: $\theta^{(0)}(t) = P^3(4\sin 5t),\ N = 3$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$
0:	10.318073	2.00000	2.85404	2.04115
1:	0.000319	0.24502	0.06980	0.00527
2:	0.000179	0.26077	0.05294	0.01625
3:	0.000177	0.26321	0.05177	0.01668

Table 4: $\theta^{(0)}(t) = P^8(4\sin 5t),\ N = 8$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$	$\Delta_4^{(k)}$	$\Delta_5^{(k)}$	$\Delta_6^{(k)}$	$\Delta_7^{(k)}$	$\Delta_8^{(k)}$
0:	11.807231	2.00000	1.86142	4.83139	0.39971	2.18554	4.55861	0.51786	2.04115
1:	0.055042	0.04142	0.03820	0.01805	0.23000	0.22969	0.52617	0.04923	0.59118
2:	0.000001	0.05690	0.02693	0.03152	0.01420	0.00792	0.00952	0.00417	0.00684

Example 3.3.2 In this example we consider again the IVP (3.3.1)-(3.3.2), where now we suppose φ and θ are defined by (3.3.3), and we consider ξ in (3.3.1) as an unknown parameter function defined on the interval $[0,3]$. We use the same measurement generated by the "true solution" (3.3.4) which was used in Example 3.3.1. The derivative of the solution $x(t,\xi)$ of IVP (3.3.1)–(3.3.2) with respect to ξ applied to a fixed function $h \in W^{1,\infty}([0,3],\mathbb{R})$ is denoted by $z(t) := z(t,\xi,h) = D_2x(t,\xi)h$, and it satisfies the variational equation

$$\begin{aligned}\dot z(t) &= \theta(t)\Big[-\dot x\Big(t-\xi^2(t)x^2(t)-1\Big)\Big(\xi^2(t)2x(t)z(t)+2\xi(t)x^2(t)h(t)\Big)\\ &\quad +z\Big(t-\xi^2(t)x^2(t)-1\Big)\Big], \qquad t\in[0,3], \qquad (3.3.7)\\ z(t) &= 0, \qquad t\in[-2,0]. \qquad (3.3.8)\end{aligned}$$

We used the numerical solution of the IVP (3.3.7)-(3.3.8) to compute the quasilinearization sequence (3.1.8). We generated the sequence starting from the initial parameter value $\xi^{(0)}(t) = 1$. The first several terms of the corresponding sequence is illustrated in Figures 5 and 6 and in Tables 5 and 6 using $N = 3$ and $N = 8$ dimensional spline approximation, respectively.

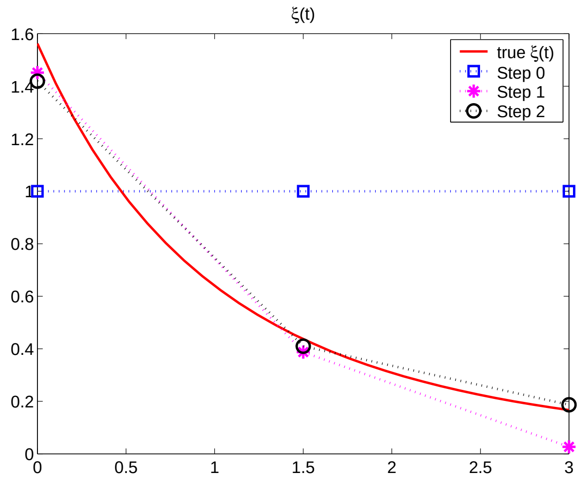

Figure 5: $\xi^{(0)}(t) = 1,\ N = 3$

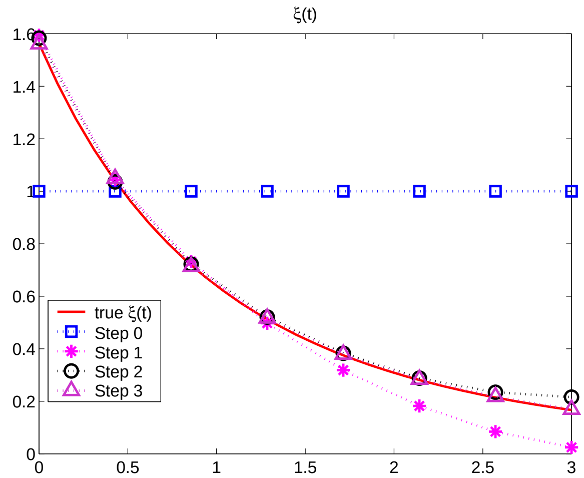

Figure 6: $\xi^{(0)}(t) = 1,\ N = 8$

Table 5: $\xi^{(0)}(t) = 1,\ N = 3$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$
0:	1.419877	0.56250	0.56287	0.83340
1:	0.080676	0.11016	0.04972	0.13968
2:	0.000964	0.14078	0.02789	0.01848
3:	0.000219	0.14846	0.02439	0.00513

Table 6: $\xi^{(0)}(t) = 1$, $N = 8$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$	$\Delta_4^{(k)}$	$\Delta_5^{(k)}$	$\Delta_6^{(k)}$	$\Delta_7^{(k)}$	$\Delta_8^{(k)}$
0:	1.419877	0.56250	0.03993	0.28132	0.48756	0.62484	0.71908	0.78550	0.83340
1:	0.078229	0.03357	0.00237	0.01607	0.01850	0.05421	0.09934	0.12863	0.14326
2:	0.001305	0.02226	0.00555	0.00493	0.00522	0.00288	0.01240	0.01409	0.06391
3:	0.000049	0.00075	0.00574	0.00230	0.00027	0.00042	0.00531	0.00153	0.00614

Example 3.3.3 Now consider again the IVP (3.3.1)-(3.3.2), where the coefficients θ and ξ are defined by (3.3.3), and in this example we consider the initial function φ as the unknown parameter in the equation. We use the same measurements that was used in Examples 3.3.1 and 3.3.2, therefore the true parameter value will be the function φ defined in (3.3.3).

Note that the difficulty to estimate the initial function in SD-DDEs is that the size of the initial interval depends on the solution, therefore it is not known in advance. One simple trick is to handle this difficulty numerically is to modify the initial condition in the computation of the numerical solution of (3.3.1). Using the measurements X_i at the time mesh points t_i and the formula of the delay function we select r so that $-r \geq \max(\xi^2(t_i)X_i^2 + 1)$, consider a function $\varphi \in W^{1,\infty}([-r, 0], \mathbb{R})$, and we replace (3.3.2) by the initial condition

$$x(t) = \begin{cases} \varphi(t), & t \in [-r, 0], \\ \varphi(-r), & t < -r. \end{cases}$$

The derivative of the solution $x(t, \varphi)$ of IVP (3.3.1)–(3.3.2) with respect to φ applied to a fixed function $h \in W^{1,\infty}([-r, 0], \mathbb{R})$ is denoted by $z(t) := z(t, \varphi, h) = D_2 x(t, \varphi)h$, and it satisfies the variational equation

$$\begin{aligned} \dot{z}(t) &= \theta(t)\Big[-\dot{x}\Big(t - \xi^2(t)x^2(t) - 1\Big)\xi^2(t)2x(t)z(t) \\ &\quad + z\Big(t - \xi^2(t)x^2(t) - 1\Big)\Big], \qquad t \in [0, 3], && (3.3.9) \\ z(t) &= h(t), \qquad t \in [-r, 0]. && (3.3.10) \end{aligned}$$

Again, in the numerical computation we replace (3.3.10) by

$$z(t) = \begin{cases} h(t), & t \in [-r, 0], \\ h(-r), & t < -r. \end{cases}$$

In the generation of the iteration (3.1.8) below we used $r = 2$ and the projection of the function $\cos t$ to the space of linear spline functions as the initial parameter value. The numerical results can be seen in Figures 7 and 8 and in Tables 7 and 8 for $N = 3$ and $N = 8$. We note that in this example the convergence of the iteration scheme was much more sensitive to the selection of the initial parameter value than in the previous two examples. For this particular values of the initial function both iteration sequences were convergent. We observe quick convergence of the approximating sequences to the true parameter function φ.

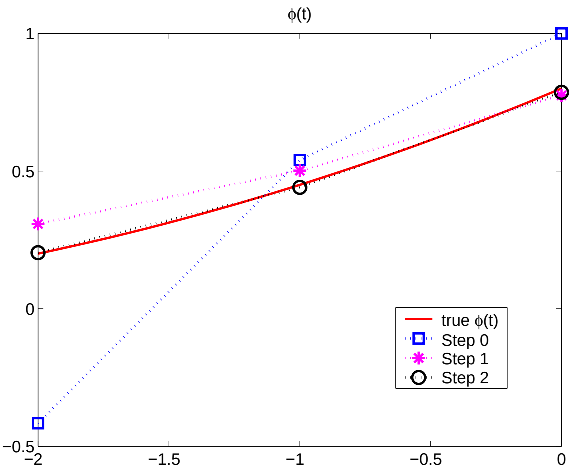

Figure 7: $\varphi^{(0)}(t) = P^3(\cos t)$, $N = 3$

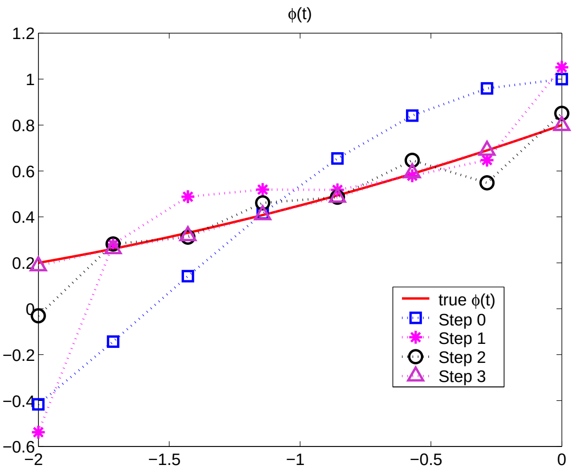

Figure 7: $\varphi^{(0)}(t) = P^8(\cos t)$, $N = 8$

Table 5: $\varphi^{(0)}(t) = P^3(\cos t)$, $N = 3$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$
0:	0.082319	0.61615	0.09030	0.20000
1:	0.108323	0.10783	0.05159	0.02523
2:	0.000084	0.00364	0.00916	0.01367
3:	0.000011	0.00592	0.01128	0.00583
4:	0.000005	0.00828	0.01205	0.00373

Table 6: $\varphi^{(0)}(t) = P^8(\cos t)$, $N = 8$

k	$J(\theta^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$	$\Delta_4^{(k)}$	$\Delta_5^{(k)}$	$\Delta_6^{(k)}$	$\Delta_7^{(k)}$	$\Delta_8^{(k)}$
0:	0.172338	0.61615	0.40422	0.18887	0.00683	0.16072	0.25337	0.26966	0.20000
1:	0.110547	0.73788	0.01933	0.15739	0.11087	0.02379	0.00866	0.04256	0.25172
2:	0.001212	0.23078	0.02075	0.01854	0.05279	0.00820	0.05878	0.14140	0.05103
3:	0.000005	0.01346	0.00017	0.01250	0.00098	0.00847	0.00407	0.00027	0.00237

We we refer to [46] for more numerical examples of the quasilinearization method (3.1.8) for SD-DDEs. We note that the parameter estimation problem for several classes of state-dependent and also for state-independent delay and neutral equations was studied in [6, 7, 17, 51, 52, 54, 55, 59, 79] using direct finite dimensional optimization methods. Finally note that the identifiability of parameters, i.e., the uniqueness of the parameter value which generate the same solution is an important issue in the theory of parameter estimation. It is studied for FDEs, e.g., in [76, 80], but similar studies are missing for SD-FDEs. We refer to Example 5.4 in [55], where the parameter estimation was numerically investigated in a case when the uniqueness of the parameter value failed.

Chapter 4

Neutral FDEs with state-dependent delays

4.1 Introduction

In this chapter we consider SD-NFDEs of the form

$$\frac{d}{dt}\Big(x(t) - g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda)\Big) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \theta\Big) \qquad t \in [0, T], \tag{4.1.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{4.1.2}$$

Here $\theta \in \Theta$, $\xi \in \Xi$, $\lambda \in \Lambda$ and $\chi \in X$ represent parameters in the functions f, τ, g and ρ, where Θ, Ξ, Λ and X are normed linear spaces with norms $|\cdot|_\Theta$, $|\cdot|_\Xi$, $|\cdot|_\Lambda$ and $|\cdot|_X$, respectively. See Section 4.2 below for the detailed assumptions on the IVP (4.1.1)-(4.1.2). By a solution of the IVP (4.1.1)-(4.1.2) we mean a continuous function defined on an interval $[-r, \alpha]$, such that (i) $t \mapsto x(t) - g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda)$ is differentiable for $t \in [0, \alpha]$, (at the ends of the interval one sided derivatives exist); (ii) x satisfies (4.1.1) for $t \in [0, \alpha]$, and (iii) x satisfies the initial condition (4.1.2).

The study of SD-DDEs, i.e., the case when $g \equiv 0$ in (4.1.1) is an active research area (see [56] and its references). Much less work is devoted to SD-NFDEs, see, [3, 4, 5, 11, 12, 25, 29, 32, 34, 39, 50, 49, 54, 61, 68, 92, 93, 94, 95] and their references. Most of the above papers deal with SD-NFDEs of the form

$$x'(t) = h\Big(t, x(t), x(t - \tau(t, x(t))), x'(t - \eta(t, x(t)))\Big). \tag{4.1.3}$$

This equation is called in [75, 92, 93] as "explicit" SD-NFDE contrary to the "implicit" SD-NFDE (4.1.1). Well-posedness of such "explicit" SD-NFDEs was investigated in [38, 67].

Equation (4.1.1) can be considered as a natural "generalization" of NFDEs of the form

$$\frac{d}{dt}G(t, x_t) = f(t, x_t), \tag{4.1.4}$$

but (4.1.4) may also contain (4.1.1) depending on appropriate conditions on G and f, see assumtions on f in [56] for SD-DDEs, and [92] and [93] for similar conditions on "implicit" SD-NFDEs. Existence, uniqueness, stability and numerical approximation of special classes of (4.1.1) was studied in [5, 50, 53, 75]. Similar classes of abstract implicit SD-NFDEs were investigated in [20, 26, 74, 83].

In a recent paper [93] Walter studied continuous semiflows generated by "explicit" SD-NFDEs in the space of continuously differentiable functions, and differentiability and continuity of derivatives with respect to initial data. Differentiability wrt parameters of "implicit" SD-NFDEs was proved in [48] for the case when the delay ρ in (4.1.1) is only time-dependent, and there are no parameters in the neutral term. The proof was based on the assumption that the parameters satisfy a compatibility condition similarly to (1.1.4) in the SD-DDE case. In this chapter we extend this result for (4.1.1), where state-dependent delay and parameters are included in the neutral term, as well. In Theorem 4.2.2 below we discuss the well-posedness of the IVP (4.1.1)-(4.1.2), and in Theorem 4.3.4 and Corollary 4.3.5 below we show the differentiability of solutions of the IVP (4.1.1)-(4.1.2) wrt the parameters $(\varphi, \xi, \theta, \lambda, \chi)$ in a pointwise sense and also using the C-norm.

The organization of the chapter is the following. In Section 4.2 we list our assumptions, and discuss well-posedness of the IVP (4.1.1)-(4.1.2), and then in Section 4.3, using and improving the method of [48], we study differentiability of solutions wrt parameters. Note that for simplicity we present our results for the single state-dependent delay case, but all our results can be easily extended to the case when both g and f contain multiple state-dependent delays.

4.2 Well-posedness and continuous dependence on parameters

Consider the SD-NFDE

$$\frac{d}{dt}\Big(x(t) - g(t, x_t, x(t-\rho(t, x_t, \chi)), \lambda)\Big) = f\Big(t, x_t, x(t-\tau(t, x_t, \xi)), \theta\Big) \qquad t \in [0, T], \tag{4.2.1}$$

and the initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{4.2.2}$$

Next we list our assumptions on the SD-NFDE (4.2.1) we will use throughout this paper. Let Θ, Ξ, Λ and X be normed linear spaces with norms $|\cdot|_\Theta$, $|\cdot|_\Xi$, $|\cdot|_\Lambda$ and $|\cdot|_X$, respectively, and let $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset \Xi$, $\Omega_5 \subset \mathbb{R}^n$, $\Omega_6 \subset \Lambda$ and $\Omega_7 \subset X$ be open subsets of the respective spaces. Let $0 < r_0 < r$ be fixed constants, and $T > 0$ be finite or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$. In addition to assumptions (A1) (i)–(iii) and (A2) (i)–(iii) introduced in Section 2.2 we assume:

(A3) (i) $g\colon \mathbb{R} \times C \times \mathbb{R}^n \times \Lambda \supset [0, T] \times \Omega_1 \times \Omega_5 \times \Omega_6 \to \mathbb{R}^n$ is continuous;

(ii) g is locally Lipschitz continuous in the following sense: for every $\alpha \in (0, T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_5 \subset \Omega_5$ of $\mathbb{R}^n$ and closed and bounded subset $M_6 \subset \Omega_6$ of Λ there exists $L_3 = L_3(\alpha, M_1, M_5, M_6)$ such that

$$|g(t,\psi,u,\lambda)-g(\bar t,\bar\psi,\bar u,\bar\lambda)| \le L_3\Big(|t-\bar t|+\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)|+|u-\bar u|+|\lambda-\bar\lambda|_\Lambda\Big),$$

for $t, \bar t \in [0, \alpha]$, $\psi, \bar\psi \in M_1$, $u, \bar u \in M_5$, $\lambda, \bar\lambda \in M_6$;

(iii) g is continuously differentiable wrt its second, third and fourth arguments;

(iv) $D_2 g$, $D_3 g$ and $D_4 g$ are locally Lipschitz continuous wrt its first three variables in the following sense: for every $\alpha \in (0, T]$, closed subsets $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_5 \subset \Omega_5$ of $\mathbb{R}^n$ and closed and bounded subset $M_6 \subset \Omega_6$ of Λ there exist $L_4 = L_4(\alpha, M_1, M_5, M_6)$ and $L_5 = L_5(\alpha, M_1, M_5, M_6)$ such that

$$\begin{aligned}
&|D_2 g(t,\psi,u,\lambda)h - D_2 g(\bar t,\bar\psi,\bar u,\lambda)h| \\
&\quad\le\ L_4\Big(|t-\bar t| + \max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)| + |u-\bar u|\Big)\max_{\zeta\in[-r,-r_0]}|h(\zeta)|, \\
&\qquad\quad + L_4\max\Big\{|h(\zeta)-h(\bar\zeta)|\colon\ \zeta,\bar\zeta\in[-r,-r_0],\ |\zeta-\bar\zeta|\le L_5|t-\bar t|\Big\}, \\
&|D_3 g(t,\psi,u,\lambda) - D_3 g(\bar t,\bar\psi,\bar u,\lambda)| \\
&\quad\le\ L_4\Big(|t-\bar t| + \max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)| + |u-\bar u|\Big), \\
&|D_4 g(t,\psi,u,\lambda) - D_4 g(\bar t,\bar\psi,\bar u,\lambda)|_{\mathcal{L}(\Lambda,\mathbb{R}^n)} \\
&\quad\le\ L_4\Big(|t-\bar t| + \max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)| + |u-\bar u|\Big),
\end{aligned}$$

for $t, \bar t \in [0, \alpha]$, $\psi, \bar\psi \in M_1$, $u, \bar u \in M_5$, $\lambda \in M_6$, $h \in C$;

(A4) (i) $\rho\colon \mathbb{R} \times C \times X \supset [0, T] \times \Omega_1 \times \Omega_7 \to \mathbb{R}$ is continuous, and

$$0 < r_0 \le \rho(t,\psi,\chi) \le r, \qquad t\in[0,T], \quad \psi\in\Omega_1, \quad \chi\in\Omega_7;$$

(ii) ρ is locally Lipschitz continuous in the following sense: for every $\alpha \in (0, T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and bounded and closed subset $M_7 \subset \Omega_7$ of X there exists $L_6 = L_6(\alpha, M_1, M_7)$ such that

$$|\rho(t,\psi,\chi) - \rho(\bar{t},\bar{\psi},\bar{\chi})| \le L_6\Big(|t-\bar{t}| + \max_{\zeta\in[-r,-r_0]} |\psi(\zeta) - \bar{\psi}(\zeta)| + |\chi - \bar{\chi}|_X\Big)$$

for $t, \bar{t} \in [0, \alpha]$, $\psi, \bar{\psi} \in M_1$, and $\chi, \bar{\chi} \in M_7$;

(iii) ρ is continuously differentiable wrt its second and third arguments;

(iv) $D_2\rho$ and $D_3\rho$ are locally Lipschitz continuous wrt its first and second variables in the following sense: for every $\alpha \in (0, T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$ and bounded and closed subset $M_7 \subset \Omega_7$ of X there exist $L_7 = L_7(\alpha, M_1, M_7)$ and $L_8 = L_8(\alpha, M_1, M_7)$ such that

$$\begin{aligned} &|D_2\rho(t,\psi,\chi)h - D_2\rho(\bar{t},\bar{\psi},\chi)h| \\ &\quad\le\ L_7\Big(|t-\bar{t}| + \max_{\zeta\in[-r,-r_0]} |\psi(\zeta) - \bar{\psi}(\zeta)|\Big) \max_{\zeta\in[-r,-r_0]} |h(\zeta)| \\ &\qquad\quad +L_7 \max\{|h(\zeta) - h(\bar{\zeta})| \colon\ \zeta, \bar{\zeta} \in [-r, -r_0],\ |\zeta - \bar{\zeta}| \le L_8|t-\bar{t}|\}, \end{aligned}$$

and

$$|D_3\rho(t,\psi,\chi) - D_3\rho(\bar{t},\bar{\psi},\chi)|_{\mathcal{L}(X,\mathbb{R})} \le L_7\Big(|t-\bar{t}| + \max_{\zeta\in[-r,-r_0]} |\psi(\zeta) - \bar{\psi}(\zeta)|\Big)$$

for $t, \bar{t} \in [0, \alpha]$, $\psi, \bar{\psi} \in M_1$, $\chi \in M_7$, $h \in C$.

It is easy to see that (A3) (ii) and (A4) (ii) yield that $g(t, \psi, u, \lambda)$ and $\rho(t, \psi, \chi)$ depend only on the restriction of ψ to the interval $[-r, -r_0]$, since if $\psi(\zeta) = \bar{\psi}(\zeta)$ for $\zeta \in [-r, -r_0]$, then $g(t, \psi, u, \lambda) = g(t, \bar{\psi}, u, \lambda)$ and $\rho(t, \psi, \chi) = \rho(t, \bar{\psi}, \chi)$. It also follows from (A3) (ii), (iii) and (A4) (ii), (iii) that

$$|D_2 g(t,\psi,u,\lambda)h| \le |D_2 g(t,\psi,u,\lambda)|_{\mathcal{L}(C,\mathbb{R}^n)} \max_{\zeta\in[-r,-r_0]} |h(\zeta)|$$

and

$$|D_2\rho(t,\psi,\chi)h| \le |D_2\rho(t,\psi,\chi)|_{\mathcal{L}(C,\mathbb{R})} \max_{\zeta\in[-r,-r_0]} |h(\zeta)|$$

hold for $t \in [0, T]$, $\psi \in \Omega_1$, $u \in \Omega_5$, $\lambda \in \Omega_6$, $\chi \in \Omega_7$ and $h \in C$.

It follows from the assumptions on M_1 in (A1) (ii), (A2) (ii), (A3) (ii), (iv) and (A4) (ii), (iv) that it has no interior in C. Note that assumptions (A1) and (A2) are practically identical to those used in [58] for SD-DDEs, i.e., for the case when $g \equiv 0$. (See also [27] or [58] for well-posedness of SD-DDEs.) The key assumptions in this paper are that ρ is bounded below by $r_0 > 0$ (see (A4) (i)) and $g(t, \psi, u, \lambda)$ and $\rho(t, \psi, \chi)$ depend only on the restriction of ψ to the interval $[-r, -r_0]$. Similar assumption is used for SD-NFDEs in [50], see condition (g1) in [92], [93], and for PDEs with state-dependent delays in [82]. The particular form of the Lipschitz continuity assumed in (A3) (ii), (iv) and (A4) (ii), (iv) is motivated by the specific form (4.2.3) and (4.2.4) of the functions g and ρ, respectively (see Lemma 4.2.1 below). We comment that Arzelà-Ascoli theorem yields that closed subsets of C which are bounded subsets of $W^{1,\infty}$ are compact in C.

Assumptions (A3) and (A4) are naturally satisfied, e.g., in the case when $\Lambda = X = W^{1,\infty}([0,T], \mathbb{R})$, and g and ρ have the form

$$g(t, \psi, u, \lambda) = \bar{g}\Big(t, \psi(-\eta^1(t)), \ldots, \psi(-\eta^k(t)), \int_{-r}^{-r_0} A(t, \zeta)\psi(\zeta)\, d\zeta, u, \lambda(t)\Big) \tag{4.2.3}$$

and

$$\rho(t, \psi, \chi) = \bar{\rho}\Big(t, \psi(-\nu^1(t)), \ldots, \psi(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t, \zeta)\psi(\zeta)\, d\zeta, \chi(t)\Big), \tag{4.2.4}$$

where $t \in [0, T]$, $\psi \in C$, $u \in \mathbb{R}^n$, $\lambda \in \Lambda$, $\chi \in X$ and $0 < r_0 < r$. The next lemma shows that assumption (A4) is satisfied under natural assumptions on $\bar{\rho}$. Clearly, (A3) can be also satisfied under similar assumptions on $\bar{g}$.

Lemma 4.2.1 *Assume $X = W^{1,\infty}([0,T], \mathbb{R})$, and ρ has the form* (4.2.4)*, where*

(i) $\bar{\rho}\colon [0,T] \times \mathbb{R}^{n\times(\ell+1)} \times \mathbb{R} \to \mathbb{R}$ *is continuous,* $\nu^1, \ldots, \nu^\ell\colon [0,T] \to \mathbb{R}$ *are continuous,* $B\colon [0,T] \times [-r, -r_0] \to \mathbb{R}^{n\times n}$ *is continuous, and*

$$0 < r_0 \le \bar{\rho}(t, u_1, \ldots, u_{\ell+1}, v) \le r, \qquad t \in [0,T], \quad u_1, \ldots, u_{\ell+1} \in \mathbb{R}^n, \quad v \in \mathbb{R},$$

and

$$0 < r_0 \le \nu^i(t) \le r, \qquad i = 1, \ldots, \ell, \quad t \in [0,T];$$

(ii) $\bar{\rho}$ *is twice continuously differentiable;*

(iii) $\nu^1, \ldots, \nu^\ell : [0,T] \to \mathbb{R}$ *and* $B : [0,T] \times [-r, -r_0] \to \mathbb{R}^{n\times n}$ *are locally Lipschitz continuous wrt t, i.e., for every $\alpha \in (0,T]$ there exist $L_9 = L_9(\alpha)$ and $L_{10} = L_{10}(\alpha)$ such that*

$$|\nu^i(t) - \nu^i(\bar{t})| \le L_9|t - \bar{t}|, \qquad t, \bar{t} \in [0, \alpha], \quad i = 1, \ldots, \ell,$$

and

$$|B(t, \zeta) - B(\bar{t}, \zeta)| \le L_{10}|t - \bar{t}|, \qquad t, \bar{t} \in [0, \alpha], \quad \zeta \in [-r, -r_0].$$

Then ρ satisfies assumptions (A4) (i)–(iv).

Moreover, if in addition $\bar{\chi}, \nu^1, \ldots, \nu^\ell \in C^1([0,T], \mathbb{R})$ and B is continuously differentiable wrt its first argument, then $\rho(t, \psi, \bar{\chi})$ is differentiable wrt t for $t \in [0,T]$ and $\psi \in C^1$, and the map $[0,T] \times C^1 \to \mathbb{R}$, $(t, \psi) \mapsto D_1\rho(t, \psi, \bar{\chi})$ is continuous.

Proof (A4) (i) is clearly satisfied under the assumptions of the lemma with $\Omega_1 = C$ and $\Omega_7 = X$. Suppose $\alpha \in (0, T]$, M_1 is a closed subset of C which is also a bounded subset of $W^{1,\infty}$, and $M_7 \subset X$ is closed and bounded. Then there exists $R_1 > 0$ and $R_2 > 0$ such that $M_1 \subset \overline{\mathcal{B}}_{W^{1,\infty}}(0;\, R_1)$ and $M_7 \subset \overline{\mathcal{B}}_X(0;\, R_2)$. We have

$$\Big|\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta\Big| \le b_{max}R_1 r, \qquad t \in [0,\alpha],\ \psi \in M_1,$$

where

$$b_{max} = b_{max}(\alpha) := \max\{|B(t,\zeta)|\colon\ t \in [0,\alpha],\ \zeta \in [-r,-r_0]\}. \tag{4.2.5}$$

Let

$$\begin{aligned} L_{11} \;:=\; \max_{i=1,\dots,\ell+3} \max\Big\{ &|D_i\bar\rho(t,u_1,\dots,u_{\ell+1},v)|\colon\ t\in[0,\alpha],\ u_1,\dots,u_\ell \in \overline{\mathcal{B}}_{\mathbb{R}^n}(0;\,R_1),\\ &u_{\ell+1} \in \overline{\mathcal{B}}_{\mathbb{R}^n}(0;\,b_{max}R_1 r),\ v \in \overline{\mathcal{B}}_{\mathbb{R}}(0;\,R_2)\Big\}. \end{aligned}$$

Then Lemma 1.2.5 yields for $t \in [0,\alpha]$, $\psi, \bar\psi \in M_1$, and $\chi, \bar\chi \in M_7$

$$\begin{aligned} &|\rho(t,\psi,\chi) - \rho(t,\bar\psi,\bar\chi)|\\ &\quad= \Big|\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\\ &\qquad\quad -\bar\rho\Big(t,\bar\psi(-\nu^1(t)),\dots,\bar\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\bar\psi(\zeta)\,d\zeta,\bar\chi(t)\Big)\Big|\\ &\quad\le L_{11}\Big(\sum_{i=1}^{\ell}|\psi(-\nu^i(t)) - \bar\psi(-\nu^i(t))| + \int_{-r}^{-r_0}|B(t,\zeta)||\psi(\zeta)-\bar\psi(\zeta)|\,d\zeta + |\chi(t)-\bar\chi(t)|\Big)\\ &\quad\le L_{11}(\ell + r b_{max})\Big(\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)| + |\chi-\bar\chi|_X\Big). \end{aligned}$$

To show the Lipschitz continuity of ρ wrt t consider for $t, \bar t \in [0,\alpha]$, $\psi \in M_1$, $\chi \in M_7$

$$\begin{aligned} &|\rho(t,\psi,\chi) - \rho(\bar t,\psi,\chi)|\\ &\le \Big|\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\\ &\qquad -\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0} B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big|\\ &\le L_{11}\Big(|t-\bar t| + \sum_{i=1}^{\ell}|\psi(-\nu^i(t)) - \psi(-\nu^i(\bar t))| + \int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||\psi(\zeta)|\,d\zeta\\ &\qquad\quad + |\chi(t)-\chi(\bar t)|\Big)\\ &\le L_{11}\Big(|t-\bar t| + \sum_{i=1}^{\ell}|\dot\psi|_{L^\infty}|\nu^i(t)-\nu^i(\bar t)| + L_{10}r|\psi|_C|t-\bar t| + sup_{s\in[0,\alpha]}|\dot\chi(s)||t-\bar t|\Big). \end{aligned}$$

Therefore (A4) (ii) holds with $L_6 := \max\{L_{11}(\ell + rb_{max}), L_{11}(1 + \ell R_1 L_9 + L_{10} r R_1 + R_2)\}$.

The differentiability of $\bar\rho$ yields for $t \in [0,T]$, $\psi \in C$, $\chi \in X$, $h \in C$ and $\eta \in X$

$$\begin{aligned} &D_2\rho(t,\psi,\chi)h \\ &\quad = \sum_{i=1}^{\ell} D_{i+1}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)h(-\nu^i(t)) \\ &\qquad + D_{\ell+2}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\int_{-r}^{-r_0} B(t,\zeta)h(\zeta)\,d\zeta \end{aligned}$$

and

$$D_3\rho(t,\psi,\chi)\eta = D_{\ell+3}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\eta(t),$$

and clearly, $D_2\rho(t,\psi,\chi) \in \mathcal{L}(C,\mathbb{R})$ and $D_3\rho(t,\psi,\chi) \in \mathcal{L}(X,\mathbb{R})$ are continuous in t, ψ and χ.

Similarly, if $\psi \in C^1$, $\nu^i \in C^1$ $(i = 1,\ldots,\ell)$, B is continuously differentiable wrt t, and $\chi \in C^1([0,T],\mathbb{R})$, then for $t \in [0,T]$

$$\begin{aligned} &D_1\rho(t,\psi,\chi) \\ &\quad = D_1\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big) \\ &\qquad - \sum_{i=1}^{\ell} D_{i+1}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\dot\psi(-\nu^i(t))\dot\nu^i(t) \\ &\qquad + D_{\ell+2}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\int_{-r}^{-r_0} D_1B(t,\zeta)\psi(\zeta)\,d\zeta \\ &\qquad + D_{\ell+3}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\dot\chi(t). \end{aligned}$$

Moreover, it is easy to see that the function $[0,T] \times C^1 \to \mathbb{R}$, $(t,\psi) \mapsto D_1\rho(t,\psi,\chi)$ is continuous.

Let

$$\begin{aligned} L_{12} := \max_{i,j=1,\ldots,\ell+3} \max\Big\{ &|D_jD_i\bar\rho(t,u_1,\ldots,u_{\ell+1},v)|\colon\ t \in [0,\alpha],\ u_1,\ldots,u_\ell \in \overline{\mathcal{B}}_{\mathbb{R}^n}(0;\,R_1), \\ &u_{\ell+1} \in \overline{\mathcal{B}}_{\mathbb{R}^n}(0;\,b_{max}R_1 r),\ v \in \overline{\mathcal{B}}_{\mathbb{R}}(0;\,R_2)\Big\}. \end{aligned}$$

Then for $t \in [0,\alpha]$, $\psi, \bar\psi \in M_1$, $\chi, \bar\chi \in M_7$ and $h \in C$ we get

$$
\begin{aligned}
&|D_2\rho(t,\psi,\chi)h - D_2\rho(t,\bar\psi,\bar\chi)h| \\
&= \Big|\sum_{i=1}^{\ell} D_{i+1}\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)h(-\nu^i(t)) \\
&\quad + D_{\ell+2}\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\int_{-r}^{-r_0} B(t,\zeta)h(\zeta)\,d\zeta \\
&\quad - \sum_{i=1}^{\ell} D_{i+1}\bar\rho\Big(t,\bar\psi(-\nu^1(t)),\dots,\bar\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\bar\psi(\zeta)\,d\zeta,\bar\chi(t)\Big)h(-\nu^i(t)) \\
&\quad - D_{\ell+2}\bar\rho\Big(t,\bar\psi(-\nu^1(t)),\dots,\bar\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\bar\psi(\zeta)\,d\zeta,\bar\chi(t)\Big)\int_{-r}^{-r_0} B(t,\zeta)h(\zeta)\,d\zeta\Big| \\
&\le L_{12}\Big(\sum_{j=1}^{\ell}|\psi(-\nu^j(t))-\bar\psi(-\nu^j(t))| + \int_{-r}^{r_0}|B(t,\zeta)||\psi(\zeta)-\bar\psi(\zeta)|\,d\zeta + |\chi(t)-\bar\chi(t)|\Big) \\
&\quad \times\Big(\sum_{i=1}^{\ell}|h(-\nu^i(t))| + \int_{-r}^{-r_0}|B(t,\zeta)||h(\zeta)|\,d\zeta\Big) \\
&\le L_7^*\big(\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)| + |\chi-\bar\chi|_X\big)\max_{\zeta\in[-r,-r_0]}|h(\zeta)|
\end{aligned}
$$

with $L_7^* := L_{12}(\ell + rb_{max})^2$.

Similarly, for $t \in [0,\alpha]$, $\psi, \bar\psi \in M_1$, $\chi, \bar\chi \in M_7$, $\eta \in X$ we have

$$
\begin{aligned}
&|D_3\rho(t,\psi,\chi)\eta - D_3\rho(t,\bar\psi,\bar\chi)\eta| \\
&= \Big|D_{\ell+3}\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\eta(t) \\
&\quad - D_{\ell+3}\bar\rho\Big(t,\bar\psi(-\nu^1(t)),\dots,\bar\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\bar\psi(\zeta)\,d\zeta,\bar\chi(t)\Big)\eta(t)\Big| \\
&\le L_{12}\Big(\sum_{i=1}^{\ell}|\psi(-\nu^i(t))-\bar\psi(-\nu^i(t))| + \int_{-r}^{-r_0}|B(t,\zeta)||\psi(\zeta)-\bar\psi(\zeta)|\,d\zeta + |\chi(t)-\bar\chi(t)|\Big)|\eta|_X \\
&\le L_7^*\big(\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)| + |\chi-\bar\chi|_X\big)|\eta|_X.
\end{aligned}
$$

For $t, \bar t \in [0,\alpha]$, $\psi \in M_1$, $\chi \in M_7$ and $h \in C$ we have

$$
\begin{aligned}
&|D_2\rho(t,\psi,\chi)h - D_2\rho(\bar t,\psi,\chi)h| \\
&\le \sum_{i=1}^{\ell}\Big|D_{i+1}\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big) \\
&\qquad - D_{i+1}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0} B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big||h(-\nu^i(t))|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|D_{\ell+2}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\\
&-D_{\ell+2}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\ldots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big|\int_{-r}^{-r_0}|B(t,\zeta)||h(\zeta)|d\zeta\\
&+\sum_{i=1}^{\ell}\Big|D_{i+1}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\ldots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big|\\
&\qquad\times|h(-\nu^i(t))-h(-\nu^i(\bar t))|\\
&+\Big|D_{\ell+2}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\ldots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big|\\
&\qquad\times\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||h(\zeta)|\,d\zeta\\
\le\;& L_{12}\Big(|t-\bar t|+\sum_{j=1}^{\ell}|\psi(-\nu^j(t))-\psi(-\nu^j(\bar t))|+\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||\psi(\zeta)|\,d\zeta\\
&\qquad+|\chi(t)-\chi(\bar t)|\Big)\Big(\sum_{i=1}^{\ell}|h(-\nu^i(t))|+\int_{-r}^{-r_0}|B(t,\zeta)||h(\zeta)|\,d\zeta\Big)\\
&+L_{11}\Big(\sum_{i=1}^{\ell}|h(-\nu^i(t))-h(-\nu^i(\bar t))|+\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||h(\zeta)|\,d\zeta\Big)\\
\le\;& (L_{12}(1+\ell R_1L_9+rL_{10}R_1+R_2)(\ell+rb_{max})+rL_{11}L_{10})|t-\bar t|\max_{\zeta\in[-r,-r_0]}|h(\zeta)|\\
&+L_{11}l\max\{|h(\zeta)-h(\bar\zeta)|:\ \zeta,\bar\zeta\in[-r,-r_0],\ |\zeta-\bar\zeta|\le L_9|t-\bar t|\}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&|D_3\rho(t,\psi,\chi)\eta-D_3\rho(\bar t,\psi,\chi)\eta|\\
&\quad\le\;\Big|D_{\ell+3}\bar\rho\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\eta(t)\\
&\qquad-D_{\ell+3}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\ldots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\eta(t)\Big|\\
&\qquad+\Big|D_{\ell+3}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\ldots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big||\eta(t)-\eta(\bar t)|\\
&\quad\le\;L_{12}\Big(|t-\bar t|+\sum_{i=1}^{\ell}|\psi(-\nu^i(t))-\psi(-\nu^i(\bar t))|+\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||\psi(\zeta)|\,d\zeta\\
&\qquad+|\chi(t)-\chi(\bar t)|\Big)|\eta|_X+L_{11}|\eta|_X|t-\bar t|,
\end{aligned}
$$

so (A4) (iv) holds with $L_7:=\max\{L_7^*,L_{12}(1+\ell R_1L_9+rL_{10}R_1+R_2)(\ell+rb_{max})+rL_{11}L_{10}+L_{11},L_{11}\ell\}$ and $L_8=L_9$. □

We define the parameter space $\Gamma := W^{1,\infty} \times \Xi \times \Theta \times \Lambda \times X$, and use the notation $\gamma = (\varphi, \xi, \theta, \lambda, \chi)$ or $\gamma = (\gamma^\varphi, \gamma^\xi, \gamma^\theta, \gamma^\lambda, \gamma^\chi)$ for the components of $\gamma \in \Gamma$, and $|\gamma|_\Gamma := |\varphi|_{W^{1,\infty}} + |\xi|_\Xi + |\theta|_\Theta + |\lambda|_\Lambda + |\chi|_X$ for the norm on Γ. We introduce the set of feasible parameters

$$\begin{aligned}\Pi \;:=\; \Big\{&(\varphi,\xi,\theta,\lambda,\chi) \in \Gamma\colon\ \varphi \in \Omega_1,\ \ \varphi(-\tau(0,\varphi,\xi)) \in \Omega_2,\ \ \theta \in \Omega_3,\ \ \xi \in \Omega_4,\\ &\varphi(-\rho(0,\varphi,\chi)) \in \Omega_5,\ \ \lambda \in \Omega_6,\ \ \chi \in \Omega_7,\Big\}.\end{aligned}$$

We will show in Theorem 4.2.2 below that Π is an open subset of Γ. Next define the special parameter set

$$\begin{aligned}\mathcal{P} := \Big\{&(\varphi,\xi,\theta,\lambda,\chi) \in \Pi\ :\ g(t,\psi,u,\lambda) \text{ and } \rho(t,\psi,\chi) \text{ are differentiable wrt } t,\\ &\text{and the maps } (t,\psi,u) \mapsto D_1 g(t,\psi,u,\lambda) \text{ and } (t,\psi) \mapsto D_1\rho(t,\psi,\chi)\\ &\text{are continuous for } t \in [0,T],\ \psi \in \Omega_1,\ u \in \Omega_2; \qquad \varphi \in C^1;\\ &\dot\varphi(0-) = D_1 g(0,\varphi,\varphi(-\rho(0,\varphi,\chi)),\lambda) + D_2 g(0,\varphi,\varphi(-\rho(0,\varphi,\chi)),\lambda)\dot\varphi\\ &+D_3 g(0,\varphi,\varphi(-\rho(0,\varphi,\chi)),\lambda)\dot\varphi(-\rho(0,\varphi,\chi))\\ &\times(1 - D_1\rho(0,\varphi,\chi) - D_2\rho(0,\varphi,\chi)\dot\varphi) + f(0,\varphi,\varphi(-\tau(0,\varphi,\xi)),\theta)\Big\}.\end{aligned}$$

Note that an analogous set was used for neutral FDEs in order to guarantee the existence of a continuous semiflow on a subset of C^1 in [72].

Next we show that under the assumptions listed in the beginning of this section the IVP (4.2.1)-(4.2.2) has a unique solution which depends continuously on the parameter $\gamma = (\varphi, \xi, \theta, \lambda, \chi)$ in the C-norm. The solution of the IVP (4.2.1)-(4.2.2) corresponding to a parameter γ and its segment function at t are denoted by $x(t,\gamma)$ and $x_t(\cdot,\gamma)$, respectively.

Theorem 4.2.2 *Assume (A1) (i), (ii), (A2) (i), (ii), (A3) (i), (ii) and (A4) (i)–(ii), and let $\widehat\gamma \in \Pi$. Then there exist $\delta > 0$ and $0 < \alpha \le T$ finite numbers such that*

(i) $P := \mathcal{B}_\Gamma(\widehat\gamma;\,\delta) \subset \Pi$;

(ii) the IVP (4.2.1)-(4.2.2) has a unique solution $x(t,\gamma)$ on $[-r,\alpha]$ for all $\gamma \in P$;

(iii) there exist a closed subset $M_1 \subset C$ which is also a bounded and convex subset of $W^{1,\infty}$, $M_2 \subset \Omega_2$ and $M_5 \subset \Omega_5$ compact and convex subsets of $\mathbb{R}^n$, such that $x(t) := x(t,\gamma)$ satisfies

$$x_t \in M_1, \qquad x(t-\tau(t,x_t,\xi)) \in M_2, \quad \text{and} \quad x(t-\rho(t,x_t,\chi)) \in M_5 \tag{4.2.6}$$

for $t \in [0,\alpha]$ and $\gamma = (\varphi,\xi,\theta,\lambda,\chi) \in P$;

(iv) $x_t(\cdot,\gamma)\in W^{1,\infty}$ *for* $t\in[0,\alpha]$, $\gamma\in P$, *and there exist* $N=N(\alpha,\delta)$ *and* $L=L(\alpha,\delta)$ *such that*

$$|x_t(\cdot,\gamma)|_{W^{1,\infty}}\le N,\qquad t\in[0,\alpha],\quad \gamma\in P,\tag{4.2.7}$$

and

$$|x_t(\cdot,\gamma)-x_t(\cdot,\bar\gamma)|_C\le L|\gamma-\bar\gamma|_\Gamma,\qquad t\in[0,\alpha],\quad \gamma,\bar\gamma\in P.\tag{4.2.8}$$

(v) *Moreover, the function* $x(\cdot,\gamma)\colon[-r,\alpha]\to\mathbb{R}^n$ *is continuously differentiable for* $\gamma\in\mathcal{P}\cap P$.

Proof (i) Let $\widehat\gamma=(\widehat\varphi,\widehat\xi,\widehat\theta,\widehat\lambda,\widehat\chi)\in\Pi$. Since $\Omega_1,\dots,\Omega_7$ are open subsets of their respective spaces, there exists $\delta_1>0$ such that $\overline{\mathcal{B}}_C(\widehat\varphi;\delta_1)\subset\Omega_1$, $\overline{\mathcal{B}}_\Theta\Big(\widehat\theta;\delta_1\Big)\subset\Omega_3$, $\overline{\mathcal{B}}_\Xi\Big(\widehat\xi;\delta_1\Big)\subset\Omega_4$, $\overline{\mathcal{B}}_\Lambda\Big(\widehat\lambda;\delta_1\Big)\subset\Omega_6$ and $\overline{\mathcal{B}}_X(\widehat\chi;\delta_1)\subset\Omega_7$. Introduce the vectors $w_1:=\widehat\varphi(-\tau(0,\widehat\varphi,\widehat\xi))$ and $w_2:=\widehat\varphi(-\rho(0,\widehat\varphi,\widehat\chi))$. Let $\varepsilon_1>0$ be such that $\overline{\mathcal{B}}_{\mathbb{R}^n}(w_1;\varepsilon_1)\subset\Omega_2$ and $\overline{\mathcal{B}}_{\mathbb{R}^n}(w_2;\varepsilon_1)\subset\Omega_5$. The map

$$\mathbb{R}\times C\times\Xi\supset[0,T]\times\Omega_1\times\Omega_4\to\mathbb{R}^n,\quad (t,\psi,\xi)\mapsto\psi(-\tau(t,\psi,\xi))$$

is continuous, since

$$\begin{aligned}
&|\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(\bar t,\bar\psi,\bar\xi))|\\
&\qquad\le\ |\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(t,\psi,\xi))|+|\bar\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(\bar t,\bar\psi,\bar\xi))|\\
&\qquad\le\ |\psi-\bar\psi|_C+|\bar\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(\bar t,\bar\psi,\bar\xi))|\\
&\qquad\to\ 0,\qquad \text{as } t\to\bar t,\ \psi\to\bar\psi,\ \xi\to\bar\xi.
\end{aligned}$$

Similarly, the map $\mathbb{R}\times C\times\Xi\supset[0,T]\times\Omega_1\times\Omega_7\to\mathbb{R}^n$, $\quad(t,\psi,\chi)\mapsto\psi(-\rho(t,\psi,\chi))$ is also continuous, therefore there exist $\delta_2\in(0,\delta_1]$ and $T_1\in(0,T]$ such that

$$|\psi(-\tau(t,\psi,\xi))-w_1|<\varepsilon_1,\qquad |\psi(-\rho(t,\psi,\chi))-w_2|<\varepsilon_1\tag{4.2.9}$$

for $t\in[0,T_1]$, $\psi\in\mathcal{B}_C(\widehat\varphi;\delta_2)$, $\xi\in\mathcal{B}_\Xi\Big(\widehat\xi;\delta_2\Big)$ and $\chi\in\mathcal{B}_X(\widehat\chi;\delta_2)$.

Let $\varepsilon_0>0$ be fixed. The continuity of the map $(t,\psi,\xi,\theta)\mapsto f(t,\psi,\psi(-\tau(t,\psi,\xi)),\theta)$ yields that there exist $\delta_3\in(0,\delta_2]$ and $T_2\in(0,T_1]$ such that

$$|f(t,\psi,\psi(-\tau(t,\psi,\xi)),\theta)-f(0,\widehat\varphi,\widehat\varphi(-\tau(0,\widehat\varphi,\widehat\xi)),\widehat\theta)|<\varepsilon_0$$

for $t\in[0,T_2]$, $\psi\in\mathcal{B}_C(\widehat\varphi;\delta_3)$, $\xi\in\mathcal{B}_\Xi\Big(\widehat\xi;\delta_3\Big)$ and $\theta\in\mathcal{B}_\Theta\Big(\widehat\theta;\delta_3\Big)$.

Define the sets

$$M_2:=\overline{\mathcal{B}}_{\mathbb{R}^n}(w_1;\varepsilon_1),\quad M_3:=\overline{\mathcal{B}}_\Theta\Big(\widehat\theta;\delta_3\Big),\quad M_4:=\overline{\mathcal{B}}_\Xi\Big(\widehat\xi;\delta_3\Big)$$

and

$$M_5:=\overline{\mathcal{B}}_{\mathbb{R}^n}(w_2;\varepsilon_1),\quad M_6:=\overline{\mathcal{B}}_\Lambda\Big(\widehat\lambda;\delta_3\Big),\quad M_7:=\overline{\mathcal{B}}_X(\widehat\chi;\delta_3).$$

Throughout this proof the extension of the function $\psi \in C$ to the interval $[-r,\infty)$ by the constant value $\psi(0)$ will be denoted by

$$\widetilde{\psi}(t) := \begin{cases} \psi(t), & t \in [-r,0], \\ \psi(0), & t \geq 0. \end{cases}$$

We define the following constants and sets

$$\begin{aligned} K_2 &:= |f(0,\widehat{\varphi},\widehat{\varphi}(-\tau(0,\widehat{\varphi},\widehat{\xi})),\widehat{\theta})| + \varepsilon_0, \\ \beta_1 &:= \frac{\delta_3}{3}, \\ \delta &:= \min\Big\{\frac{\delta_3}{3}, \frac{\varepsilon_1}{2}\Big\}, \\ a_0 &:= |\widehat{\varphi}|_{W^{1,\infty}} + \delta, \\ M_{1,0} &:= \{\psi \in W^{1,\infty} \colon |\psi - \widehat{\varphi}|_C \leq \delta_3,\ |\dot{\psi}|_{L^\infty} \leq a_0\}, \end{aligned}$$

It is easy to check that $M_{1,0}$ is closed in C and it is bounded in $W^{1,\infty}$, so let

$$\begin{aligned} L_{3,0} &:= L_3(T_2, M_{1,0}, M_5, M_6) \text{ be the Lipschitz constant defined by (A3) (ii)}, \\ L_{6,0} &:= L_6(T_2, M_{1,0}, M_7) \text{ be the Lipschitz constant defined by (A4) (ii)}, \\ K_{1,1} &:= L_{3,0}(1 + a_0(2 + L_{6,0}(1 + a_0))), \\ a_1 &:= \max\{a_0, K_{1,1} + K_2\}, \\ \alpha_1 &:= \min\Big\{\frac{\beta_1}{a_1}, \frac{\varepsilon_1}{2a_0}, T_2, r_0\Big\}, \\ E_1 &:= \Big\{y \in C([-r,\alpha_1],\mathbb{R}^n) \colon y(s) = 0 \text{ for } s \in [-r,0] \text{ and } |y(s)| \leq \beta_1 \text{ for } s \in [0,\alpha_1]\Big\}. \end{aligned}$$

We have $|\dot{\varphi}|_{L^\infty} \leq |\varphi|_{W^{1,\infty}} \leq |\widehat{\varphi}|_{W^{1,\infty}} + |\varphi - \widehat{\varphi}|_{W^{1,\infty}} \leq a_0$ for $\varphi \in \mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\,\delta)$, and so $\mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\,\delta) \subset M_{1,0}$. Then for $y \in E_1$, $\varphi \in \mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\,\delta)$, $t \in [0,\alpha_1]$ and $\zeta \in [-r,0]$ we get

$$\begin{aligned} |y(t+\zeta) + \widetilde{\varphi}(t+\zeta) - \widehat{\varphi}(\zeta)| &\leq |y(t+\zeta)| + |\widetilde{\varphi}(t+\zeta) - \widetilde{\varphi}(\zeta)| + |\varphi(\zeta) - \widehat{\varphi}(\zeta)| \\ &< \beta_1 + t|\dot{\varphi}|_{L^\infty} + \delta \\ &\leq \beta_1 + \alpha_1 a_0 + \delta \\ &\leq \delta_3, \end{aligned} \tag{4.2.10}$$

and hence $|y_t + \widetilde{\varphi}_t - \widehat{\varphi}|_C < \delta_3$. Consequently, $y_t + \widetilde{\varphi}_t \in \mathcal{B}_C(\widehat{\varphi};\,\delta_3) \subset \Omega_1$, and so

$$\Big| f\Big(t, y_t + \widetilde{\varphi}_t, y(t - \tau(t, y_t + \widetilde{\varphi}_t, \xi)) + \widetilde{\varphi}(t - \tau(t, y_t + \widetilde{\varphi}_t, \xi)), \theta\Big)\Big| \leq K_1,$$

and $\psi = y_t + \widetilde{\varphi}_t$ satisfies (4.2.9) for $y \in E_1$, $\varphi \in \mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\,\delta)$, $\xi \in \mathcal{B}_\Xi\Big(\widehat{\xi};\,\delta\Big)$, $\theta \in \mathcal{B}_\Theta(\widehat{\varphi};\,\delta)$ and $t \in [0,\alpha_1]$. Therefore the definitions of M_2, M_5 and (4.2.9) yield

$$(y_t + \widetilde{\varphi}_t)(-\tau(t,\psi,\xi)) \in M_2, \qquad (y_t + \widetilde{\varphi}_t)(-\rho(t,\psi,\chi)) \in M_5 \tag{4.2.11}$$

for $t \in [0, \alpha_1]$, $y \in E_1$, $\varphi \in \mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\, \delta)$, $\chi \in \mathcal{B}_X(\widehat{\chi};\, \delta)$ and $\xi \in \mathcal{B}_\Xi\Big(\widehat{\xi};\, \delta\Big)$.

Fix $\gamma = (\varphi, \theta, \xi, \lambda, \chi) \in \mathcal{B}_\Gamma(\bar{\gamma};\, \delta)$. Then $\varphi \in \mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\, \delta)$, $\theta \in \mathcal{B}_\Theta\Big(\widehat{\theta};\, \delta\Big)$, $\chi \in \mathcal{B}_X(\widehat{\chi};\, \delta)$, $\lambda \in \mathcal{B}_\Lambda\Big(\widehat{\lambda};\, \delta\Big)$ and $\chi \in \mathcal{B}_X(\widehat{\chi};\, \delta)$. We can use the method of steps to show that the IVP (4.2.1)-(4.2.2) corresponding to γ has a solution. First note that a solution will satisfy $x_t(\zeta) = x(t + \zeta) = \varphi(t + \zeta) = \widetilde{\varphi}_t(\zeta)$ for $t \in [0, r_0]$ and $\zeta \in [-r, -r_0]$. We have $t - \rho(t, \widetilde{\varphi}_t, \chi) \le t - r_0 \le 0$ for $t \in [0, r_0]$, so $y_t(-\rho(t, \widetilde{\varphi}_t, \chi)) = 0$ for $t \in [0, r_0]$. Hence (4.2.11) yields that $\varphi[t - \rho(t, \widetilde{\varphi}_t, \chi)] \in M_5$ for $t \in [0, r_0]$. An estimate similar to (4.2.10) gives $|\widetilde{\varphi}_t - \widehat{\varphi}|_C < \delta_3$ for $t \in [0, r_0]$. Therefore, the function

$$\mu^1(t) := g\{t, \widetilde{\varphi}_t, \varphi[t - \rho(t, \widetilde{\varphi}_t, \chi)], \lambda\}, \qquad t \in [0, r_0] \tag{4.2.12}$$

is well-defined. Then (A3) (ii), (A4) (ii), Lemma 1.2.5, $|\dot{\varphi}|_{L^\infty} \le a_0$, $\widetilde{\varphi}_t \in M_{1,0}$ for $t \in [0, r_0]$, and the definition of $K_{1,1}$ yield

$$\begin{aligned} |\mu^1(t) - \mu^1(\bar{t})| &\le L_{3,0}\Big\{|t - \bar{t}| + \max_{\zeta \in [-r, -r_0]} |\varphi(t + \zeta) - \varphi(\bar{t} + \zeta)| \\ &\qquad + \Big|\varphi[t - \rho(t, \widetilde{\varphi}_t, \chi)] - \varphi[\bar{t} - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \chi)]\Big|\Big\} \\ &\le L_{3,0}\Big\{|t - \bar{t}| + |\dot{\varphi}|_{L^\infty}|t - \bar{t}| + |\dot{\varphi}|_{L^\infty}[1 + L_{6,0}(1 + |\dot{\varphi}|_{L^\infty})]|t - \bar{t}|\Big\} \\ &\le K_{1,1}|t - \bar{t}|, \qquad t, \bar{t} \in [0, r_0]. \end{aligned} \tag{4.2.13}$$

On the interval $[0, r_0]$ Equation (4.2.1) is equivalent to

$$\frac{d}{dt}\Big(x(t) - \mu^1(t)\Big) = f(t, x_t, x(t - \tau(t, x_t, \xi)), \theta), \qquad t \in [0, r_0].$$

Therefore, (4.2.1) is equivalent to

$$x(t) = \mu^1(t) + \varphi(0) - \mu^1(0) + \int_0^t f(s, x_s, x(s - \tau(s, x_s, \xi)), \theta)\, ds, \qquad t \in [0, r_0]. \tag{4.2.14}$$

We introduce the new variable $y(t) := x(t) - \widetilde{\varphi}(t)$, and we define the operator

$$T^1(y, \gamma)(t) := \begin{cases} \mu^1(t) - \mu^1(0) + \displaystyle\int_0^t f\Big(s, y_s + \widetilde{\varphi}_s, (y + \widetilde{\varphi})(s - \tau(s, y_s + \widetilde{\varphi}_s, \xi)), \theta\Big) ds, & t \in [0, \alpha_1], \\ 0, & t \in [-r, 0]. \end{cases}$$

Then in the new variable y, on the interval $[-r, \alpha_1]$ the IVP (4.2.1)-(4.2.2) is equivalent to the fixed point problem

$$y = T^1(y, \gamma).$$

It is easy to check that $T^1(\cdot,\gamma)$ maps the closed, bounded and convex subset E_1 of C into E_1 for all $\gamma\in\mathcal{B}_\Gamma(\widehat{\gamma};\,\delta)$. Therefore, Schauder's Fixed Point Theorem yields the existence of a fixed point $y=y(\cdot,\gamma)$ of $T^1(\cdot,\gamma)$, and therefore, (4.2.1) has a solution $x=x(\cdot,\gamma)=y(\cdot,\gamma)+\widetilde{\varphi}$ on the interval $[-r,\alpha_1]$. Estimate (4.2.13) yields that μ^1 is Lipschitz continuous, and therefore, it is a.e. differentiable, and $|\dot\mu^1(t)|\le K_{1,1}$ for a.e. $t\in[0,\alpha_1]$. Hence y, and so, x is also a.e. differentiable on $t\in[-r,\alpha_1]$, and (4.2.14) implies $|\dot x(t)|=|\dot y(t)|\le K_{1,1}+K_2$ for a.e. $t\in[0,\alpha_1]$, and so $|\dot x(t)|\le a_1$ for a.e. $t\in[-r,\alpha_1]$.

(ii) Next we show by iteration that the solution obtained in part (i) of the proof can be extended to a larger interval so that estimate (4.2.7) remains to hold with some N independent of the selection of γ from $\mathcal{B}_\Gamma(\widehat{\gamma};\,\delta)$. Let $j:=2$, and let $x=x(\cdot,\gamma)$ be the solution of (4.2.1)-(4.2.2) on $[-r,\alpha_{j-1}]$, $\varphi^j:=x_{\alpha_{j-1}}$ and

$$\mu^j(t):=g\Big(t+\alpha_{j-1},\widetilde{\varphi^j_t},\varphi^j[t-\rho(t+\alpha_{j-1},\widetilde{\varphi^j_t},\chi)],\lambda\Big),\qquad t\in[0,r_0],$$

where $\widetilde{\varphi^j_t}$ denotes the segment function of $\widetilde{\varphi^j}$ at t. If $\alpha_{j-1}<T_2$, repeating the first part of the proof, we are looking for an extension of the solution of the IVP (4.2.1)-(4.2.2) by solving the fixed point equation

$$y=T^j(y,\gamma),$$

where $y(t):=x(t+\alpha_{j-1})-\widetilde{\varphi^j}(t)$, and

$$T^j(y,\gamma)(t)$$
$$:=\begin{cases}\mu^j(t)-\mu^j(0)\\ +\displaystyle\int_0^t f(s+\alpha_{j-1},y_s+\widetilde{\varphi^j_s},(y+\widetilde{\varphi^j})(s-\tau(s+\alpha_{j-1},y_s+\widetilde{\varphi^j_s},\xi)),\theta)\,ds, & t\in[0,\Delta\alpha_j],\\ 0, & t\in[-r,0]\end{cases}$$

for some $\Delta\alpha_j\in(0,T_2-\alpha_{j-1}]$. Relation (4.2.10) yields that $|\varphi^j-\widehat{\varphi}|_C<\delta_3$. Therefore, there exists $\varepsilon_j>0$ such that $\overline{\mathcal{B}}_C(\varphi^j;\,\varepsilon_j)\subset\mathcal{B}_C(\widehat{\varphi};\,\delta_3)$. Define the constants and sets

$$\begin{aligned}
\beta_j &:= \frac{\varepsilon_j}{2},\\
M_{1,j-1} &:= \{\psi\in W^{1,\infty}\colon\ |\psi-\widehat{\varphi}|_C\le\delta_3,\ |\dot\psi|_{L^\infty}\le a_{j-1}\},\\
L_{3,j-1} &:= L_3(T_2,M_{1,j-1},M_5,M_6)\text{ be the Lipschitz constant defined by (A3) (ii)},\\
L_{6,j-1} &:= L_6(T_2,M_{1,j-1},M_7)\text{ be the Lipschitz constant defined by (A4) (ii)},\\
K_{1,j} &:= L_{3,j-1}(1+a_{j-1}(2+L_{6,j-1}(1+a_{j-1}))),\\
a_j &:= \max\{a_{j-1},K_{1,j}+K_2\},\\
\Delta\alpha_j &:= \min\Big\{\frac{\beta_j}{a_j},\ \frac{\varepsilon_j}{2a_{j-1}},\ T_2-\alpha_{j-1},\ r_0\Big\},\\
\alpha_j &:= \alpha_{j-1}+\Delta\alpha_j,\\
E_j &:= \Big\{y\in C([-r,\Delta\alpha_j],\mathbb{R}^n)\colon\ y(s)=0,\ s\in[-r,0]\text{ and }|y(s)|\le\beta_j,\ s\in[0,\Delta\alpha_j]\Big\}.
\end{aligned}$$

Since $|\dot{\varphi}^j|_{L^\infty} \le a_{j-1}$, it is easy to check that $|y_t + \widetilde{\varphi_t^j} - \varphi^j|_C \le \varepsilon_j$ for $t \in [0, \Delta\alpha_j]$, $y \in E_j$, and hence $\alpha_j \le T_2$ and (4.2.9) imply $(y_t + \widetilde{\varphi_t^j})(-\tau(t+\alpha_{j-1}, y_t + \widetilde{\varphi}_t^j, \xi)) \in M_2$ and $(y_t + \widetilde{\varphi_t^j})(-\rho(t+\alpha_{j-1}, y_t + \widetilde{\varphi_t^j}, \chi)) \in M_5$ for $t \in [0, \Delta\alpha_j]$, $y \in E_j$. Also, one can check that $|\mu^j(t) - \mu^j(\bar{t})| \le K_{1,j}|t-\bar{t}|$ for $t, \bar{t} \in [0, r_0]$, and the operator $T^j(\cdot, \gamma)$ maps E_j into E_j for all $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma};\, \delta)$. Hence Schauder's Fixed Point Theorem yields the existence of a fixed point y of $T^j(\cdot, \gamma)$ in E_j, and hence the function $x(t) := y(t-\alpha_{j-1}) + \widetilde{\varphi}^j(t-\alpha_{j-1})$, $t \in [\alpha_{j-1}, \alpha_j]$ gives an extension of the solution of the IVP (4.2.1)-(4.2.2) from the interval $[-r, \alpha_{j-1}]$ to the interval $[-r, \alpha_j]$. Moreover, for the extended solution we have $|\dot{x}(t)| \le a_j$ for a.e. $t \in [-r, \alpha_j]$. If $\alpha_j < T_2$, by repeating the previous iteration, we can extend the solution to a larger interval. In case of an infinite iteration, we stop it after finitely many steps to guarantee the boundedness of the sequence a_j. Suppose we repeat the iteration k times. Then let $\alpha := \alpha_k$. This completes the proof of the existence of a solution $x = x(\cdot, \gamma)$ of the IVP (4.2.1)-(4.2.2) on $[-r, \alpha]$ for any $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma};\, \delta)$, which satisfies $|\dot{x}(t)| \le a_k$ for a.e. $t \in [-r, \alpha]$. The estimate

$$|x(t)| \le |\varphi(0)| + \int_0^t |\dot{x}(s)|\, ds \le a_0 + a_k\alpha, \qquad t \in [0, \alpha]$$

yields that x satisfies (4.2.7) with $N := \max\{a_k, a_0 + a_k\alpha\}$. Define the set

$$M_1 := M_{1,k} = \Big\{\psi \in W^{1,\infty}\colon\ |\psi - \widehat{\varphi}|_C \le \delta_3, \quad |\dot{\psi}|_{L^\infty} \le a_k\Big\}.$$

Then $M_{1,j} \subset M_1$ for all $j = 0, \dots, k$, and $x_t \in M_1$ for $t \in [0, \alpha]$. The Arzelà-Ascoli Theorem implies that M_1 is a compact subset of C, and hence the solution $x = x(\cdot, \gamma)$ constructed by the above argument satisfies (4.2.6) for $t \in [0, \alpha]$ and $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma};\, \delta)$.

(iii) The uniqueness of the solution will follow from (4.2.8). To show (4.2.8) suppose $\gamma = (\varphi, \xi, \theta, \lambda, \chi)$ and $\bar{\gamma} = (\bar{\varphi}, \bar{\xi}, \bar{\theta}, \bar{\lambda}, \bar{\chi})$ are fixed parameters in $\mathcal{B}_\Gamma(\widehat{\gamma};\, \delta)$, and let x be *any* fixed solution of the IVP (4.2.1)-(4.2.2) corresponding to γ, and let $\bar{x} := x(\cdot; \bar{\gamma})$ be the solution of the IVP (4.2.1)-(4.2.2) obtained by the argument of part (i) of the proof on the interval $[-r, \alpha]$. Then part (i) of the proof yields $|\bar{x}_t|_{W^{1,\infty}} \le N$ and

$$|\bar{x}_t - \widehat{\varphi}|_C < \delta_3, \quad |\bar{x}(t - \tau(t, \bar{x}_t, \bar{\xi})) - w_1| < \varepsilon_1, \quad |\bar{x}(t - \rho(t, \bar{x}_t, \bar{\chi})) - w_2| < \varepsilon_1 \quad (4.2.15)$$

for $t \in [0, \alpha]$, and therefore $\bar{x}(t - \tau(t, \bar{x}_t, \bar{\xi})) \in M_2$ and $\bar{x}(t - \rho(t, \bar{x}_t, \bar{\chi})) \in M_5$ for $t \in [0, \alpha]$. Since $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma};\, \delta)$, it follows that $\varphi \in \mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\, \delta)$, $\xi \in \mathcal{B}_\Xi\Big(\widehat{\xi};\, \delta\Big)$, $\theta \in \mathcal{B}_\Theta\Big(\widehat{\theta};\, \delta\Big)$, $\lambda \in \mathcal{B}_\Lambda\Big(\widehat{\lambda};\, \delta\Big)$ and $\chi \in \mathcal{B}_X(\widehat{\chi};\, \delta)$. Hence $\delta < \delta_3$ and (4.2.9) yield $|\varphi - \widehat{\varphi}|_C < \delta_3$, $|\varphi(-\tau(0, \varphi, \xi)) - w_1| < \varepsilon_1$ and $|\varphi(-\rho(0, \varphi, \chi)) - w_2| < \varepsilon_1$. Therefore the continuity of x implies that the above inequalities are preserved for small t. Let $\alpha_\gamma \in (0, \alpha]$ be the largest number for which

$$|x_t - \widehat{\varphi}|_C < \delta_3, \qquad |x(t - \tau(t, x_t, \xi)) - w_1| < \varepsilon_1, \qquad |x(t - \rho(t, x_t, \chi)) - w_2| < \varepsilon_1 \quad (4.2.16)$$

hold for $t \in [0, \alpha^\gamma)$. Then $x(t - \tau(t, x_t, \xi)) \in M_2$ and $x(t - \rho(t, x_t, \chi)) \in M_5$ also hold for $t \in [0, \alpha^\gamma]$.

Next we show that $x_t \in M_1$ for $t \in [0, \alpha^\gamma]$. It is enough to show that $|\dot{x}_t|_{L^\infty} \le a_k$ for a.e. $t \in [0, \alpha^\gamma]$. Let $m = [\alpha^\gamma / r_0]$, where here $[\cdot]$ is the greatest integer part function. Note that $m \le k$ since $m r_0 \le \alpha^\gamma \le \alpha = \alpha_k \le k r_0$. Let $t_j := j r_0$ for $j = 0, \dots, m$, and $t_{m+1} := \alpha^\gamma$. Suppose first that $t_0 \le \bar{t} \le t \le t_1$. Then integrating (4.2.1) from $\bar{t}$ to t and using (A3) (ii), (A4) (i), (ii), (4.2.16), $|\dot{\varphi}|_{L^\infty} \le a_0$ and the definitions of $L_{3,0}$, $L_{6,0}$, K_2, $K_{1,1}$ and a_1 we get

$$\begin{aligned}
|x(t) - x(\bar{t})| &\le |g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda) - g(\bar{t}, x_{\bar{t}}, x(\bar{t} - \rho(\bar{t}, x_{\bar{t}}, \chi)), \lambda)| \\
&\quad + \int_{\bar{t}}^{t} |f(s, x_s, x(s - \tau(s, x_s, \xi)), \theta)|\, ds \\
&= |g(t, \widetilde{\varphi}_t, \varphi(t - \rho(t, \widetilde{\varphi}_t, \chi)), \lambda) - g(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \varphi(\bar{t} - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \chi)), \lambda)| \\
&\quad + \int_{\bar{t}}^{t} |f(s, x_s, x(s - \tau(s, x_s, \xi)), \theta)|\, ds \\
&\le L_{3,0}\Big(|t - \bar{t}| + \max_{\zeta \in [-r, -r_0]} |\varphi(t + \zeta) - \varphi(\bar{t} + \zeta)| \\
&\quad + |\varphi(t - \rho(t, \widetilde{\varphi}_t, \chi)) - \varphi(\bar{t} - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \chi))|\Big) + K_2 |t - \bar{t}| \\
&\le \Big(L_{3,0}(1 + a_0(2 + L_{6,0}(1 + a_0))) + K_2\Big)|t - \bar{t}| \\
&\le a_1 |t - \bar{t}|, \qquad t, \bar{t} \in [t_0, t_1].
\end{aligned}$$

Then $a_0 \le a_1$ implies $|x(t) - x(\bar{t})| \le a_1 |t - \bar{t}|$ for $t, \bar{t} \in [-r, t_1]$.

Suppose now that $|x(t) - x(\bar{t})| \le a_j |t - \bar{t}|$ holds for $t, \bar{t} \in [-r, t_j]$ for some $j \le m$. Then for $t, \bar{t} \in [-r, t_{j+1}]$ we get easily that

$$\begin{aligned}
|x(t) - x(\bar{t})| &\le \Big(L_{3,j}(1 + a_j(2 + L_{6,j}(1 + a_j))) + K_2\Big)|t - \bar{t}| \\
&\le a_{j+1} |t - \bar{t}|, \qquad t, \bar{t} \in [t_0, t_{j+1}].
\end{aligned}$$

This shows that $|x(t) - x(\bar{t})| \le a_k |t - \bar{t}|$ for $t, \bar{t} \in [-r, \alpha^\gamma]$, hence $|\dot{x}_t|_{L^\infty} \le a_k$ for $t \in [0, \alpha^\gamma]$, and therefore $x_t \in M_1$ for $t \in [0, \alpha^\gamma]$.

Let $L_1 = L_1(\alpha, M_1, M_2, M_3)$, $L_2 = L_2(\alpha, M_1, M_4)$, $L_3 = L_3(\alpha, M_1, M_5, M_6)$ and $L_6 = L_6(\alpha, M_1, M_7)$ be the Lipschitz constants from (A1) (ii), (A2) (ii), (A3) (ii) and (A4) (ii), respectively. Integrating (4.2.1) from 0 to t we get for $t \in [0, \alpha^\gamma]$

$$\begin{aligned}
&|x(t) - \bar{x}(t)| \\
&\quad \le |g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda) - g(t, \bar{x}_t, \bar{x}(t - \rho(t, \bar{x}_t, \bar{\chi})), \bar{\lambda})| + |\varphi(0) - \bar{\varphi}(0)| \\
&\quad\quad + |g(0, \varphi, \varphi(-\rho(0, \varphi, \chi)), \lambda) - g(0, \bar{\varphi}, \bar{\varphi}(-\rho(0, \bar{\varphi}, \bar{\chi})), \bar{\lambda})| \\
&\quad\quad + \int_0^t \Big| f(s, x_s, x(s - \tau(s, x_s, \xi)), \theta) - f(s, \bar{x}_s, \bar{x}(s - \tau(s, \bar{x}_s, \bar{\xi})), \bar{\theta}) \Big|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\le\ L_3\Big(\max_{\zeta\in[-r,-r_0]}|x(t+\zeta)-\bar{x}(t+\zeta)|+|x(t-\rho(t,x_t,\chi))-\bar{x}(t-\rho(t,\bar{x}_t,\bar{\chi}))|\\
&\qquad +|\lambda-\bar{\lambda}|\Big)+|\varphi-\bar{\varphi}|_C\\
&\quad +L_3\big(|\varphi-\bar{\varphi}|_C+|\varphi(-\rho(0,\varphi,\chi))-\bar{\varphi}(-\rho(0,\bar{\varphi},\bar{\chi}))|+|\lambda-\bar{\lambda}|\big)\\
&\quad +L_1\int_0^t\Big(|x_s-\bar{x}_s|_C+|x(s-\tau(s,x_s,\xi))-\bar{x}(s-\tau(s,\bar{x}_s,\bar{\xi}))|+|\theta-\bar{\theta}|_\Theta\Big)\,ds.
\end{aligned}
$$

Lemma 1.2.5, $|\bar{x}_t|_{W^{1,\infty}}\le N$ for $t\in[0,\alpha]$ and (A2) (ii) yield

$$
\begin{aligned}
&|x(s-\tau(s,x_s,\xi))-\bar{x}(s-\tau(s,\bar{x}_s,\bar{\xi}))|\\
&\qquad\le\ |\bar{x}(s-\tau(s,x_s,\xi))-\bar{x}(s-\tau(s,\bar{x}_s,\bar{\xi}))|+|x(s-\tau(s,x_s,\xi))-\bar{x}(s-\tau(s,x_s,\xi))|\\
&\qquad\le\ N|\tau(s,x_s,\xi)-\tau(s,\bar{x}_s,\bar{\xi})|+|x_s-\bar{x}_s|_C\\
&\qquad\le\ L_2N(|x_s-\bar{x}_s|_C+\ |\xi-\bar{\xi}|_\Xi)+|x_s-\bar{x}_s|_C,\qquad s\in[0,\alpha^\gamma].
\end{aligned}\tag{4.2.17}
$$

Define $\mu(t):=\max\{|x(s)-\bar{x}(s)|:\ -r\le s\le t\}$ for $t\in[0,\alpha^\gamma]$. Assumption (A4) (i), Lemma 1.2.5, $|\bar{x}_t|_{W^{1,\infty}}\le N$ for $t\in[0,\alpha]$ and (A4) (ii) imply

$$
\begin{aligned}
&|x(t-\rho(t,x_t,\chi))-\bar{x}(t-\rho(t,\bar{x}_t,\bar{\chi}))|\\
&\qquad\le\ |x(t-\rho(t,x_t,\chi))-\bar{x}(t-\rho(t,x_t,\chi))|+|\bar{x}(t-\rho(t,x_t,\chi))-\bar{x}(t-\rho(t,\bar{x}_t,\bar{\chi}))|\\
&\qquad\le\ \mu(t-r_0)+N|\rho(t,x_t,\chi)-\rho(t,\bar{x}_t,\bar{\chi})|\\
&\qquad\le\ (1+NL_6)\mu(t-r_0)+NL_6|\chi-\bar{\chi}|_X,\qquad t\in[0,\alpha^\gamma].
\end{aligned}
$$

Similarly, $|\varphi(-\rho(0,\varphi,\chi))-\bar{\varphi}(-\rho(0,\bar{\varphi},\bar{\chi}))|\le(1+NL_6)|\varphi-\bar{\varphi}|_C+NL_6|\chi-\bar{\chi}|_X$. Therefore

$$
\begin{aligned}
|x(t)-\bar{x}(t)|\ \le\ &K_3\mu(t-r_0)+(K_3+1)|\varphi-\bar{\varphi}|_{W^{1,\infty}}+2L_3|\lambda-\bar{\lambda}|+2NL_3L_6|\chi-\bar{\chi}|_X\\
&+\ L_1\int_0^t\Big((2+L_2N)\mu(s)+L_2N|\xi-\bar{\xi}|_\Xi+|\theta-\bar{\theta}|_\Theta\Big)\,ds,\quad t\in[0,\alpha^\gamma],
\end{aligned}
$$

where $K_3:=L_3(2+NL_6)$. Lemma 1.2.2 yields

$$
\mu(t)\le K_3\mu(t-r_0)+K_4|\gamma-\bar{\gamma}|_\Gamma+K_5\int_0^t\mu(s)\,ds,\quad t\in[0,\alpha^\gamma],
$$

where $K_4:=K_3+1+2L_3+2NL_3L_6+L_1(L_2N+1)\alpha$ and $K_5:=L_1(2+L_2N)$. Applying Lemma 1.2.3 we get

$$
|x(t)-\bar{x}(t)|\le\mu(t)\le de^{ct},\qquad t\in[-r,\alpha^\gamma],\tag{4.2.18}
$$

where $c>0$ is the solution of $cK_3e^{-cr_0}+K_5=c$, and $d=d(\gamma,\bar{\gamma})$ is defined by

$$
d:=\max\left\{\frac{K_4|\gamma-\bar{\gamma}|_\Gamma}{1-K_3e^{-cr_0}},e^{cr}|\varphi-\bar{\varphi}|_C\right\}.
$$

Therefore there exists $K_6 > 0$ such that $d(\gamma,\bar\gamma) \le K_6|\gamma-\bar\gamma|_\Gamma$, so, combining this with (4.2.18), we get

$$|x(t)-\bar x(t)| \le L|\gamma-\bar\gamma|_\Gamma, \qquad t\in[-r,\alpha^\gamma], \quad \gamma\in\mathcal{B}_\Gamma(\bar\gamma;\,\delta), \tag{4.2.19}$$

where $L = K_6 e^{c\alpha}$. Note that the Lipschitz-constant L is independent of the selection of $\gamma,\bar\gamma\in P$. This concludes the proof of (4.2.8) on $[-r,\alpha^\gamma]$.

Hence if $\gamma=\bar\gamma$, then (4.2.19) yields that $x(t)=\bar x(t)$ for $t\in[0,\alpha^\gamma]$. But then (4.2.15) and the definition of α^γ yield that $\alpha^\gamma=\alpha$. This concludes the proof of the uniqueness of the solution of the IVP (4.2.1)-(4.2.2) on the interval $[-r,\alpha]$ for all $\gamma\in\mathcal{B}_\Gamma(\widehat\gamma;\,\delta)$. This completes the proof of part (iv) of the theorem.

(iv) For $\gamma\in P\cap\mathcal{P}$ the definition of $\mathcal{P}$ gives that the function μ^1 defined in (4.2.12) is continuously differentiable on $[0,r_0]$, since $\widetilde\varphi_t$ is continuously differentiable on $[-r,-r_0]$. Therefore (4.2.14) implies that x is continuously differentiable on $[0,r_0]$, and the compatibiliy condition in the definition of $\mathcal{P}$ yields $\varphi(0-)=x(0+)$, so x is continuously differentiable on $[-r,r_0]$. Hence $g(t,x_t,x(t-\rho(t,x_t,\chi)),\lambda)$ is differentiable wrt t for $t\in[0,r_0]$, and therefore on $[0,r_0]$ the IVP (4.2.1)-(4.2.2) is equivalent to

$$\begin{aligned}\dot x(t) \;&=\; D_1g(t,x_t,x(v(t)),\lambda)+D_2g(t,x_t,x(v(t)),\lambda)\dot x_t+D_3g(t,x_t,x(v(t)),\lambda)\\ &\quad\times\dot x(v(t))\{1-D_1\rho(t,x_t,\chi)-D_2\rho(t,x_t,\chi)\dot x_t\}+f(t,x_t,x(u(t)),\theta),\end{aligned} \tag{4.2.20}$$

where $v(t) := t-\rho(t,x_t,\chi)$ and $u(t) := t-\tau(t,x_t,\xi)$. (A1)–(A4) imply that the right-hand side of (4.2.20) is continuous in t, therefore the definition of $\mathcal{P}$ yields that $\dot x$ is continuous on $[-r,r_0]$. Now the continuity of $\dot x$ follows from (4.2.20) and the definition of $\mathcal{P}$, using the method of steps with the intervals $[ir_0,(i+1)r_0]$, $i=0,1,2,\ldots$.

□

4.3 Differentiability wrt the parameters

In this section we study differentiability of solutions of the IVP (4.2.1)-(4.2.2) wrt the initial function, φ, the parameters ξ, θ, λ and χ of the functions τ, f, g and ρ, respectively.

Let the positive constants α and δ, the parameter set P, and the compact and convex sets M_1, M_2 and M_5 be defined by Theorem 4.2.2. Let

$$M_3 := \overline{\mathcal{B}}_\Theta\Big(\widehat\theta;\,\delta\Big), \quad M_4 := \overline{\mathcal{B}}_\Xi\Big(\widehat\xi;\,\delta\Big), \quad M_6 := \overline{\mathcal{B}}_\Lambda\Big(\widehat\lambda;\,\delta\Big) \quad \text{and} \quad M_7 := \overline{\mathcal{B}}_X(\widehat\chi;\,\delta), \tag{4.3.1}$$

as in the proof of Theorem 4.2.2.

First we define a few notations will be used throughout this section. Introduce

$$\begin{aligned}\omega_f(t,\bar\psi,\bar u,\bar\theta,\psi,u,\theta) \;\; := \;\; & f(t,\psi,u,\theta)-f(t,\bar\psi,\bar u,\bar\theta)-D_2f(t,\bar\psi,\bar u,\bar\theta)(\psi-\bar\psi)\\ & -D_3f(t,\bar\psi,\bar u,\bar\theta)(u-\bar u)-D_4f(t,\bar\psi,\bar u,\bar\theta)(\theta-\bar\theta)\end{aligned}$$

for $t\in[0,T]$, $\bar\psi,\psi\in M_1$, $\bar u,u\in M_2$, and $\bar\theta,\theta\in M_3$. Lemma 1.2.4, assumption (A1) (iii) and the convexity of M_1, M_2 and M_3 yield

$$\begin{aligned}&|\omega_f(t,\bar\psi,\bar u,\bar\theta,\psi,u,\theta)|\\ &\le \sup_{0<\nu<1}\Big(\Big|D_2f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta))-D_2f(t,\bar\psi,\bar u,\bar\theta)\Big|_{\mathcal{L}(C,\mathbb{R}^n)}\\ &\qquad\qquad \times|\psi-\bar\psi|_C\\ &\quad+\Big|D_3f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta))-D_3f(t,\bar\psi,\bar u,\bar\theta)\Big||u-\bar u|\\ &\quad+\Big|D_4f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta))-D_4f(t,\bar\psi,\bar u,\bar\theta)\Big|_{\mathcal{L}(\Theta,\mathbb{R}^n)}|\theta-\bar\theta|_\Theta\Big)\end{aligned}$$

for $t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$, $u,\bar u\in M_2$ and $\theta,\bar\theta\in M_3$. Then

$$|\omega_f(t,\bar\psi,\bar u,\bar\theta,\psi,u,\theta)|\le\Omega_f\Big(|\psi-\bar\psi|_C+|u-\bar u|+|\theta-\bar\theta|_\Theta\Big)\Big(|\psi-\bar\psi|_C+|u-\bar u|+|\theta-\bar\theta|_\Theta\Big)\quad(4.3.2)$$

for $t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$, $u,\bar u\in M_2$ and $\theta,\bar\theta\in M_3$, where

$$\begin{aligned}\Omega_f(\varepsilon)\;\;:=\;\;\sup\Big\{\max\Big(&|D_2f(t,\psi,u,\theta)-D_2f(t,\bar\psi,\bar u,\bar\theta)|_{\mathcal{L}(C,\mathbb{R}^n)},\\ &|D_3f(t,\psi,u,\theta)-D_3f(t,\bar\psi,\bar u,\bar\theta)|,\\ &|D_4f(t,\psi,u,\theta)-D_4f(t,\bar\psi,\bar u,\bar\theta)|_{\mathcal{L}(\Theta,\mathbb{R}^n)}\Big):\\ &\quad|\psi-\bar\psi|_C+|u-\bar u|+|\theta-\bar\theta|_\Theta\le\varepsilon,\\ &\quad t\in[0,\alpha],\;\psi,\bar\psi\in M_1,\;u,\bar u\in M_2,\;\theta,\bar\theta\in M_3\Big\}.\end{aligned}$$

Similarly, we define

$$\omega_\tau(t,\bar\psi,\bar\xi,\psi,\xi):=\tau(t,\psi,\xi)-\tau(t,\bar\psi,\bar\xi)-D_2\tau(t,\bar\psi,\bar\xi)(\psi-\bar\psi)-D_3\tau(t,\bar\psi,\bar\xi)(\xi-\bar\xi)$$

for $t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$ and $\xi,\bar\xi\in M_4$. Then Lemma 1.2.4 and (A2) (iii) give that

$$|\omega_\tau(t,\bar\psi,\bar\xi,\psi,\xi)|\le\Omega_\tau(|\psi-\bar\psi|_C+|\xi-\bar\xi|)(|\psi-\bar\psi|_C+|\xi-\bar\xi|)\qquad(4.3.3)$$

for $t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$ and $\xi,\bar\xi\in M_4$, where

$$\begin{aligned}\Omega_\tau(\varepsilon)\;\;:=\;\;\sup\Big\{\max\Big(&|D_2(t,\psi,\xi)-D_2(t,\bar\psi,\bar\xi)|_{\mathcal{L}(C,\mathbb{R}^n)},|D_3(t,\psi,\xi)-D_3(t,\bar\psi,\bar\xi)|_{\mathcal{L}(\Xi,\mathbb{R}^n)}\Big):\\ &t\in[0,\alpha],\;\psi,\bar\psi\in M_1,\;\xi,\bar\xi\in M_4,\;|\psi-\bar\psi|_C+|\xi-\bar\xi|\le\varepsilon\Big\}.\end{aligned}$$

We introduce the function

$$\begin{aligned}\omega_g(t,\bar\psi,\bar u,\bar\lambda,\psi,u,\lambda) \;:=\;& g(t,\psi,u,\lambda)-g(t,\bar\psi,\bar u,\bar\lambda)-D_2g(t,\bar\psi,\bar u,\bar\lambda)(\psi-\bar\psi)\\ &-D_3g(t,\bar\psi,\bar u,\bar\lambda)(u-\bar u)-D_4g(t,\bar\psi,\bar u,\bar\lambda)(\lambda-\bar\lambda)\end{aligned}$$

for $t\in[0,\alpha]$, $\bar\psi,\psi\in M_1$, $\bar u,u\in M_5$, $\bar\lambda,\lambda\in M_6$, and let $L_4=L_4(\alpha,M_1,M_5,M_6)$ be the Lipschitz constant from (A3) (iv). Then Lemma 1.2.4 yields

$$|\omega_g(t,\bar\psi,\bar u,\bar\lambda,\psi,u,\lambda)|\le L_4\big(\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)|+|u-\bar u|+|\lambda-\bar\lambda|_\Lambda\big)^2, \qquad (4.3.4)$$

for $t\in[0,\alpha]$, $\bar\psi,\psi\in M_1$, $u,\bar u\in M_5$, $\bar\lambda,\lambda\in M_6$.

Let $\bar\gamma=(\bar\varphi,\bar\xi,\bar\theta,\bar\lambda,\bar\chi)\in P\cap\mathcal{P}$, and $x(t):=x(t,\bar\gamma)$ be the corresponding solution of the IVP (4.2.1)-(4.2.2) on $[-r,\alpha]$. Note that Theorem 4.2.2 yields that x is continuously differentiable on $[-r,\alpha]$. Fix $h=(h^\varphi,h^\xi,h^\theta,h^\lambda,h^\chi)\in\Gamma$, and consider the variational equation

$$\begin{aligned}\frac{d}{dt}\Big(&z(t)-D_2g(t,x_t,x(t-\rho(t,x_t,\bar\chi)),\bar\lambda)z_t-D_3g(t,x_t,x(t-\rho(t,x_t,\bar\chi)),\bar\lambda)\\ &\times\Big[-\dot x(t-\rho(t,x_t,\bar\chi))\Big\{D_2\rho(t,x_t,\bar\chi)z_t+D_3\rho(t,x_t,\bar\chi)h^\chi\Big\}+z(t-\rho(t,x_t,\bar\chi))\Big]\\ &-D_4g(t,x_t,x(t-\rho(t,x_t,\bar\chi)),\bar\lambda)h^\lambda\Big)\\ =\;& D_2f(t,x_t,x(t-\tau(t,x_t,\bar\xi)),\bar\theta)z_t+D_3f(t,x_t,x(t-\tau(t,x_t,\bar\xi)),\bar\theta)\\ &\times\Big[-\dot x(t-\tau(t,x_t,\bar\xi))\Big\{D_2\tau(t,x_t,\bar\xi)z_t+D_3\tau(t,x_t,\bar\xi)h^\xi\Big\}+z(t-\tau(t,x_t,\bar\xi))\Big]\\ &+D_4f(t,x_t,x(t-\tau(t,x_t,\bar\xi)),\theta)h^\theta,\qquad t\in[0,\alpha] \qquad (4.3.5)\\ z(t)\;=\;& h^\varphi(t),\qquad t\in[-r,0]. \qquad (4.3.6)\end{aligned}$$

This is an inhomogeneous linear time-dependent but state-independent NFDE for z with continuous coefficients, therefore this IVP has a unique solution, $z(t)=z(t,\bar\gamma,h)$, which depends linearly on h. The boundedness of the map $\Gamma\to\mathbb{R}^n$, $h\mapsto z(t,\bar\gamma,h)$ for each $t\in[0,\alpha]$ follows from Theorem 4.3.1 below.

For a fixed $t\in[0,\alpha]$ we introduce the linear operator $L(t,x)\colon C\times\Xi\times\Theta\to\mathbb{R}^n$ defined by

$$\begin{aligned}&L(t,x)(\psi,h^\xi,h^\theta)\\ &\quad:= D_2f(t,x_t,x(t-\tau(t,x_t,\bar\xi)),\bar\theta)\psi+D_3f(t,x_t,x(t-\tau(t,x_t,\bar\xi)),\bar\theta)\\ &\qquad\times\Big[-\dot x(t-\tau(t,x_t,\bar\xi))\Big\{D_2\tau(t,x_t,\bar\xi)\psi+D_3\tau(t,x_t,\bar\xi)h^\xi\Big\}+\psi(-\tau(t,x_t,\bar\xi))\Big]\\ &\qquad+D_4f(t,x_t,x(t-\tau(t,x_t,\bar\xi)),\bar\theta)h^\theta \qquad (4.3.7)\end{aligned}$$

and the linear operator $G(t,x)\colon C\times\Lambda\times X\to\mathbb{R}^n$ defined by

$$\begin{aligned}
&G(t,x)(\psi,h^\lambda,h^\chi)\\
&\quad:= D_2g(t,x_t,x(t-\rho(t,x_t,\bar\chi)),\bar\lambda)\psi + D_3g(t,x_t,x(t-\rho(t,x_t,\bar\chi)),\bar\lambda)\\
&\qquad\times\Big[-\dot x(t-\rho(t,x_t,\bar\chi))\Big\{D_2\rho(t,x_t,\bar\chi)\psi + D_3\rho(t,x_t,\bar\chi)h^\chi\Big\} + \psi(-\rho(t,x_t,\bar\chi))\Big]\\
&\qquad+D_4g(t,x_t,x(t-\rho(t,x_t,\bar\chi)),\bar\lambda)h^\lambda. \qquad (4.3.8)
\end{aligned}$$

With these notations (4.3.5) can be rewritten as

$$\frac{d}{dt}\Big(z(t)-G(t,x)(z_t,h^\lambda,h^\chi)\Big) = L(t,x)(z_t,h^\xi,h^\theta),\qquad t\in[0,\alpha]. \qquad (4.3.9)$$

Let $L_1=L_1(\alpha,M_1,M_2,M_3)$ and $L_2=L_2(\alpha,M_1,M_4)$ be the Lipschitz constants from (A1) (ii) and (A2) (ii), respectively. Then (A1) (ii), (A2) (ii) and (4.2.7) yield

$$\begin{aligned}
&|L(t,x)(\psi,h^\xi,h^\theta)|\\
&\quad\le L_1|\psi|_C + L_1\Big(NL_2(|\psi|_C+|h^\xi|_\Xi)+|\psi|_C\Big)+L_1|h^\theta|_\Theta\\
&\quad\le N_0\Big(|\psi|_C+|h^\xi|_\Xi+|h^\theta|_\Theta\Big),\qquad t\in[0,\alpha],\ \psi\in C,\ h^\xi\in\Xi,\ h^\theta\in\Theta, \quad (4.3.10)
\end{aligned}$$

where $N_0:=L_1(2NL_2+2)$.

Let $L_3=L_3(\alpha,M_1,M_5,M_6)$, $L_6=L_6(\alpha,M_1,M_7)$ be defined by (A3) (ii) and (A4) (ii), respectively. Then we have by (A3) (ii) and (A4) (ii) that

$$|G(t,x)(\psi,h^\lambda,h^\chi)|\le N_1\Big(\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)|+|h^\lambda|_\Lambda+|h^\chi|_X\Big),\qquad t\in[0,\alpha], \qquad (4.3.11)$$

for $\psi\in C,\ h^\lambda\in\Lambda,\ h^\chi\in X$, where $N_1:=L_3(2NL_6+2)$.

Theorem 4.3.1 *Assume (A1) (i)–(iii), (A2) (i)–(iii), (A3) (i)–(iv) and (A4) (i)–(iv), let $\alpha>0$ and $P\subset\Pi$ be defined by Theorem 4.2.2. There exists $N_2\ge 0$ such that the solution of the IVP* (4.3.5)-(4.3.6) *satisfies*

$$|z(t,\gamma,h)|\le N_2|h|_\Gamma,\qquad t\in[-r,\alpha],\quad h\in\Gamma,\quad \gamma\in P\cap\mathcal{P}. \qquad (4.3.12)$$

Moreover, for $\bar\gamma\in P\cap\mathcal{P}$ there exists a monotone increasing function $A=A(\bar\gamma)$ such that $A\colon[0,\infty)\to[0,\infty)$, $A(u)\to 0$ as $u\to 0$, and

$$|z(t,\bar\gamma,h)-z(\bar t,\bar\gamma,h)|\le A(|t-\bar t|)|h|_\Gamma,\qquad t,\bar t\in[-r,\alpha],\quad h\in\Gamma. \qquad (4.3.13)$$

Proof (i) Let $\gamma\in P\cap\mathcal{P}$. For simplicity we use the notations $h=(h^\varphi,h^\xi,h^\theta,h^\lambda,h^\chi)\in\Gamma$, $x(t):=x(t,\gamma)$ and $z(t):=z(t,\gamma,h)$. Let δ,M_1,M_2 and M_5 be defined by Theorem 4.2.2, M_3, M_4, M_6 and M_7 be defined by (4.3.1), $L_1,\dots,L_8$ be the corresponding Lipschitz constants form (A1)–(A4), and let N_0 and N_1 be corresponding constants defined by (4.3.10) and (4.3.11), respectively. Integrating (4.3.9) from 0 to t we get

$$|z(t)| \leq |G(t,x)(z_t, h^\lambda, h^\chi)| + |h^\varphi(0)| + |G(0,x)(h^\varphi, h^\lambda, h^\chi)| + \int_0^t |L(s,x)(z_s, h^\xi, h^\theta)|\, ds$$

for $t \in [0,\alpha]$, and therefore (4.3.10) and (4.3.11) yield

$$\begin{aligned} |z(t)| \;\leq\; & N_1 \max_{\zeta\in[-r,-r_0]} |z(t+\zeta)| + (1+N_1)|h^\varphi|_C + 2N_1(|h^\lambda|_\Lambda + |h^\chi|_X) \\ & + N_0 \int_0^t (|z_s|_C + |h^\xi|_\Xi + |h^\theta|_\Theta)\, ds, \qquad t \in [0,\alpha]. \end{aligned}$$

An application of Lemma 1.2.2 implies

$$\mu(t) \leq N_1 \mu(t-r_0) + K_7 |h|_\Gamma + N_0 \int_0^t \mu(s)\, ds, \qquad t \in [0,\alpha],$$

where $\mu(t) := \max\{|z(s)| : s \in [-r,t]\}$ and $K_7 := \max\{N_0\alpha, 1+N_1, 2N_1\}$. Then Lemma 1.2.3 yields

$$|z(t)| \leq \mu(t) \leq N_2 |h|_\Gamma, \qquad t \in [0,\alpha],$$

where

$$N_2 := \max\left\{ \frac{K_7}{1 - N_1 e^{-cr_0}}, e^{cr} \right\} e^{c\alpha}$$

and c is the positive solution of $cN_1 e^{-cr_0} + N_0 = c$. Moreover, $\mu(0) \leq N_2|h|_\Gamma$ yields that (4.3.12) holds for $t \in [-r,0]$, as well. This concludes the proof of (4.3.12).

(ii) Let $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P \cap \mathcal{P}$, $x(t) := x(t,\bar\gamma)$, $h = (h^\varphi, h^\xi, h^\theta, h^\lambda, h^\chi) \in \Gamma$, $z(t) := z(t,\bar\gamma,h)$, $v(t) := t - \rho(t, x_t, \bar\chi)$. Let $t, \bar t \in [0,\alpha]$, and consider

$$\begin{aligned} & G(t,x)(z_t, h^\lambda, h^\chi) - G(\bar t, x)(z_{\bar t}, h^\lambda, h^\chi) \\ &= D_2 g(t, x_t, x(v(t)), \bar\lambda) z_t - D_2 g(\bar t, x_{\bar t}, x(v(\bar t)), \bar\lambda) z_t + D_2 g(\bar t, x_{\bar t}, x(v(\bar t)), \bar\lambda)(z_t - z_{\bar t}) \\ &\quad + \Big[D_3 g(t, x_t, x(v(t)), \bar\lambda) - D_3 g(\bar t, x_{\bar t}, x(v(\bar t)), \bar\lambda) \Big] \\ &\qquad \times \Big[-\dot x(v(t)) \Big\{ D_2\rho(t, x_t, \bar\chi) z_t + D_3\rho(t, x_t, \bar\chi) h^\chi \Big\} + z(v(t)) \Big] \\ &\quad + D_3 g(\bar t, x_{\bar t}, x(v(\bar t)), \bar\lambda) \Big[-\Big(\dot x(v(t)) - \dot x(v(\bar t)) \Big) \Big\{ D_2\rho(t, x_t, \bar\chi) z_t + D_3\rho(t, x_t, \bar\chi) h^\chi \Big\} \Big] \\ &\quad - D_3 g(\bar t, x_{\bar t}, x(v(\bar t)), \bar\lambda) \dot x(v(\bar t)) \Big[D_2\rho(t, x_t, \bar\chi) z_t - D_2\rho(\bar t, x_{\bar t}, \bar\chi) z_t + D_2\rho(\bar t, x_{\bar t}, \bar\chi)(z_t - z_{\bar t}) \\ &\qquad + D_3\rho(t, x_t, \bar\chi) h^\chi - D_3\rho(\bar t, x_{\bar t}, \bar\chi) h^\chi \Big] + D_3 g(\bar t, x_{\bar t}, x(v(\bar t)), \bar\lambda) \Big[z(v(t)) - z(v(\bar t)) \Big] \\ &\quad + \Big[D_4 g(t, x_t, x(v(t)), \bar\lambda) - D_4 g(\bar t, x_{\bar t}, x(v(\bar t)), \bar\lambda) \Big] h^\lambda. \end{aligned} \tag{4.3.14}$$

Let N be defined by (4.2.7), and the Lipschitz constants $L_6 = L_6(\alpha, M_1, M_7)$, $L_7 = L_7(\alpha, M_1, M_7)$ and $L_8 = L_8(\alpha, M_1, M_7)$ be defined by (A4) (ii) and (iv), respectively. Then (A4) (ii) and (4.2.7) yield

$$\begin{aligned} |v(t) - v(\bar t)| &= |\rho(t, x_t, \bar\chi) - \rho(\bar t, x_{\bar t}, \bar\chi)| \\ &\le L_6(|t-\bar t| + |x_t - x_{\bar t}|_C) \\ &\le L_6(1+N)|t-\bar t|, \qquad t, \bar t \in [0,\alpha], \end{aligned} \tag{4.3.15}$$

and hence

$$|x(v(t)) - x(v(\bar t))| \le N L_6(1+N)|t-\bar t|, \qquad t,\bar t \in [0,\alpha]. \tag{4.3.16}$$

Define the function

$$\Omega_{\dot x}(\varepsilon) := \sup\Big\{ |\dot x(u) - \dot x(\bar u)| \colon\ |u-\bar u| \le \varepsilon, \quad u, \bar u \in [-r,\alpha]\Big\}. \tag{4.3.17}$$

Since $\bar\gamma \in \mathcal{P}$, x is continuously differentiable on $[-r,\alpha]$, hence $\Omega_{\dot x}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore (A3) (ii), (iv), (A4) (ii) and (4.2.7) imply for $t, \bar t \in [0,\alpha]$

$$\begin{aligned} &|G(t,x)(z_t, h^\lambda, h^\chi) - G(\bar t, x)(z_{\bar t}, h^\lambda, h^\chi)| \\ &\quad\le L_4\Big(|t-\bar t| + |x_t - x_{\bar t}|_C + |x(v(t)) - x(v(\bar t))|\Big)|z_t| \\ &\qquad + L_4 \max\{|z(t+\zeta) - z(t+\bar\zeta)| \colon\ \zeta,\bar\zeta \in [-r,-r_0],\ |\zeta-\bar\zeta| \le L_5|t-\bar t|\} \\ &\qquad + L_3 \max_{\zeta\in[-r,-r_0]} |z(t+\zeta) - z(\bar t+\zeta)| \\ &\qquad + L_4\Big(|t-\bar t| + |x_t - x_{\bar t}|_C + |x(v(t)) - x(v(\bar t))|\Big)\Big(N L_6(|z_t|_C + |h^\chi|_X) + |z(v(t))|\Big) \\ &\qquad + L_3 \Omega_{\dot x}(|v(t) - v(\bar t)|) L_6(|z_t|_C + |h^\chi|_X) + L_3 N \Big(L_7(|t-\bar t| + |x_t - x_{\bar t}|_C)|z_t|_C \\ &\qquad + L_7 \max\{|z(t+\zeta) - z(t+\bar\zeta)| \colon\ \zeta,\bar\zeta \in [-r,-r_0],\ |\zeta-\bar\zeta| \le L_8|t-\bar t|\} \\ &\qquad + L_6 \max_{\zeta\in[-r,-r_0]} |z(t+\zeta) - z(\bar t+\zeta)| + L_7(|t-\bar t| + |x_t - x_{\bar t}|_C)|h^\chi|_X\Big) \\ &\qquad + L_3|z(v(t)) - z(v(\bar t))| + L_4\Big(|t-\bar t| + |x_t - x_{\bar t}|_C + |x(v(t)) - x(v(\bar t))|\Big)|h^\lambda|_\Lambda. \end{aligned} \tag{4.3.18}$$

Let

$$w(t,\varepsilon) := \max\{|z(s) - z(\bar s)| \colon\ s, \bar s \in [-r,t],\ |s-\bar s| \le \varepsilon\}, \quad t \in [0,\alpha],\ \varepsilon \in [0,\infty).$$

Note that $w(t_1,\varepsilon_1) \le w(t_2,\varepsilon_2)$ for $0 \le t_1 \le t_2 \le \alpha$ and $0 \le \varepsilon_1 \le \varepsilon_2$. Then using (4.2.7), (4.3.12), (4.3.15), (4.3.16) and the definition of w we get for $0 \le \bar t \le t \le \alpha$

$$\begin{aligned} &|G(t,x)(z_t, h^\lambda, h^\chi) - G(\bar t, x)(z_{\bar t}, h^\lambda, h^\chi)| \\ &\quad\le L_4(1 + N + N L_6(1+N))N_2|t-\bar t||h|_\Gamma + L_4 w(t-r_0, L_5|t-\bar t|) + L_3 w(t-r_0, |t-\bar t|) \\ &\qquad + L_4(1+N+NL_6(1+N))(NL_6(N_2+1) + N_2)|t-\bar t||h|_\Gamma \\ &\qquad + L_3\Omega_{\dot x}\Big(L_6(1+N)|t-\bar t|\Big) L_6(N_2+1)|h|_\Gamma + L_3 N\Big(L_7(1+N)N_2|t-\bar t||h|_\Gamma \end{aligned}$$

$$
\begin{aligned}
&+L_7 w(t-r_0, L_8|t-\bar t|) + L_6 w(t-r_0, |t-\bar t|) + L_7(1+N)|t-\bar t||h|_\Gamma\Big) \\
&+L_3 w(t-r_0, L_6(1+N)|t-\bar t|) + L_4(1+N+NL_6(1+N))|t-\bar t||h|_\Gamma \\
\leq\ & a^0(|t-\bar t|)|h|_\Gamma + K_{11} w(t-r_0, K_{12}|t-\bar t|), \qquad (4.3.19)
\end{aligned}
$$

where $a^0(u) := K_8 u + K_9 \Omega_{\dot x}(K_{10} u)$ with appropriate nonnegative constants K_8, K_9, K_{10}, K_{11}, and $K_{12} := \max\{1, L_5, L_8, L_6(1+N)\}$.

Integrating (4.3.9) from $\bar t$ to t we get

$$
z(t) - z(\bar t) = G(t,x)(z_t, h^\lambda, h^\chi) - G(\bar t, x)(z_{\bar t}, h^\lambda, h^\chi) + \int_{\bar t}^t L(s,x)(z_s, h^\xi, h^\theta)\, ds.
$$

Hence (4.3.10), (4.3.12) and (4.3.19) yield for $0 \leq \bar t \leq t \leq \alpha$

$$
|z(t) - z(\bar t)| \ \leq\ a^1(|t-\bar t|)|h|_\Gamma + K_{11} w(t-r_0, K_{12}|t-\bar t|) \qquad (4.3.20)
$$

with $a^1(u) := a^0(u) + N_0(N_2+1)u$.

Let $m := [\alpha/r_0]$ (here $[\cdot]$ denotes the greatest integer part), and $t_j := jr_0$, $j = 0, 1, \ldots, m$, $t_{m+1} := \alpha$. First suppose $t, \bar t \in [t_0, t_1]$. Then $|\dot h^\varphi|_{L^\infty} \leq |h^\varphi|_{W^{1,\infty}} \leq |h|_\Gamma$ and Lemma 1.2.5 yield

$$
|z(t) - z(\bar t)| = |h^\varphi(t) - h^\varphi(\bar t)| \leq |t-\bar t||h|_\Gamma, \qquad t, \bar t \in [-r, 0].
$$

Therefore (4.3.20) and the definition of w imply for $t, \bar t \in [t_0, t_1]$

$$
|z(t) - z(\bar t)| \leq a^1(|t-\bar t|)|h|_\Gamma + K_{11} w(t_0, K_{12}|t-\bar t|) \leq a^1(|t-\bar t|)|h|_\Gamma + K_{11}K_{12}|t-\bar t||h|_\Gamma.
$$

For $-r \leq \bar t \leq t_0 \leq t \leq t_1$ the above inequalities yield

$$
\begin{aligned}
|z(t) - z(\bar t)| \ &\leq\ |z(t) - z(t_0)| + |z(t_0) - z(\bar t)| \\
&\leq\ a^1(t)|h|_\Gamma + K_{11}K_{12} t |h|_\Gamma + |\bar t||h|_\Gamma \\
&\leq\ a^1(|t-\bar t|)|h|_\Gamma + (1 + K_{11}K_{12})|t-\bar t||h|_\Gamma. \qquad (4.3.21)
\end{aligned}
$$

But now it is easy to see that (4.3.21) holds for all $-r \leq \bar t \leq t \leq t_1$, and therefore,

$$
w(t_1, \varepsilon) \leq a^1(\varepsilon)|h|_\Gamma + (1 + K_{11}K_{12})\varepsilon |h|_\Gamma, \qquad \varepsilon > 0. \qquad (4.3.22)
$$

If $t, \bar t \in [t_1, t_2]$, then (4.3.20) and (4.3.22) yield

$$
\begin{aligned}
|z(t) - z(\bar t)| \ &\leq\ a^1(|t-\bar t|)|h|_\Gamma + K_{11} w(t_1, K_{12}|t-\bar t|) \\
&\leq\ a^1(|t-\bar t|)|h|_\Gamma + K_{11} a^1(K_{12}|t-\bar t|)|h|_\Gamma + (K_{11}K_{12} + K_{11}^2 K_{12}^2)|t-\bar t||h|_\Gamma \\
&\leq\ (1 + K_{11}) a^2(|t-\bar t|)|h|_\Gamma + (K_{11}K_{12} + K_{11}^2 K_{12}^2)|t-\bar t||h|_\Gamma\Big),
\end{aligned}
$$

where $a^2(u) := a^1(K_{12}u)$. But then for $-r \le \bar t \le t_1 \le t \le t_2$ we have

$$\begin{aligned} |z(t) - z(\bar t)| &\le |z(t) - z(t_1)| + |z(t_1) - z(\bar t)| \\ &\le (2 + K_{11})a^2(|t - \bar t|)|h|_\Gamma + (1 + 2K_{11}K_{12} + K_{11}^2K_{12}^2)|t - \bar t||h|_\Gamma. \end{aligned} \tag{4.3.23}$$

Again, it follows that (4.3.23) holds for all $t, \bar t \in [-r, t_2]$.

Repeating the previous steps for the intervals $[-r, t_j]$ for $j = 2, \dots, m+1$, we get that

$$|z(t) - z(\bar t)| \le A(|t - \bar t|)|h|_\Gamma$$

for $t, \bar t \in [-r, \alpha]$ with an appropriate function A satisfying $A(s) \to 0$ as $s \to 0+$, which proves (4.3.13). □

We need the following estimates in the proof of the next theorem.

Lemma 4.3.2 *Assume (A3) (i)–(iv), (A4) (i)–(iv). Suppose* $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P \cap \mathcal{P}$, $h_k = (h_k^\varphi, h_k^\xi, h_k^\theta, h_k^\lambda, h_k^\chi) \in \Gamma$ *is such that* $\bar\gamma + h_k \in P$ *for* $k \in \mathbb{N}$, *and* $|h_k|_\Gamma \to 0$ *as* $k \to \infty$. *Let* $x(t) := x(t, \bar\gamma)$, $x^k(t) := x(t, \bar\gamma + h_k)$, $z^k(t) := z(t, \bar\gamma, h_k)$, $v^k(t) := t - \rho(t, x_t^k, \bar\chi + h_k^\chi)$ *and* $v(t) := t - \rho(t, x_t, \bar\chi)$. *Then there exist a nonnegative constant* N_4 *and a nonnegative sequence* $A_k = A_k(\bar\gamma, h_k)$ *such that* $A_k \to 0$ *as* $k \to \infty$, *and for* $k \in \mathbb{N}$

$$\begin{aligned} &|g(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda) - g(t, x_t, x(v(t)), \bar\lambda) - G(t, x)(z_t^k, h_k^\lambda, h_k^\chi)| \\ &\quad\le A_k|h_k|_\Gamma + N_4 \max_{\zeta \in [-r, -r_0]} |x^k(t+\zeta) - x(t+\zeta) - z^k(t+\zeta)|, \quad t \in [0, \alpha]. \end{aligned} \tag{4.3.24}$$

Proof Let α, M_1 and M_5 be defined by Theorem 4.2.2, M_6 and M_7 be defined by (4.3.1), and $L_3, \dots, L_7$ be the corresponding Lipschitz constants from (A3)–(A4). Simple manipulations yield

$$\begin{aligned} &g(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda) - g(t, x_t, x(v(t)), \bar\lambda) - G(t, x)(z_t^k, h_k^\lambda, h_k^\chi) \\ &= g(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda) - g(t, x_t, x(v(t)), \bar\lambda) \\ &\quad - D_2g(t, x_t, x(v(t)), \bar\lambda)(x_t^k - x_t) + D_2g(t, x_t, x(v(t)), \bar\lambda)(x_t^k - x_t - z_t^k) \\ &\quad - D_3g(t, x_t, x(v(t)), \bar\lambda)\Big[x^k(v^k(t)) - x(v(t))\Big] - D_4g(t, x_t, x(v(t)), \bar\lambda)h_k^\lambda \\ &\quad + D_3g(t, x_t, x(v(t)), \bar\lambda)\Big[x^k(v^k(t)) - x(v^k(t)) - z^k(v^k(t))\Big] \\ &\quad + D_3g(t, x_t, x(v(t)), \bar\lambda)\Big[x(v^k(t)) - x(v(t)) - \dot x(v(t))(v^k(t) - v(t))\Big] \\ &\quad + D_3g(t, x_t, x(v(t)), \bar\lambda)\dot x(v(t))\Big[v^k(t) - v(t) + D_2\rho(t, x_t, \bar\chi)(x_t^k - x_t) + D_3\rho(t, x_t, \bar\chi)h_k^\chi\Big] \\ &\quad - D_3g(t, x_t, x(v(t)), \bar\lambda)\dot x(v(t))D_2\rho(t, x_t, \bar\chi)\Big[x_t^k - x_t - z_t^k\Big] \\ &\quad + D_3g(t, x_t, x(v(t), \bar\lambda))\Big[z^k(v^k(t)) - z^k(v(t))\Big], \qquad t \in [0, \alpha], \quad k \in \mathbb{N}. \end{aligned} \tag{4.3.25}$$

Using the definition of ω_g, and applying (A3) (iv), (A4) (ii), (4.2.7), (4.2.8) and (4.3.4) we have

$$
\begin{aligned}
&|\omega_g(t, x_t, x(v(t)), \bar\lambda, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda)| \\
&\quad\le L_4\Big(|x_t^k - x_t|_C + |x^k(v^k(t)) - x(v(t))| + |h_k^\lambda|_\Lambda\Big)^2 \\
&\quad\le L_4\Big(|x_t^k - x_t|_C + |x^k(v^k(t)) - x(v^k(t))| + |x(v^k(t)) - x(v(t))| + |h_k^\lambda|_\Lambda\Big)^2 \\
&\quad\le L_4\Big(2|x_t^k - x_t|_C + |\dot x_t|_{L^\infty}|v^k(t) - v(t)| + |h_k^\lambda|_\Lambda\Big)^2 \\
&\quad\le L_4\Big((2+NL_6)|x_t^k - x_t|_C + NL_6|h_k^\chi|_X + |h_k^\lambda|_\Lambda\Big)^2 \\
&\quad\le L_4\Big((2+NL_6)L + NL_6 + 1\Big)^2 |h_k|_\Gamma^2, \qquad t \in [0,\alpha], \quad k \in \mathbb{N}.
\end{aligned}
$$

Lemma 1.2.4, (A4) (iv) and (4.2.8) imply

$$
\begin{aligned}
&|v^k(t) - v(t) + D_2\rho(t, x_t, \bar\chi)(x_t^k - x_t) + D_3\rho(t, x_t, \bar\chi)h_k^\chi| \\
&\quad\le |x_t^k - x_t|_C \max_{0<\nu<1} |D_2\rho(t, x_t + \nu(x_t^k - x_t), \bar\chi) - D_2\rho(t, x_t, \bar\chi)|_{\mathcal{L}(C,\mathbb{R}^n)} \\
&\qquad + |h_k^\chi|_X \max_{0<\nu<1} |D_3\rho(t, x_t, \bar\chi + \nu h_k^\chi) - D_3\rho(t, x_t, \bar\chi)|_{\mathcal{L}(X,\mathbb{R}^n)} \\
&\quad\le L_7|x_t^k - x_t|_C^2 + L_7|h_k^\chi|_X^2 \\
&\quad\le L_7(L^2+1)|h_k|_\Gamma^2, \qquad t \in [0,\alpha], \quad k \in \mathbb{N}.
\end{aligned}
$$

Relations (4.2.8), (4.3.13) and (A4) (ii) yield

$$
\begin{aligned}
|z^k(v^k(t)) - z^k(v(t))| &\le A\Big(|v^k(t) - v(t)|\Big)|h_k|_\Gamma \\
&\le A\Big(L_6(|x_t^k - x_t|_C + |h_k^\chi|_X)\Big)|h_k|_\Gamma \\
&\le A\Big(L_6(L+1)|h_k|_\Gamma\Big)|h_k|_\Gamma, \qquad t \in [0,\alpha], \quad k \in \mathbb{N}.
\end{aligned}
$$

Relations (A4) (ii), (4.2.8), (4.3.13), (4.3.17) and Lemma 1.2.4 imply

$$
\begin{aligned}
&|x(v^k(t)) - x(v(t)) - \dot x(v(t))(v^k(t) - v(t))| \\
&\quad\le |v^k(t) - v(t)| \sup_{0<\nu<1} \{|\dot x(v(t) + \nu(v^k(t) - v(t))) - \dot x(v(t))|\} \\
&\quad\le L_6(L+1)|h_k|\Omega_{\dot x}\Big(L_6(L+1)|h_k|_\Gamma\Big), \qquad t \in [0,\alpha], \quad k \in \mathbb{N}.
\end{aligned}
$$

Combining the above estimates, $t - r \le v^k(t) \le t - r_0$ together with (4.3.25), we get (4.3.24) with $A_k := L_3L_6(L+1)\Omega_{\dot x}\Big(L_6(L+1)|h_k|\Big) + L_3A\Big(L_6(L+1)|h_k|_\Gamma\Big) + K_{13}|h_k|_\Gamma$ and with appropriate constants N_4 and K_{13}. $\square$

Lemma 4.3.3 *Suppose (A1) (i)–(iii), (A2) (i)–(iii), and let* $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P \cap \mathcal{P}$, $h_k = (h_k^\varphi, h_k^\xi, h_k^\theta, h_k^\lambda, h_k^\chi) \in \Gamma$ *be such that* $\bar\gamma + h_k \in P$ *for* $k \in \mathbb{N}$ *and* $|h_k|_\Gamma \to 0$ *as* $k \to \infty$. *Let* $x(t) := x(t, \bar\gamma)$, $x^k(t) := x(t, \bar\gamma + h_k)$, $z^k(t) := z(t, \bar\gamma, h_k)$, $u(t) := t - \tau(t, x_t, \bar\xi)$, *and* $u^k(t) := t - \tau(t, x_t^k, \bar\xi + h_k^\xi)$. *Then there exist a nonnegative constant* N_5 *and a nonnegative sequence* $B_k = B_k(\bar\gamma, h_k)$ *such that* $B_k \to 0$ *as* $k \to \infty$, *and*

$$\begin{aligned} &|f(s, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta) - f(s, x_s, x(u(s)), \bar\theta) - L(s, x)(z_s^k, h_k^\xi, h_k^\theta)| \\ &\qquad \leq \; B_k|h_k|_\Gamma + N_5|x_s^k - x_s - z_s^k|_C, \qquad t \in [0, \alpha], \quad k \in \mathbb{N}. \end{aligned} \tag{4.3.26}$$

Proof Let α, M_1 and M_2 be defined by Theorem 4.2.2, M_3 and M_4 be defined by (4.3.1), and L_1 and L_2 be the corresponding Lipschitz constants from (A1) (ii) and (A4) (ii), respectively. The definitions of ω_f and ω_τ yield

$$\begin{aligned} &f(s, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta) - f(s, x_s, x(u(s)), \bar\theta) - L(s, x)(z_s^k, h_k^\xi, h_k^\theta) \\ &= \; \omega_f(s, x_s, x(u(s)), \bar\theta, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta) + D_2 f(s, x_s, x(u(s)), \bar\theta)\Big[x_s^k - x_s - z_s^k\Big] \\ &\quad + D_3 f(s, x_s, x(u(s)), \bar\theta)\Big\{x^k(u^k(s)) - x(u^k(s)) - z^k(u^k(s)) \\ &\qquad + x(u^k(s)) - x(u(s)) - \dot x(u(s))(u^k(s) - u(s)) - \dot x(u(s))\omega_\tau(s, x_s, \bar\xi, x_s^k, \bar\xi + h_k^\xi) \\ &\qquad + \dot x(u(s)) D_2\tau(s, x_s, \bar\xi)\Big[x_s^k - x_s - z_s^k\Big] + z^k(u^k(s)) - z^k(u(s))\Big\}. \end{aligned}$$

Using (4.2.17) we have that

$$\begin{aligned} &|x_s^k - x_s|_C + |x^k(u^k(s)) - x(u(s))| + |h_k^\theta|_\Theta \\ &\quad \leq \; 2|x_s^k - x_s|_C + L_2 N(|x_s^k - x_s|_C + |h_k^\xi|_\Xi) + |h_k^\theta|_\Theta \\ &\quad \leq \; K_{14}|h_k|_\Gamma, \qquad s \in [0, \alpha], \quad k \in \mathbb{N}, \end{aligned}$$

where $K_{14} := 2L + L_2 N(L+1) + 1$. Hence (4.3.2) implies

$$|\omega_f(s, x_s, x(u(s)), \bar\theta, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta)| \leq \Omega_f(K_{14}|h_k|_\Gamma) K_{14}|h_k|_\Gamma, \quad s \in [0, \alpha], \; k \in \mathbb{N}.$$

Similarly,

$$|\omega_\tau(s, x_s, \bar\xi, x_s^k, \bar\xi + h_k^\xi)| \leq \Omega_\tau((L+1)|h_k|_\Gamma)(L+1)|h_k|_\Gamma, \qquad s \in [0, \alpha], \; k \in \mathbb{N}.$$

Using (A2) (ii), (4.2.8) we get

$$|u^k(s) - u(s)| = |\tau(s, x_s^k, \bar\xi + h_k^\xi) - \tau(s, x_s, \bar\xi)| \leq L_2\Big(|x_s^k - x_s|_C + |h_k^\xi|_\Xi\Big) \leq L_2(L+1)|h_k|_\Gamma,$$

and therefore the definition of $\Omega_{\dot x}$ and (4.3.13) yield

$$|x(u^k(s)) - x(u(s)) - \dot x(u(s))(u^k(s) - u(s))| \leq \Omega_{\dot x}\Big(L_2(L+1)|h_k|_\Gamma\Big) L_2(L+1)|h_k|_\Gamma$$

and

$$|z^k(u^k(s)) - z^k(u(s))| \leq A\Big(|u^k(s) - u(s)|\Big)|h_k|_\Gamma \leq A\Big(L_2(L+1)|h_k|_\Gamma\Big)|h_k|_\Gamma$$

for $s \in [0,\alpha]$ and $k \in \mathbb{N}$. Therefore, combining the above estimates we get

$$\begin{aligned}
&|f(s, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta) - f(s, x_s, x(u(s)), \bar\theta) - L(s,x)(z_s^k, h_k^\xi, h_k^\theta)| \\
&\leq \Omega_f\Big(K_{14}|h_k|_\Gamma\Big)K_{14}|h_k|_\Gamma + L_1|x_s^k - x_s - z_s^k|_C + L_1\Big\{|x^k(u^k(s)) - x(u^k(s)) - z^k(u^k(s))| \\
&\quad +\Omega_{\dot x}\Big(L_2(L+1)|h_k|_\Gamma\Big)L_2(L+1)|h_k|_\Gamma + N\Omega_\tau\Big((L+1)|h_k|_\Gamma\Big)(L+1)|h_k|_\Gamma \\
&\quad +NL_2|x_s^k - x_s - z_s^k|_C + A\Big(L_2(L+1)|h_k|_\Gamma\Big)|h_k|_\Gamma\Big\}.
\end{aligned}$$

Hence (4.3.26) holds with the sequence

$$\begin{aligned}
B_k \;:=\; & \Omega_f(K_{14}|h_k|_\Gamma)K_{14} + L_1\Omega_{\dot x}(L_2(L+1)|h_k|_\Gamma)L_2(L+1) \\
& +L_1N\Omega_\tau((L+1)|h_k|_\Gamma)(L+1) + L_1A(L_2(L+1)|h_k|_\Gamma)
\end{aligned}$$

and with the constant $N_5 := L_1(2 + NL_2)$. □

Next we study differentiability of the function $x(t,\gamma)$ wrt γ. We denote this differentiation by D_2x.

Theorem 4.3.4 *Assume (A1) (i)–(iii), (A2) (i)–(iii), (A3) (i)–(iv) and (A4) (i)–(iv), and let P and $\alpha > 0$ be defined by Theorem 4.2.2, $\bar\gamma \in P \cap \mathcal{P}$, and $x(t;\gamma)$ be the solution of the IVP* (4.2.1)-(4.2.2) *on $[-r,\alpha]$ for $\gamma \in \mathcal{B}_\Gamma(\bar\gamma;\delta)$. Then the function $x(t,\cdot)\colon \Gamma \supset P \to \mathbb{R}^n$ is differentiable at $\bar\gamma$ for $t \in [0,\alpha]$, and*

$$D_2x(t,\bar\gamma)h = z(t,\bar\gamma,h), \qquad h \in \Gamma, \quad t \in [0,\alpha],$$

where z is the solution of the IVP (4.3.5)-(4.3.6).

Proof Let $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P$ be fixed, and α, δ, M_1, M_2 and M_5 be defined by Theorem 4.2.2, M_3, M_4, M_6 and M_7 be defined by (4.3.1). Let $h_k = (h_k^\varphi, h_k^\xi, h_k^\theta, h_k^\lambda, h_k^\chi) \in \Gamma$ be a sequence such that $|h_k|_\Gamma \to 0$ as $k \to \infty$. We may assume that $|h_k|_\Gamma \leq \delta$, hence $\bar\gamma + h_k \in P$ for $k \in \mathbb{N}$. For brevity, we use the notations $x(t) := x(t,\bar\gamma)$, $x^k(t) := x(t,\bar\gamma + h_k)$, $z^k(t) := z(t,\bar\gamma,h_k)$, $u(t) := t - \tau(t,x_t,\bar\xi)$, $u^k(t) := t - \tau(t, x_t^k, \bar\xi + h_k^\xi)$, $v(t) := t - \rho(t,x_t,\bar\chi)$ and $v^k(t) := t - \rho(t, x_t^k, \bar\chi + h_k^\chi)$.

Integrating (4.2.1) and (4.3.5) we get for $t \in [0,\alpha]$

$$\begin{aligned}
x^k(t) &= g\Big(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda\Big) + \bar\varphi(0) + h_k^\varphi(0) \\
&\quad -g\Big(0, \bar\varphi + h_k^\varphi, \bar\varphi(v^k(0)) + h_k^\varphi(v^k(0)), \bar\lambda + h_k^\lambda\Big) + \int_0^t f(s, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta)\,ds, \\
x(t) &= g(t, x_t, x(v(t)), \bar\lambda) + \bar\varphi(0) - g(0, \bar\varphi, \bar\varphi(v(0)), \bar\lambda) + \int_0^t f(s, x_s, x(u(s)), \bar\theta)\,ds, \\
z^k(t) &= G(t,x)(z_t^k, h_k^\lambda, h_k^\chi) + h_k^\varphi(0) - G(0,x)(h_k^\varphi, h_k^\lambda, h_k^\chi) + \int_0^t L(s,x)(z_s^k, h_k^\xi, h_k^\theta)\,ds.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
x^k(t) - x(t) - z^k(t) & \\
= \;\; & g(t, x_t^k, x^k(v^k(t)), \bar{\lambda} + h_k^\lambda) - g(t, x_t, x(v(t)), \bar{\lambda}) - G(t,x)(z_t^k, h_k^\lambda, h_k^\chi) \\
& - \Big[g\Big(0, \bar{\varphi} + h_k^\varphi, \bar{\varphi}(v^k(0)) + h_k^\varphi(v^k(0)), \bar{\lambda} + h_k^\lambda\Big) - g(0, \bar{\varphi}, \bar{\varphi}(-v(0)), \bar{\lambda}) \\
& \qquad -G(0,x)(h_k^\varphi, h_k^\lambda, h_k^\chi)\Big] \\
& + \int_0^t \Big[f(s, x_s^k, x^k(u^k(s)), \bar{\theta} + h_k^\theta) - f(s, x_s, x(u(s)), \bar{\theta}) - L(s,x)(z_s^k, h_k^\xi, h_k^\theta) \Big] ds.
\end{aligned}
$$

Define the function $w^k(t) := x^k(t) - x(t) - z^k(t)$. Then Lemmas 4.3.2 and 4.3.3 yield for $t \in [0, \alpha]$

$$
|w^k(t)| \;\; \leq \;\; C_k |h_k|_\Gamma + N_4 \max_{\zeta \in [-r, -r_0]} |w^k(t+\zeta)| + N_5 \int_0^t |w_s^k|_C \, ds, \tag{4.3.27}
$$

where $C_k := 2A_k + B_k \alpha \to 0$ as $k \to \infty$. Let $\mu^k(t) := \max\{|w^k(s)| \colon -r \leq s \leq t\}$. We have $w^k(t) = 0$ for $t \in [-r, 0]$. Therefore Lemma 1.2.2 implies from (4.3.27) that

$$
\mu^k(t) \;\; \leq \;\; C_k |h_k|_\Gamma + N_4 \mu^k(t - r_0) + N_5 \int_0^t \mu^k(s) \, ds, \quad t \in [0, \alpha]. \tag{4.3.28}
$$

Therefore Lemma 1.2.3 and $\mu^k(t) = 0$ for $t \in [-r, 0]$ yield

$$
|x^k(t) - x(t) - z(t)| \leq \mu^k(t) \leq \frac{C_k}{1 - N_4 e^{-cr_0}} e^{c\alpha} |h_k|_\Gamma, \qquad t \in [0, \alpha], \tag{4.3.29}
$$

where c is the unique positive solution of $cN_4 e^{-cr_0} + N_5 = c$. Hence the claim of the theorem follows, since $C_k \to 0$ as $k \to \infty$.

The proof of the theorem is complete. □

The proof immediately implies differentiability of the parameter map in the C-norm:

Corollary 4.3.5 *Assume the conditions of Theorem 4.3.4. Then the function*

$$
\Gamma \supset P \to C, \qquad \gamma \mapsto x(\cdot, \gamma)_t
$$

is differentiable at $\bar{\gamma} \in P \cap \mathcal{P}$ *for* $t \in [0, \alpha]$*, and its derivative is given by*

$$
D_2 x_t(\cdot, \bar{\gamma}) h = z_t(\cdot, \bar{\gamma}, h), \qquad h \in \Gamma, \quad t \in [0, \alpha].
$$

We remark that the proof of Theorem 4.3.1 relies on the compatibility assumption $\gamma \in \mathcal{P}$. To prove the existence of higher order derivatives wrt the parameters we would need to get rid of this assumption. Also, to extend the quasilinearization method of Chapter 3 to SD-NFDEs it is necessary to omit the compatibility assumption from the assumptions of Theorem 4.3.1. We comment that numerical experiments show that the quasilinearization method works for NFDEs also in cases when the compatibility assumption fails.

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