ON CLASSES OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS

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In this dissertation we study a class of nonlinear functional differential equations (FDEs) with state-dependent delays given by

$$\dot{x}(t) = f\left(t, x(t), \int_{-r}^{0} d_s\mu(s,t,x_t)\, x(t+s)\right), \qquad t \geq 0, \tag{1}$$

where the term describing the delay dependence is a Stieltjes-integral of the solution segment $x(t+\cdot)$ with respect to $\mu(\cdot, t, x_t)$, which is a matrix valued function of bounded variations depending on time, t, and the state of the equation, x_t.

The main objective of this work is to extend the basic theory of delay equations for the type of FDEs described by (1). We establish well-posedness of the initial value problem corresponding to (1) in the state-space C, and we discuss other potential state-spaces, namely, $W^{1,\infty}$ and $W^{1,p}$. Then we investigate differentiability of solutions with respect to parameters in these state-spaces. We can summarize our findings as follows: In special cases we prove differentiability of solutions in the state-space $W^{1,\infty}$, but in order to obtain differentiability under less restrictive assumptions on the class of equations under consideration we study differentiability in a weaker sense, i.e., in the norm $|\cdot|_{W^{1,p}}$. In particular, we define a special norm on the set $W^{1,\infty}$, which is weaker than the $|\cdot|_{W^{1,\infty}}$ norm and stronger than the $|\cdot|_{W^{1,p}}$ norm, and consider $W^{1,\infty}$ equipped with this norm. The resulting normed linear space is a so-called quasi-Banach space, and using a modified version of the Uniform Contraction Principle (which was generalized for quasi-Banach spaces by Hale and Ladeira) we obtain differentiability of solutions wrt parameters in this norm, and therefore in the weaker norm, $|\cdot|_{W^{1,p}}$, as well.

In the second part of the dissertation we discuss three important issues of applications. First we obtain stability results for the autonomous version of (1) using a linearization technique. Then we formulate an Euler's scheme for computing approximate solutions of (1), and present a new proof for convergence of this method using equations with piecewise constant arguments. As an application of this numerical scheme, we discuss the problem of parameter identification for equation (1). In all applications we present several examples to illustrate the theoretical results.

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Chapter 1
INTRODUCTION

In the past several decades delay differential equations appeared in an increasing number of system models in biology, engineering, physics, etc. In most of these applications it is assumed that the delay (discrete or distributed) is either constant or at most time-dependent. (See e.g. [40] for a discussion on the importance of delays, especially distributed delays in biological models.) However, in recent years there are several proposed models in which it is assumed that the delay depends not only on time, but on the state as well (see e.g. [1], [6], [18], [20] and [45]).

Motivated by these developments, in this paper we study a general nonlinear distributed state-dependent delay equation of the form

$$\dot{x}(t) = f\Big(t, x(t), \Lambda(t, x_t)\Big), \quad t \in [0, T], \tag{1.1}$$

where the term Λ, describing the delay dependencies, has the form

$$\Lambda(t, x_t) \equiv \int_{-r}^{0} d_s\mu(s, t, x_t)\, x(t+s). \tag{1.2}$$

Here r is a positive constant, $x(\cdot) \in \mathbb{R}^n$, x_t denotes the segment $x_t(s) \equiv x(t+s)$ for $s \in [-r, 0]$. $\mu(\cdot, t, \psi)$ is an $n \times n$ matrix valued function of bounded variation on $[-r, 0]$ for all $t \in [0, T]$, $\psi \in C \equiv C([-r, 0]; \mathbb{R}^n)$, and the integral is the Stieltjes-integral of $x(t + \cdot)$ with respect to $\mu(\cdot, t, x_t)$.

The main goal of this work is to extend the basic theory of delay differential equations for the class of delay equations with distributed state-dependent delays described by (1.1)-(1.2).

Note that representation (1.2), describing the dependence on the past, is natural for linear delay equations of the form $\dot{x}(t) = Lx_t$, where L is a bounded linear operator on C (see e.g. [31]). In Example 1.1 and 1.2 we show the construction of μ for the constant and time-varying delay cases. Example 1.3 and 1.4 show that we can use representation (1.2) to describe delayed arguments for equations including pointwise and distributed state-dependent delays as well. These examples illustrate that equations (1.1) and (1.2) describe a large class of delay systems important in applications.

Example 1.1 Consider the linear system with constant delays

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k), \tag{1.3}$$

where A_k $(k = 0, 1, \ldots, m)$ are constant $n \times n$ matrices, τ_k $(k = 1, 2, \ldots, m)$ are positive constants.

Define $r \equiv \max\{\tau_1, \ldots, \tau_m\}$, and

$$\mu(s) \equiv \sum_{k=1}^{m} A_k \chi_{[-\tau_k,0]}(s) \qquad \text{for} \quad s \in [-r, 0],$$

where $\chi_{[-\tau_k,0]}(s)$ is the characteristic function of the interval $[-\tau_k, 0]$. It is easy to see that the total variation of μ satisfies that $\operatorname*{Var}_{[-r,0]}[\mu] \leq \sum_{k=1}^{m} \|A_k\|$, and therefore μ is of bounded variation on $[-r, 0]$. With this μ equation (1.3) is equivalent to

$$\dot{x}(t) = A_0 x(t) + \int_{-r}^{0} d\mu(s) x(t+s),$$

therefore (1.3) has the form (1.1)-(1.2) with $\mu(s, t, \psi) = \mu(s)$, and $f(t, x, y) = A_0 x + y$.

Example 1.2 Consider the time-varying linear delay system

$$\dot{x}(t) = A_0(t)x(t) + \sum_{k=1}^{m} A_k(t) x(t - \tau_k(t)) + \int_{-\tau_0}^{0} G(s,t) x(t+s)\, ds, \qquad t \in [0, T], \tag{1.4}$$

where $A_k(t)$ $(k = 0, 1, \ldots, m)$ are $n \times n$ matrices, $\tau_k(t) \geq 0$ $(k = 1, 2, \ldots, m)$ are bounded functions, and $G : [-\tau_0, 0] \times [0, T] \to \mathbb{R}^{n \times n}$.

Define $r \equiv \max\Big\{\tau_0, \sup\{\tau_k(t) \ : \ t \in [0, T], k = 1, \ldots, m\}\Big\}$, and, for $t \in [0, T]$,

$$\mu(s, t) \equiv \sum_{k=1}^{m} A_k(t) \chi_{[-\tau_k(t),0]}(s) + \tilde{\mu}(s, t), \qquad s \in [-r, 0],$$

where

$$\tilde{\mu}(s,t) \equiv \begin{cases} 0, & s \in [-r, -\tau_0], \\ \int_{-\tau_0}^{s} G(u,t)\, du, & s \in (-\tau_0, 0]. \end{cases}$$

Clearly, for all $t \in [0, T]$ the function $\mu(\cdot, t)$ is of bounded variation, and (1.4) is equivalent to

$$\dot{x}(t) = A_0(t)x(t) + \int_{-r}^{0} d_s\mu(s,t) x(t+s), \qquad t \in [0, T],$$

and therefore (1.4) has the form (1.1)-(1.2) with $f(t, x, y) = A_0(t)x + y$.

Example 1.3 Consider the delay system with a point state-dependent delay term

$$\dot{x}(t) = f\Big(t, x(t), x(t - \tau(t, x_t))\Big), \qquad t \in [0, T], \tag{1.5}$$

where $f : [0, T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$, (where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$), $\tau : [0, T] \times \Omega_3 \to \mathbb{R}^+$ (where Ω_3 is an open subset of C) is a bounded function, and $r > 0$ is such that $r \geq \sup\{\tau(t, \psi) : t \in [0, T],\ \psi \in \Omega_3\}$.

Define

$$\mu(s, t, \psi) \equiv \chi_{[-\tau(t,\psi),0]}(s) I, \qquad s \in [-r, 0],$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix. It is easy to see that $\operatorname*{Var}_{[-r,0]}[\mu(\cdot, t, \psi)] = 1$ for all $t \in [0, T]$ and $\psi \in \Omega_3$, so $\mu(\cdot, t, \psi)$ is of bounded variation for all $t \in [0, T]$, $\psi \in \Omega_3$, and (1.5) has the form (1.1)-(1.2).

Example 1.4 Consider the system with discrete and distributed state-dependent delays

$$\dot{x}(t) = A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t - \tau_k(t, x_t)) + \int_{-\tau_0}^{0} G(s, t, x_t)x(t + s)\, ds, \qquad t \in [0, T], \qquad (1.6)$$

where $A_k(t)$ $(k = 0, 1, \ldots, m)$ are $n \times n$ matrices, $\tau_k : [0, T] \times \Omega_3 \to \mathbb{R}^+$ are bounded, nonnegative functions for $k = 1, 2, \ldots, m$, and $G\ [-\tau_0, 0] \times [0, T] \times \Omega_3 \to \mathbb{R}^{n\times n}$, where Ω_3 is an open subset of $\mathbb{R}^n$. We assume that $r > 0$ is such that $r \geq \max\Big\{\tau_0, \sup\{\tau_k(t, \psi) : t \in [0, T], \psi \in \Omega_3, k = 1, \ldots, m\}\Big\}$.

Similarly to Example 1.2 we get the following representation of the delay equation. For $t \geq 0$, $\psi \in \Omega_3$ define

$$\mu(s, t, \psi) \equiv \sum_{k=1}^{m} A_k(t)\chi_{[-\tau_k(t,\psi),0]}(s) + \tilde{\mu}(s, t, \psi), \qquad s \in [-r, 0],$$

with

$$\tilde{\mu}(s, t, \psi) \equiv \begin{cases} 0, & s \in [-r, -\tau_0], \\ \int_{-\tau_0}^{s} G(u, t, \psi)\, du, & s \in (-\tau_0, 0]. \end{cases}$$

Then it is easy to see that $\mu(\cdot, t, \psi)$ is of bounded variation for all $t \in [0, T]$ and $\psi \in \Omega_3$, and (1.6) can be rewritten as (1.1)-(1.2) with $f(t, x, y) = A_0(t)x + y$.

The remaining part of this dissertation is organized as follows:

In Chapter 2 we introduce some basic notations and definitions used throughout this paper, and recall some results for future reference.

In Chapter 3 we study well-posedness of the initial value problem corresponding to (1.1)-(1.2). In Sections 3.1–3.3 we present conditions implying existence, uniqueness and continuous dependence of solutions on parameters. In Section 3.4 we discuss other potential state-spaces, $W^{1,\infty}$ and $W^{1,p}$.

Chapter 4 contains the main results of this work, here we investigate differentiability of solutions with respect to parameters. This issue has not been discussed yet for delay equations with state-dependent delays in the literature, not even for the point delay case. By applying an extension of the Uniform Contraction Principle developed by Hale and Ladeira in [33], we are able to show differentiability of solutions with respect to parameters in the state-space $W^{1,p}$. We consider three cases: differentiability with respect to initial functions, a (vector) parameter in the delay term (μ), and a (vector) parameter in the equation (f).

In Chapters 5, 6 and 7 we address some basic issues of applications. In Chapter 5 we investigate stability of solutions by linearization technique.

In Chapter 6 we present a simple numerical scheme to approximate solutions of (1.1)-(1.2).

In Chapter 7 we apply the numerical scheme defined in Chapter 6 for parameter identification.

Finally, in Chapter 8 we briefly discuss the well-posedness of (1.1)-(1.2) in L^p-spaces.

Chapter 2
NOTATIONS, PRELIMINARIES

In this chapter we introduce some basic notations, definitions of function spaces and norms we shall use in the sequel, and recall some results for future references.

2.1 Function spaces and norms

We denote the set of real numbers, nonnegative real numbers and positive integers by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively.

Throughout this paper $|\cdot|$ and $\|\cdot\|$ denote a norm on $\mathbb{R}^n$ and the corresponding matrix norm on $\mathbb{R}^{n\times n}$, respectively. (The constant n is fixed throughout this paper.) In the case when we use dimension different from n, we shall use the notation $|\cdot|_{\mathbb{R}^m}$ for the norm in $\mathbb{R}^m$.

The notation $f : \Big(A \subset X\Big) \to Y$ will be used to denote that the function maps the subset A of the normed linear space X to Y. This notation emphasizes that the topology on A is defined by the norm of X.

We denote the open ball around a point x_0 with radius R in a normed linear space $(X, |\cdot|_X)$ by $\mathcal{G}_X(x_0;\, R)$, i.e., $\mathcal{G}_X(x_0;\, R) \equiv \{x \in X \,:\, |x - x_0|_X < R\}$, and the corresponding closed ball by $\overline{\mathcal{G}}_X(x_0;\, R)$. If the ball is centered at the origin, we use simply $\mathcal{G}_X(R)$ and $\overline{\mathcal{G}}_X(R)$, respectively.

We shall use the following standard function spaces and norms:

$C\Big([a,b];\ \mathbb{R}^n\Big)$ is the Banach-space of continuous functions $\psi : [a,b] \to \mathbb{R}^n$ with the norm $|\psi|_{C([a,b];\mathbb{R}^n)} \equiv \sup\{|\psi(t)| \,:\, t \in [a,b]\}$.

$L^p([a,b];\mathbb{R}^n)$ $(1 \le p < \infty)$ is the Banach space of measurable functions $\psi : [a,b] \to \mathbb{R}^n$ such that $\int_a^b |\psi(u)|^p\,du < \infty$, with the norm $|\psi|_{L^p([a,b];\mathbb{R}^n)} \equiv \Big(\int_a^b |\psi(u)|^p\,du\Big)^{1/p}$.

$L^\infty([a,b];\mathbb{R}^n)$ is the Banach space of essentially bounded measurable functions $\psi : [a,b] \to \mathbb{R}^n$ with the norm $|\psi|_{L^\infty([a,b];\mathbb{R}^n)} \equiv \operatorname*{ess\,sup}_{a\le u\le b} |\psi(u)|$.

$BC\Big([0,T]\times\Omega_1\times\Omega_2;\ \mathbb{R}^n\Big)$ is the Banach-space of bounded continuous functions $f : [0,T]\times \Omega_1\times\Omega_2 \to \mathbb{R}^n$ (where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$) with norm $\|f\| \equiv \sup\{|f(t,x,y)| \,:\, t \in [0,T],\ x \in \Omega_1,\ y \in \Omega_2\}$.

$W^{1,p}([a,b];\mathbb{R}^n)$ $(1 \le p \le \infty)$ denotes the Sobolev-space of order 1 of functions $\psi : [a,b] \to \mathbb{R}^n$, where ψ and its generalized derivative belong to $L^p([a,b];\mathbb{R}^n)$. It is well-known that $W^{1,p}([a,b];\mathbb{R}^n)$ is a Banach-spaces with norm

$$|\psi|_{W^{1,p}([a,b];\mathbb{R}^n)} \equiv \left(\int_a^b |\psi(s)|^p + |\dot\psi(s)|^p\,ds\right)^{1/p}, \qquad 1 \le p < \infty, \tag{2.1}$$

and

$$|\psi|_{W^{1,\infty}([a,b];\mathbb{R}^n)} \equiv \max\left\{ \operatorname*{ess\,sup}_{a\le s\le b} |\psi(s)|, \operatorname*{ess\,sup}_{a\le s\le b} |\dot\psi(s)| \right\}, \qquad p=\infty, \tag{2.2}$$

respectively.

Remark 2.1 *Let $\psi \in W^{1,p}([a,b];\mathbb{R}^n)$ $(1 \le p \le \infty)$. It is known (see e.g. [36]) that ψ is a.e. equal to an absolutely continuous function, $\bar\psi$. By the notation $\psi(s)$ we mean $\bar\psi(s)$, the function value of the continuous representation of the L^p-function ψ, i.e., point evaluation of functions in $W^{1,p}([a,b];\mathbb{R}^n)$ is well-defined.*

Remark 2.2 *It is known (see e.g. [36]), that for an absolutely continuous function, ψ, the generalized derivative is a.e. equal to the classical derivative. Therefore in (2.1), (2.2), and later, $\dot\psi$, ψ' or $\frac{\partial\psi}{\partial s}$ denote the classical derivative of ψ.*

The following lemma is the Mean Value Theorem for $W^{1,\infty}([a,b];\mathbb{R}^n)$ functions.

Lemma 2.3 *Let $\psi \in W^{1,\infty}([a,b];\mathbb{R}^n) \cap C([a,b];\mathbb{R}^n)$, and $[c,d] \subset [a,b]$. Then*

$$|\psi(d)-\psi(c)| \le |\dot\psi|_{L^\infty([a,b];\mathbb{R}^n)}(d-c).$$

Proof Since by the assumptions and Remark 2.1 ψ is absolutely continuous, it follows that

$$\begin{aligned} |\psi(d)-\psi(c)| &= \left| \int_c^d \dot\psi(u)\,du \right| \\ &\le \int_c^d |\dot\psi(u)|\,du \\ &\le |\dot\psi|_{L^\infty([a,b];\mathbb{R}^n)}(d-c), \end{aligned}$$

which proves the statement. □

Remarks 2.1, 2.2 and Lemma 2.3 yield the following characterization of $W^{1,\infty}([a,b];\mathbb{R}^n)$.

Lemma 2.4 *The following statements are equivalent*

(i) $\psi \in W^{1,\infty}([a,b];\mathbb{R}^n)$,

(ii) ψ is equivalent (i.e., a.e. equal) to a Lipschitz-continuous function.

The constant $r > 0$ is fixed throughout this dissertation.

To keep the notation simple, the spaces $C([-r,0];\mathbb{R}^n)$, $L^p([-r,0];\mathbb{R}^n)$, $W^{1,p}([-r,0];\mathbb{R}^n)$ and the corresponding norms will be denoted by C, L^p, $W^{1,p}$ and $|\cdot|_C$, $|\cdot|_{L^p}$ and $|\cdot|_{W^{1,p}}$, respectively. Similarly, the spaces $C([-r,\alpha];\mathbb{R}^n)$, $L^p([-r,\alpha];\mathbb{R}^n)$, $W^{1,p}([-r,\alpha];\mathbb{R}^n)$ and the corresponding norms will be denoted by C_α, L^p_α, $W^{1,p}_\alpha$ and $|\cdot|_{C_\alpha}$, $|\cdot|_{L^p_\alpha}$ and $|\cdot|_{W^{1,p}_\alpha}$, respectively.

NBV $\equiv$ NBV $([-r,0];\ \mathbb{R}^{n\times n})$ denotes the set of functions $\eta : [-r,0] \to \mathbb{R}^{n\times n}$ which are of bounded variation, and normalized such that $\eta(s)$ is right-continuous at each $s \in (-r,0)$ and

$\eta(-r) = 0$. The space is a Banach-space with norm $\|\eta\|_{\mathrm{NBV}} \equiv \operatorname*{Var}_{[-r,0]}[\eta(s)]$ (the total variation of ν over $[-r,0]$, defined as $\operatorname*{Var}_{[-r,0]}[\eta(s)] = \sup\Big\{\sum_{k=1}^{m} \|\eta(s_i) - \eta(s_{i-1})\|\Big\}$, where the supremum is taken for all possible finite partition $-r \le s_0 < s_1 < \ldots < s_m \le 0$ of $[-r,0]$).

Let Ω_3 be an open subset of C, and $T > 0$ or $T = \infty$. In the latter case $[0,T]$ denotes the interval $[0,\infty)$. Then define $\Theta(T,\Omega_3)$ as the set of functions $\mu : [0,T] \times \Omega_3 \to \mathrm{NBV}$ such that

$$\sup\left\{\left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right| : t \in [0,T], \psi \in \Omega_3,\ \xi \in \overline{\mathcal{G}}_C(1)\right\} < \infty,$$

where $\mu(\cdot,t,\psi)$ is the image function corresponding to $t \in [0,T]$ and $\psi \in C$, and the integral $\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)$ denotes the Stieltjes-integral of the continuous function ξ over $[-r,0]$ with respect to $\mu(\cdot,t,\psi)$. Then, clearly, $\Theta(T,\Omega_3)$ is a normed linear space with the norm

$$\|\mu\| \equiv \sup\left\{\left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right| : t \in [0,T],\ \psi \in \Omega_3,\ \xi \in \overline{\mathcal{G}}_C(1)\right\}. \tag{2.3}$$

We comment, that according to the inequality

$$\left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right| \le \|\mu(\cdot,t,\psi)\|_{\mathrm{NBV}}|\xi|_C, \tag{2.4}$$

we have that for each fixed $t \in [0,T]$ and $\psi \in \Omega_3$ the map

$$C \to \mathbb{R}^n, \quad \xi \mapsto \int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)$$

is a bounded linear functional. The Riesz-representation theorem (see e.g. [47]) implies that the opposite result is also true, i.e., to an arbitrary bounded linear functional, A, on C there corresponds a unique $\eta \in \mathrm{NBV}$ such that $A\xi = \int_{-r}^{0} d\eta(s)\xi(s)$, and $\|A\| = \|\eta\|_{\mathrm{NBV}}$. From this result, and from the equality

$$\sup_{t\in[0,T],\psi\in\Omega_3,\xi\in\overline{\mathcal{G}}_C(1)} \left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right| = \sup_{t\in[0,T],\psi\in\Omega_3}\ \sup_{\xi\in\overline{\mathcal{G}}_C(1)} \left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right|,$$

we have the following remark.

Remark 2.5 *The normed linear space $\Theta(T,\Omega_3)$ is isometrically isomorphic to the space of bounded functions from $[0,T] \times \Omega_3$ to C^* (the dual space of C), i.e.,*

$$\Theta(T,\Omega_3) \simeq B\Big([0,T] \times \Omega_3; C^*\Big),$$

and

$$\|\mu\| = \sup\Big\{\|\mu(\cdot,t,\psi)\|_{\mathrm{NBV}} \ : \ t \in [0,T],\ \psi \in \Omega_3\Big\}.$$

We shall frequently use the following estimate, which easily follows from the definition of $\|\mu\|$:

$$\left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right| \le \|\mu\||\xi|_C, \qquad \text{for } t \in [0,T], \psi \in \Omega_3, \xi \in C. \tag{2.5}$$

We introduce $\Theta_C(T,\Omega_3)$, which is the subspace of $\Theta(T,\Omega_3)$ consisting of functions $\mu \in \Theta(T,\Omega_3)$ such that for all $\xi \in C$ the function

$$[0,T]\times\Omega_3 \to \mathbb{R}^n, \qquad (t,\psi)\mapsto \int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)$$

is continuous. This set is, by the next lemma, a closed subspace of $\Theta(T,\Omega_3)$ in the $\|\cdot\|_{\Theta(T,\Omega_3)}$ norm.

Lemma 2.6 *$\Theta_C(T,\Omega_3)$ is a closed subspace of $\Theta(T,\Omega_3)$ in the $\|\cdot\|_{\Theta(T,\Omega_3)}$ norm.*

Proof Let $\mu^k \in \Theta_C(T,\Omega_3)$ such that $\|\mu^k-\bar\mu\|\to 0$ as $k\to\infty$, and $\bar\mu\in\Theta(T,\Omega_3)$. Fix $\xi,\bar\psi\in\Omega_3$, $\bar t\in[0,T]$, and let $\varepsilon>0$. Then by elementary manipulations and estimate (2.5) we have for $\psi\in C$ and $t\in[0,T]$ that

$$\begin{aligned}
&\Big|\int_{-r}^{0} d_s\bar\mu(s,t,\psi)\xi(s)-\int_{-r}^{0} d_s\bar\mu(s,\bar t,\bar\psi)\xi(s)\Big|\\
&\quad\le \Big|\int_{-r}^{0} d_s\Big[\bar\mu(s,t,\psi)-\mu^k(s,t,\psi)\Big]\xi(s)\Big|+\Big|\int_{-r}^{0} d_s\Big[\mu^k(s,t,\psi)-\mu^k(s,\bar t,\bar\psi)\Big]\xi(s)\Big|\\
&\qquad +\Big|\int_{-r}^{0} d_s\Big[\bar\mu(s,\bar t,\bar\psi)-\mu^k(s,\bar t,\bar\psi)\Big]\xi(s)\Big|\\
&\quad\le 2\|\bar\mu-\mu^k\||\xi|_C+\Big|\int_{-r}^{0} d_s\Big[\mu^k(s,t,\psi)-\mu^k(s,\bar t,\bar\psi)\Big]\xi(s)\Big|. \qquad (2.6)
\end{aligned}$$

The first term on the right hand side of (2.6) is less than $\varepsilon/2$ for large enough k because $\mu^k\to\bar\mu$. For such a fixed k the second term is less than $\varepsilon/2$ if $|\psi-\bar\psi|_C+|t-\bar t|$ is small, because $\mu^k\in\Theta_C(T,\Omega_3)$, and therefore we have proved the statement of the lemma. □

Remark 2.5 and the definition of $\Theta_C(T,\Omega_3)$ yield the next remark immediately.

Remark 2.7 *The normed linear space $\Theta_C(T,\Omega_3)$ is isometrically isomorphic to the space of bounded continuous maps from $[0,T]\times\Omega_3$ to C^*, where C^* is equipped with the weak* topology, i.e.,*

$$\Theta_C(T,\Omega_3)\simeq BC_{w^*}([0,T]\times\Omega_3,C^*).$$

We introduce the functions λ and Λ defined by

$$\lambda : [0,T]\times\Omega_3\times C\to\mathbb{R}^n, \qquad \lambda(t,\psi,\xi)\equiv\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s), \qquad (2.7)$$

and

$$\Lambda : [0,T]\times\Omega_3\to\mathbb{R}^n, \qquad \Lambda(t,\psi)\equiv\lambda(t,\psi,\psi)=\int_{-r}^{0} d_s\mu(s,t,\psi)\psi(s), \qquad (2.8)$$

respectively.

If we need to emphasize that the functions λ and Λ correspond to a given μ, then we shall use the notations $\lambda_\mu(t,\psi,\xi)$ and $\Lambda_\mu(t,\psi)$, respectively.

Lemma 2.8 *Assume that $\mu \in \Theta_C(T, \Omega_3)$. Then the function $\Lambda(\cdot,\cdot)$ defined by (2.8) is continuous on $[0,T] \times \Omega_3$.*

Proof Fix $\bar{t} \in [0,T]$ and $\bar{\psi} \in \Omega_3$. Then by applying (2.5) we have for $t \in [0,T]$ and $\psi \in \Omega_3$ that

$$\begin{aligned}
|\Lambda(t,\psi) - \Lambda(\bar{t},\bar{\psi})| &= \left| \int_{-r}^{0} d_s\mu(s,t,\psi)\psi(s) - \int_{-r}^{0} d_s\mu(s,\bar{t},\bar{\psi})\bar{\psi}(s) \right| \\
&\le \left| \int_{-r}^{0} d_s\Big[\mu(s,t,\psi) - \mu(s,\bar{t},\bar{\psi})\Big]\bar{\psi}(s) \right| + \left| \int_{-r}^{0} d_s\mu(s,t,\psi)\Big[\psi(s) - \bar{\psi}(s)\Big] \right| \\
&\le \left| \int_{-r}^{0} d_s\Big[\mu(s,t,\psi) - \mu(s,\bar{t},\bar{\psi})\Big]\bar{\psi}(s) \right| + \|\mu\| |\psi - \bar{\psi}|_C.
\end{aligned}$$

In the last inequality the first term goes to 0 by the definition of $\Theta_C(T, \Omega_3)$, as $t \to \bar{t}$ and $\psi \to \bar{\psi}$, and so does the second term. □

Define $BC([0,T] \times \Omega_3\,;\,\mathrm{NBV})$ as the linear space of bounded continuous functions from $[0,T] \times \Omega_3$ to NBV with the norm $\|\mu\|_{BC([0,T]\times\Omega_3\,;\,\mathrm{NBV})} = \sup\{\|\mu(\cdot,t,\psi)\|_{\mathrm{NBV}} : t \in [0,T],\ \psi \in \Omega_3\}$.

Lemma 2.9 *Let $\Omega_3 \subset C$ and $T > 0$. Then $BC([0,T] \times \Omega_3\,;\,\mathrm{NBV}) \subset \Theta_C(T, \Omega_3)$.*

Proof The inclusion follows immediately from the inequality

$$\left| \int_{-r}^{0} d_s\Big[\mu(s,t,\psi) - \mu(s,\bar{t},\bar{\psi})\Big]\xi(s) \right| \le \|\mu(\cdot,t,\psi) - \mu(\cdot,\bar{t},\bar{\psi})\|_{\mathrm{NBV}} |\xi|_C.$$

□

2.2 Some integral inequalities and results on differentiability

The following notations will be used extensively throughout this paper.

Let $\varphi \in C$. Define the extension $\tilde{\varphi}$ of φ to $[-r,\infty)$ as

$$\tilde{\varphi}(t) \equiv \begin{cases} \varphi(t), & t \in [-r,0] \\ \varphi(0), & t \ge 0. \end{cases} \tag{2.9}$$

Clearly, the definition of $\tilde{\varphi}$ implies the inequality

$$\sup_{-r \le u \le t} |\tilde{\varphi}(u)| \le |\varphi|_C, \qquad t \ge -r. \tag{2.10}$$

For $\alpha > 0$ and $\beta > 0$ and $y \in \overline{\mathcal{G}}_{C_\alpha}(\beta)$ and $\varphi \in C$ define

$$\begin{aligned}
\omega_y(h;\alpha) &\equiv \sup\{|y(t_1) - y(t_2)| : t_1, t_2 \in [-r,\alpha],\ |t_1 - t_2| \le h\}, \\
\omega_{\tilde{\varphi}}(h) &\equiv \sup\{|\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)| : t_1, t_2 \in [-r,\infty),\ |t_1 - t_2| \le h\}, \\
\omega_{\varphi}(h) &\equiv \sup\{|\varphi(t_1) - \varphi(t_2)| : t_1, t_2 \in [-r,0],\ |t_1 - t_2| \le h\}.
\end{aligned}$$

Note that each function is nonnegative, and monotone increasing in h.

By the definition of $\tilde{\varphi}(\cdot)$ it follows that

$$|\tilde{\varphi}(t_1) - \tilde{\varphi}(t_2)| = \begin{cases} |\varphi(t_1) - \varphi(t_2)|, & t_1, t_2 \leq 0 \\ |\varphi(t_1) - \varphi(0)|, & t_1 \leq 0 \leq t_2 \\ 0, & t_1, t_2 \geq 0, \end{cases}$$

hence we have that

$$\omega_{\tilde{\varphi}}(h) = \omega_{\varphi}(h). \tag{2.11}$$

Let $x : [-r, \alpha] \to \mathbb{R}^n$ be a continuous function. For $t \in [0, \alpha]$ the segment function $x_t : [-r, 0] \to \mathbb{R}^n$ is defined as $x_t(s) \equiv x(t+s)$.

The following two results discuss the continuity of the map $t \mapsto x_t$ in different spaces.

Lemma 2.10 (see e.g. in [31]) *If $x \in C_\alpha$, then the function $[0, \alpha] \to C$, $t \mapsto x_t$ is continuous.*

Lemma 2.11 (see e.g. in [25]) *If $x \in L^p_\alpha$, then the function $[0, \alpha] \to L^p$, $t \mapsto x_t$ is continuous.*

Lemma 2.12 *Let $x : [-r, \alpha] \to \mathbb{R}^n$ be a differentiable function. Then the segment function $x_t(\cdot)$ is differentiable on $[-r, 0]$, and*

$$\frac{d}{ds} x_t(s) = (\dot{x})_t(s), \qquad s \in [-r, 0], \quad t \in [0, \alpha].$$

Proof The result follows from the elementary relations

$$\frac{d}{ds} x_t(s) = \frac{d}{ds} x(t+s) = \dot{x}(t+s) = (\dot{x})_t(s).$$

□

Next we present two results for integral inequalities. The fist lemma is the famous Gronwall-Bellman inequality.

Lemma 2.13 (see e.g. in [19]) *Let $c \geq 0$ be a constant, $f : [a, b] \to \mathbb{R}^+$ be a nonnegative continuous function, and $x : [a, b] \to \mathbb{R}^+$ be a continuous function satisfying*

$$x(t) \leq c + \int_a^t f(s) x(s)\, ds, \qquad t \in [a, b].$$

Then x satisfies

$$x(t) \leq c \exp\left(\int_a^t f(s)\, ds \right), \qquad t \in [a, b].$$

The following simple integral inequality will be used several times in our proofs later.

Lemma 2.14 *Let $f : [0,\alpha] \to \mathbb{R}^+$, and $g : [0,\alpha]\times[a,b] \to \mathbb{R}^+$ be continuous functions, such that f is nondecreasing on $[0,\alpha]$, and $g(t,u)$ is nonnegative on $[0,\alpha]\times[a,b]$, and nondecreasing for $u \in [a,b]$ for all fixed $t \in [0,\alpha]$, and let $x : [-r,\alpha] \to \mathbb{R}$ be a continuous function satisfying the inequality*

$$|x(t)| \le f(t) + \int_0^t g(s, |x_s|_C)\, ds, \qquad t \in [0,\alpha].$$

Assume that $|x_0|_C \le f(0)$, then the function $y(t) \equiv \max_{-r\le u\le t} |x(u)|$ (or the function $y(t) \equiv |x_t|_C$) satisfies the same inequality, i.e.,

$$y(t) \le f(t) + \int_0^t g(s, y(s))\, ds, \qquad t \in [0,\alpha].$$

Proof Let $t \le \bar t$. Then from the relation $|x_s|_C \le y(s)$ and the assumed monotonicity properties we get

$$\begin{aligned} |x(t)| &\le f(t) + \int_a^t g(s, |x_s|_C)\, ds \\ &\le f(t) + \int_a^t g(s, y(s))\, ds \\ &\le f(\bar t) + \int_a^{\bar t} g(s, y(s))\, ds. \end{aligned}$$

Since this is true for all $t \le \bar t$, by taking the maximum of the left-hand-side of the inequality for $t \in [0,\bar t]$ and using that $|x_0| \le f(0) \le f(t)$ for all $t \ge 0$ we prove the statement of the lemma. □

Finally, we recall some results for later reference concerning differentiability of functions. Note, that in this paper all the derivatives we use are Frechét-derivatives.

Lemma 2.15 (Chain Rule, see e.g. in [43]) *Let X, Y and Z be Banach-spaces, $F : U \to Y$ and $G : V \to Z$, where U and V are open subsets of X and Y, respectively. Then if F is differentiable at $u \in U$, and G is differentiable at $v \equiv F(u) \in V$, then $G \circ F$ is differentiable at u, and $(G\circ F)'(u) = G'(F(u))F'(u)$.*

Lemma 2.16 (see e.g. in [43]) *Suppose that X and Y are Banach-spaces, and Q is an open subset of X, and $F : Q \to Y$ is differentiable. Let $x, y \in Q$ and $y + \nu(x-y) \in Q$ for $\nu \in [0,1]$. Then*

$$|F(y) - F(x) - F'(x)(y-x)|_Y \le |x-y|_X \sup_{0<\nu<1} \|F'(y+\nu(x-y)) - F'(x)\|_{\mathcal{L}(X,Y)}$$

Lemma 2.17 (see e.g. in [16]) *Let X, Y and Z be Banach-spaces, and let Q be an open set in $X \times Y$, and let $F(x,y)$ be a continuous function from Q into Z. Then in order that F be continuously differentiable in Q, a necessary and sufficient condition is that F be continuously differentiable wrt x and y on Q, and then the derivative satisfies*

$$F'(\bar x, \bar y)(x,y) = \frac{\partial F}{\partial x}(\bar x,\bar y)x + \frac{\partial F}{\partial y}(\bar x,\bar y)y.$$

2.3 Linear delay equations and semigroups

Consider a linear delay equation with constant delays of the form:

$$\dot{x}(t) = Lx_t, \qquad t \geq 0, \tag{2.12}$$

where $L : C \to \mathbb{R}^n$ is a bounded linear operator. It is well-known (e.g. [31]), that (2.12) has a unique solution, $x(t;\varphi)$, corresponding to any initial function $\varphi \in C$, defined on $t \in [-r, \infty)$. Moreover (see e.g. [31]), the family of linear operators, $\{S(t)\}_{t\geq 0}$, given by

$$S(t)\varphi \equiv x(\cdot;\varphi)_t, \qquad t \geq 0$$

defines a strongly continuous semigroup on C.

Let define

$$\omega_0 \equiv \sup\Big\{ Re\,\lambda \ : \ det(\lambda I - Le^{\lambda\cdot}) = 0 \Big\},$$

i.e., ω_0 is the supremum of the real part of the characteristic roots of (2.12). We shall need the following lemma:

Lemma 2.18 (see e.g. in [31]) *If $\omega_0 < 0$, then for any $\omega_0 < \omega < 0$ there exists $M = M(\omega) \geq 1$ such that*

$$\|S(t)\| \leq Me^{\omega t}, \qquad t \geq 0.$$

Consider the perturbed equation

$$\dot{x}(t) = Lx_t + f(t), \qquad t \geq 0, \tag{2.13}$$

where $f \in L^1_{\text{loc}}([0,\infty), \mathbb{R}^n)$. Then (2.13) has a unique solution on $[0, \infty)$ for all $\varphi \in C$, and the solution, $x(t)$ satisfies the following abstract variation of constant formula:

Lemma 2.19 (see e.g. [30]) *The solution, $x(t)$, of (2.13), corresponding to an initial function $\varphi \in C$ has the form:*

$$x_t = S(t)\varphi + \int_0^t S(t-s)X_0 f(s)\,ds,$$

where

$$X_0 \ : \ [-r, 0] \to \mathbb{R}^{n\times n}, \qquad X_0(u) \equiv \begin{cases} 0, & u < 0, \\ I, & u = 0. \end{cases} \tag{2.14}$$

We shall need the following variation of Lemma 2.19.

Lemma 2.20 *The solution, $x(t)$, of (2.13) satisfies*

$$x_t = S(t-r)x_r + \int_0^{t-r} S(t-r-s)X_0 f(s+r)\,ds, \qquad t \geq r,$$

where X_0 is defined by (2.14).

Proof By applying Lemma 2.19, semigroup properties of $S(t)$, and change of variables we get

$$\begin{aligned} x_t &= S(t)\varphi + \int_0^t S(t-s)X_0 f(s)\,ds \\ &= S(t-r)S(r)\varphi + S(t-r)\int_0^r S(r-s)X_0 f(s)\,ds + \int_r^t S(t-s)X_0 f(s)\,ds \\ &= S(t-r)x_r + \int_0^{t-r} S(t-r-s)X_0 f(s+r)\,ds, \end{aligned}$$

which proves the lemma. □

2.4 Fixed point theorems

First we recall the Schauder fixed point theorem.

Theorem 2.21 (see e.g. in [43]) *Let U be a closed, convex and bounded subset of a Banach-space X, and $f : U \to U$ be a completely continuous map. Then the map f has a fixed point in U.*

Let Y and Z be Banach-spaces, a map $S : Y \times Z \to Y$ is called uniform contraction, if there exists $0 \le \theta < 1$ such that $|S(y,z) - S(\bar{y},z)|_Y \le \theta |y - \bar{y}|_Y$ for all $y, \bar{y} \in Y$ and $z \in Z$. The following theorem holds

Theorem 2.22 (Uniform Contraction Principle, see e.g. in [33]) *Let U and V be open sets in the Banach spaces Y and Z, respectively, and let $\bar{U}$ be the closure of U, and $S : \bar{U} \times V \to \bar{U}$ a uniform contraction on $\bar{U}$. Then for all $z \in V$ the map $S(\cdot, z)$ has a unique fixed point $g(z)$. Moreover, if $S \in C^k(\bar{U} \times V, Y)$, $0 \le k < \infty$, then $g \in C^k(V, Y)$.*

In [33], Hale and Ladeira proved a generalization of this theorem to quasi-Banach spaces. Let Y be a linear space with two norms: $|\cdot|$ and $\|\cdot\|$. We say that $(Y, |\cdot|)$ is a quasi-Banach space with respect to the norm $\|\cdot\|$, if for all $R > 0$, $\left(\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R), |\cdot|\right)$ is a complete metric space, i.e., all the closed balls of Y at the origin corresponding to the $\|\cdot\|$ norm are complete sets in the $|\cdot|$ norm. We consider Y with the topology defined by the norm $|\cdot|$, i.e., by open, closed sets in Y we mean open, closed sets of Y in the norm $|\cdot|$. Introduce $\tilde{\mathcal{L}}(Y)$, the quasi-Banach space of linear operators $S : Y \to Y$ which are bounded in both $|\cdot|$ and $\|\cdot\|$ norms. (See [33].)

The following generalization of the Uniform Contraction Principle holds for quasi-Banach spaces:

Theorem 2.23 (see in [33]) *Let Z be a normed space, $(Y, |\cdot|)$ is a quasi-Banach space with respect to the norm $\|\cdot\|$. Let $U \subset Y$ be open, and $V \subset Z$ be open, and assume that $S : \bar{U} \times V \to \bar{U}$ satisfies* *(i) S is a uniform $|\cdot|$ and $\|\cdot\|$ contraction, i.e., there exists $0 \leq \theta < 1$ such that*

$$|S(y,z) - S(\bar{y},z)| \leq \theta |y - \bar{y}|, \qquad \text{for } y, \bar{y} \in \bar{U}, z \in V,$$

and

$$\|S(y,z) - S(\bar{y},z)\| \leq \theta \|y - \bar{y}\|, \qquad \text{for } y, \bar{y} \in \bar{U}, z \in V.$$

(ii) For each $\rho > 0$ there exists $R > 0$ such that

$$S\Big((\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap \bar{U}) \times \mathcal{G}_Z(\rho)\Big) \subset (\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap \bar{U}).$$

(iii) $S \in C^k(\bar{U} \times V)$ for some $k \geq 1$.

Then for each $z \in V$, there exist a unique fixed point $g(z)$ of $S(\cdot, z)$ in $\bar{U}$, and the map g is in $C^k(V, Y)$.

Chapter 3
WELL-POSEDNESS IN C

In this chapter we study the the nonlinear state-dependent delay system

$$\dot{x}(t) = f\left(t, x(t), \int_{-r}^{0} d_s\mu(s,t,x_t)\, x(t+s)\right), \qquad t \in [0,T], \tag{3.1}$$

where: $r > 0$, $T > 0$ or $T = \infty$ (in the latter case $[0,T]$ denotes $[0,\infty)$),
Ω_1, Ω_2 are open subsets of $\mathbb{R}^n$, Ω_3 is an open subset of C,
$f : [0,T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$,
$\mu(\cdot, t, \psi) \in \mathrm{NBV}$ for all $t \in [0,T]$, $\psi \in \Omega_3$,
$x_t : [-r, 0] \to \mathbb{R}^n, \quad x_t(s) \equiv x(t+s)$ for $s \in [-r, 0]$.

In order to evaluate $x(t)$, x_t and $x(t+s)$ in (3.1) at $t = 0$ we need an initial condition for $x(\cdot)$ on $[-r, 0]$, i.e.,

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{3.2}$$

Using the simplying notation introduced by (2.8), equation (3.1) can be written as

$$\dot{x}(t) = f\Big(t, x(t), \Lambda(t, x_t)\Big), \qquad t \in [0,T]. \tag{3.3}$$

Throughout this paper we shall assume that the initial time of the equation is at $t = 0$, i.e., the solution starts at $t = 0$. An IVP of the form

$$\dot{\bar{x}}(t) = \bar{f}\Big(t, \bar{x}(t), \int_{-r}^{0} d_s\bar{\mu}(s,t,\bar{x}_t)\, \bar{x}(t+s)\Big), \qquad t \in [\sigma, \sigma+T], \tag{3.4}$$

$$\bar{x}(t) = \bar{\varphi}(t), \quad t \in [\sigma - r, \sigma] \tag{3.5}$$

where: $r > 0$, $T > 0$ (or $T = \infty$), $\sigma \in \mathbb{R}$,
$\bar{f} : [\sigma, \sigma+T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$,
$\bar{\mu}(\cdot, t, \psi) \in \mathrm{NBV}$ for all $t \in [\sigma, \sigma+T]$, $\psi \in \Omega_3$
can be rewritten in the form (3.1)-(3.2) using the transformations

$$f(t,x,y) = \bar{f}(t+\sigma,x,y),\ \mu(s,t,\psi) = \bar{\mu}(s,t+\sigma,\psi),\ \varphi(t) = \bar{\varphi}(t+\sigma) \text{ and } x(t) = \bar{x}(t+\sigma). \tag{3.6}$$

We comment, that if the dependence on the initial time σ is relevant (e.g., we study the dependence of the solution on the initial time, or the actual initial time is not known, and the problem is to identify the initial time), we may explicitly keep σ in the equation as a parameter, and consider an equation of the form

$$\dot{x}(t) = f\Big(t, x(t), \Lambda(t, x_t), \sigma\Big), \qquad t \in [0,T],$$

where $f : [0,T] \times \Omega_1 \times \Omega_2 \times [\sigma_0, \sigma_1] \to \mathbb{R}^n$.

Note, that in (3.4)-(3.5) the functions $\bar{\varphi}$, $\bar{\mu}$ and $\bar{f}$ are defined on a set which depends on σ which is also a parameter in the equation, but in IVP (3.1)-(3.2) the parameters belong to fixed spaces.

In Section 3.1 we give conditions, under which IVP (3.1)-(3.2) has a solution on an interval $[0,\alpha] \subset [0,T]$, and study continuability of solutions. In Section 3.2 we investigate the uniqueness of solutions, and in Section 3.3 we show that solutions depend continuously (in appropriate norms) on the parameters of the IVP, i.e., on φ, μ and f. In Section 3.4 we study state-space candidates, e.g., $W^{1,\infty}$ and $W^{1,p}$, and argue why $W^{1,p}$ is the best choice in the state-dependent case as the state-space of the solutions. In Section 3.5 we give some remarks how the results can be extended for more general cases.

3.1 Existence of solutions

In this section we show that the hypotheses

(A1) $f \in BC\Big([0,T] \times \Omega_1 \times \Omega_2;\ \mathbb{R}^n\Big)$,

(A2) $\mu \in \Theta_C(T, \Omega_3)$,

(A3) $\varphi \in C$

are sufficient for local existence of a solution of IVP (3.1)-(3.2).

We say that a function $x(\cdot) : [-r,T] \to \mathbb{R}^n$ is a solution of IVP (3.1)-(3.2), if it is continuous on $[-r,T]$, satisfies initial condition (3.2), differentiable on $[0,T]$, and satisfies (3.1). If we want to emphasize that the solution of IVP (3.1)-(3.2) corresponds to the parameter $\gamma \equiv (\varphi, \mu, f)$, we use the notation $x(t;\gamma)$.

First we show that the functions, μ, in Examples 1.1–1.4 satisfy assumption (A2) under natural assumptions on the original equations.

Example 3.1 Clearly $\mu(s)$ in Example 1.1 satisfies (A2), because it is independent of t and ψ.

Example 3.2 In Example 1.2 μ depends only on s and t, and it is easy to see that $\mu(\cdot,t)$ satisfies

$$\int_{-r}^{0} d_s\mu(s,t)\xi(s) = \sum_{k=1}^{m} A_k(t)\xi(-\tau_k(t)) + \int_{-\tau_0}^{0} G(s,t)\xi(s)\,ds.$$

and for $\xi \in \overline{\mathcal{G}}_C(1)$ we have

$$\left|\int_{-r}^{0} d_s\mu(s,t)\xi(s)\right| \le \sum_{k=1}^{m} \|A_k(t)\| + \int_{-\tau_0}^{0} \|G(s,t)\|\,ds. \tag{3.7}$$

Clearly, μ is in $\Theta_C(T,\Omega_3)$, if we assume that:

(i) each $A_k(t)$ and $\tau_k(t)$ are continuous on $[0,T]$, $(k = 1, \ldots, m)$,

(ii) the function G satisfies a Lipschitz-condition of the form $\|G(s,t)-G(s,\bar{t})\| \le g(s)|t-\bar{t}|$, where $g \in L^1([-\tau_0,0];\mathbb{R})$.

If $T=\infty$, to obtain boundedness of (3.7), we also need that:

(iii) $A_k(t)$ be bounded on $[0,\infty)$, $(k=1,\dots,m)$,

(iv) $\|G(s,t)\| \le g_0(s)$ for $t \in [0,\infty)$ where $g_0 \in L^1([-\tau_0,0];\mathbb{R})$.

Example 3.3 The function $\mu(\cdot,t,\psi)$ in Example 1.3 satisfies

$$\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s) = \xi(-\tau(t,\psi)),$$

for arbitrary $t \in [0,T]$ and $\psi \in \Omega_3$, and hence for $\xi \in \overline{\mathcal{G}}_C(1)$ we have

$$\left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right| \le 1.$$

Therefore (A2) is satisfied if $\tau(t,\psi)$ is continuous in t and ψ.

Example 3.4 In Example 1.4 we have that

$$\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s) = \sum_{k=1}^{m} A_k(t)\xi(-\tau_k(t,\psi)) + \int_{-\tau_0}^{0} G(s,t,\psi)\xi(s)\,ds,$$

and therefore for $\xi \in \overline{\mathcal{G}}_C(1)$ it follows that

$$\left|\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\right| \le \sum_{k=1}^{m} \|A_k(t)\| + \int_{-\tau_0}^{0} \|G(s,t,\psi)\|\,ds.$$

Similarly to that in Example 1.2, the following assumptions imply (A2):

(i) each $A_k(t)$ and $\tau_k(t,\psi)$ are continuous on $[0,T]$ and $[0,T]\times\Omega_3$, $(k=1,\dots,m)$, respectively,

(ii) the function G satisfies a Lipschitz-condition of the form

$$\|G(s,t,\psi)-G(s,\bar{t},\bar{\psi})\| \le g(s)\Big(|t-\bar{t}| + |\psi-\bar{\psi}|_C\Big), \tag{3.8}$$

for $s \in [-\tau_0,0]$, $t,\bar{t} \in [0,T]$, and $\psi,\bar{\psi} \in \Omega_3$, where $g \in L^1([-\tau_0,0];\mathbb{R})$,

and for the case $T=\infty$,

(iii) $A_k(t)$ is bounded on $[0,\infty)$, $(k=1,\dots,m)$,

(iv) $\|G(s,t,\psi)\| \le g_0(s)$, for all $t \in [0,\infty)$, $\psi \in \Omega_3$, where $g_0 \in L^1([-\tau_0,0];\mathbb{R})$.

Note, that if in Example 1.4 there is no discrete delay, (i.e., $A_k(t) = 0$ for all $t \geq 0$, $k = 1, 2, \ldots, m$), then the above Lipschitz-continuity of G implies that the corresponding $\mu \in BC([0,T] \times \Omega_3;\ \mathrm{NBV})$.

We note that the conditions in the above examples guaranteeing assumption (A2) are natural assumptions for the existence of solutions of the corresponding equations.

We introduce the notation $\Gamma_0(T, \Omega_1, \Omega_2, \Omega_3) \equiv C \times \Theta_C(T, \Omega_3) \times BC\Big([0,T] \times \Omega_1 \times \Omega_2;\ \mathbb{R}^n\Big)$ for our parameter space, which is a normed linear space using the product norm $\|\gamma\|_{\Gamma_0} \equiv |\varphi|_C + \|\mu\| + \|f\|$.

Next we shall study the existence of solutions, $x(t;\gamma)$, of IVP (3.1)-(3.2) corresponding to a given parameter $\gamma = (\varphi, \mu, f)$. Clearly a necessary condition for existence of $x(t;\gamma)$ is that the second argument of the function f in (3.1) should belong to Ω_1, the third argument to Ω_2, and the third argument of μ should be in Ω_3. Hence the initial condition for x yields that $x(0) = \varphi(0) \in \Omega_1$, $x_0 = \varphi \in \Omega_3$ and

$$\int_{-r}^{0} d_s\mu(s, 0, x_0)\, x(s) = \int_{-r}^{0} d_s\mu(s, 0, \varphi)\, \varphi(s) \in \Omega_2.$$

Therefore the feasible parameters of IVP (3.1)-(3.2) belong to the set

$$\begin{aligned} \Pi_0(T, \Omega_1, \Omega_2, \Omega_3) \quad \equiv \quad & \Big\{(\varphi, \mu, f) \in \Gamma_0(T, \Omega_1, \Omega_2, \Omega_3)\ :\ \varphi(0) \in \Omega_1,\ \varphi \in \Omega_3, \\ & \qquad \text{and} \int_{-r}^{0} d_s\mu(s, 0, \varphi)\, \varphi(s) \in \Omega_2\Big\}. \end{aligned} \tag{3.9}$$

We introduce the new variable $y(t) \equiv x(t) - \tilde{\varphi}(t)$, where $\tilde{\varphi}$ is defined by (2.9). Then IVP (3.1)-(3.2) becomes

$$\dot{y}(t) \;=\; f\left(t,\, y(t) + \tilde{\varphi}(t),\, \int_{-r}^{0} d_s\mu(s, t, y_t + \tilde{\varphi}_t)\Big[y(t+s) + \tilde{\varphi}(t+s)\Big]\right), \quad t \in [0,T], \tag{3.10}$$

$$y(t) \;=\; 0, \qquad t \in [-r, 0]. \tag{3.11}$$

Note, that by using the notation (2.7), the third argument of f in (3.10) can be written as $\Lambda(t, y_t + \tilde{\varphi}_t)$ or if we want to emphasize the μ-dependence, as $\Lambda_\mu(t, y_t + \tilde{\varphi}_t)$.

The following lemma has important consequences.

Lemma 3.5 *The function*

$$[0,\alpha] \times C \times \Theta_C(T, \Omega_3) \times \overline{\mathcal{G}}_{C_\alpha}(\beta) \to \mathbb{R}^n, \quad (u, \varphi, \mu, y) \mapsto \Lambda_\mu(u, y_u + \tilde{\varphi}_u) \tag{3.12}$$

is continuous, whenever it is defined.

Proof Fix $u^0 \in [0,\alpha]$, $\varphi^0 \in C$, $\mu^0 \in \Theta_C(T,\Omega_3)$, and $y^0 \in \overline{\mathcal{G}}_{C_\alpha}(\beta)$ for which $\Lambda_{\mu^0}(u^0, y^0_{u^0} + \tilde{\varphi}^0_u)$ is defined. By elementary manipulations and estimate (2.5) we have that

$$\begin{aligned}
&|\Lambda_\mu(u, y_u + \tilde{\varphi}_u) - \Lambda_{\mu^0}(u^0, y^0_{u^0} + \tilde{\varphi}^0_u)| \\
&\quad\le \left| \int_{-r}^0 d_s\Big[\mu^0(s,u,y_u+\tilde{\varphi}_u) - \mu^0(s,u,y^0_{u^0} + (\tilde{\varphi^0})_{u^0})\Big]\Big[y^0(u^0+s) + \tilde{\varphi}^0(u^0+s)\Big] \right| \\
&\qquad + \left| \int_{-r}^0 d_s\Big[\mu(s,u,y_u+\tilde{\varphi}_u) - \mu^0(s,u,y_u+\tilde{\varphi}_u)\Big]\Big[y(u+s) + \tilde{\varphi}(u+s)\Big] \right| \\
&\qquad + \left| \int_{-r}^0 d_s\mu^0(s,u,y_u+\tilde{\varphi}_u)\Big[y(u+s) + \tilde{\varphi}(u+s) - (y^0(u^0+s) + \tilde{\varphi}^0(u^0+s))\Big] \right| \\
&\quad\le \left| \int_{-r}^0 d_s\Big[\mu^0(s,u,y_u+\tilde{\varphi}_u) - \mu^0(s,u^0,y^0_{u^0} + (\tilde{\varphi^0})_{u^0})\Big]\Big[y^0(u^0+s) + \tilde{\varphi^0}(u^0+s)\Big] \right| \\
&\qquad + \|\mu-\mu^0\| \sup_{-r\le s\le 0}(|y(u+s)| + |\tilde{\varphi}(u+s)|) \\
&\qquad + \|\mu^0\| |y_u + \tilde{\varphi}_u - y^0_{u^0} - (\tilde{\varphi^0})_{u^0}|_C. \qquad (3.13)
\end{aligned}$$

Inequality (3.13), the definition of the norms $|\cdot|_C$, $|\cdot|_{C_\alpha}$, and inequality (2.10) imply

$$\begin{aligned}
&|\eta(u;\gamma,y) - \eta(u^0;\gamma^0,y^0)| \\
&\quad\le \left| \int_{-r}^0 d_s\Big[\mu^0(s,u,y_u+\tilde{\varphi}_u) - \mu^0(s,u^0,y^0_{u^0} + (\tilde{\varphi^0})_{u^0})\Big]\Big[y^0(u^0+s) + \tilde{\varphi^0}(u^0+s)\Big] \right| \\
&\qquad + \|\mu-\mu^0\| \sup_{-r\le s\le u}(|y(s)| + |\tilde{\varphi}(s)|) + \|\mu^0\| |y_u + \tilde{\varphi}_u - y^0_{u^0} - (\tilde{\varphi^0})_{u^0}|_C \\
&\quad\le \left| \int_{-r}^0 d_s\Big[\mu^0(s,u,y_u+\tilde{\varphi}_u) - \mu^0(s,u,y^0_{u^0} + (\tilde{\varphi^0})_{u^0})\Big]\Big[y^0(u^0+s) + \tilde{\varphi^0}(u^0+s)\Big] \right| \\
&\qquad + \|\mu-\mu^0\|(|y|_{C_\alpha} + |\varphi|_C) + \|\mu^0\| |y_u + \tilde{\varphi}_u - y^0_{u^0} - (\tilde{\varphi^0})_{u^0}|_C. \qquad (3.14)
\end{aligned}$$

In view of (3.14) and Lemma 2.8, it is enough to show for finishing the proof of the lemma, that $|y_u + \tilde{\varphi}_u - y^0_{u^0} - (\tilde{\varphi^0})_{u^0}|_C \to 0$, as $\varphi \to \varphi^0$, $\mu \to \mu^0$, $u \to u^0$, and $y \to y^0$. Consider

$$\begin{aligned}
&|y_u + \tilde{\varphi}_u - y^0_{u^0} - (\tilde{\varphi^0})_{u^0}|_C \\
&\quad = \sup_{-r\le s\le 0}\Big| y(u+s) + \tilde{\varphi}(u+s) - y^0(u^0+s) - \tilde{\varphi^0}(u^0+s)\Big| \\
&\quad \le \sup_{-r\le s\le 0}\Big(|y(u+s) - y^0(u+s)| + |\tilde{\varphi}(u+s) - \tilde{\varphi^0}(u+s)|\Big) \\
&\qquad + \sup_{-r\le s\le 0}\Big(|y^0(u+s) - y^0(u^0+s)| + |\tilde{\varphi^0}(u+s) - \tilde{\varphi^0}(u^0+s)|\Big). \qquad (3.15)
\end{aligned}$$

Inequality (2.10) yields that

$$\sup_{-r\le s\le 0} |\tilde{\varphi}(u+s) - \tilde{\varphi^0}(u+s)| \le |\varphi - \varphi^0|_C. \qquad (3.16)$$

It follows from (3.15) using the definition of ω_y, $\omega_{\tilde{\varphi}}$, relations (2.11) and (3.16) that

$$|y_u + \tilde{\varphi}_u - y^0_{u^0} - (\tilde{\varphi^0})_{u^0}|_C$$

$$\leq \sup_{-r\leq s\leq u} |y(s)-y^0(s)| + |\varphi-\varphi^0|_C + \omega_{y^0}(|u-u^0|;\,\alpha) + \omega_{\tilde{\varphi}^0}(|u-u^0|)$$
$$\leq |y-y^0|_{C_\alpha} + |\varphi-\varphi^0|_C + \omega_{y^0}(|u-u^0|;\,\alpha) + \omega_{\varphi^0}(|u-u^0|). \tag{3.17}$$

Using the uniform continuity of functions y^0 and φ^0 over intervals $[-r,\alpha]$ and $[-r,0]$, respectively, we get

$$\begin{aligned} \omega_{y^0}(h;\alpha) &\to 0 \qquad \text{as } h\to 0,\\ \omega_{\varphi^0}(h) &\to 0 \qquad \text{as } h\to 0. \end{aligned} \tag{3.18}$$

Relations (3.17) and (3.18) yield that $|y_u+\tilde{\varphi}_u - y^0_{u^0} - (\tilde{\varphi^0})_{u^0}|_C \to 0$ as $\varphi\to\varphi^0$, $\mu\to\mu^0$, $u\to u^0$, and $y\to y^0$, therefore we have finished the proof of the lemma. □

Lemma 3.5 and the continuity of f yield the next lemma immediately.

Lemma 3.6 *Fix $\gamma=(\varphi,\mu,f)\in\Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$ and $y\in\overline{\mathcal{G}}_{C_\alpha}(\beta)$. Assume that there exists $0<\alpha\leq T$ such that $y(u)+\tilde{\varphi}(u)\in\Omega_1$, $y_u+\tilde{\varphi}_u\in\Omega_3$ for $u\in[0,\alpha]$, and the function $\Lambda_\mu(u,y_u+\tilde{\varphi}_u)$ is defined and satisfies $\Lambda_\mu(u,y_u+\tilde{\varphi}_u)\in\Omega_2$ for $u\in[0,\alpha]$. Then the function*

$$[0,\alpha]\to\mathbb{R}^n, \qquad u\mapsto f\Big(u, y(u)+\tilde{\varphi}(u), \Lambda_\mu(u,y_u+\tilde{\varphi}_u)\Big)$$

is continuous on $[0,\alpha]$.

Using Lemma 3.6 we can make the following observation.

Lemma 3.7 *IVP (3.10)-(3.11) is equivalent to the integral equation*

$$y(t)=\begin{cases} 0, & t\in[-r,0],\\ \displaystyle\int_0^t f\left(u, y(u)+\tilde{\varphi}(u), \int_{-r}^0 d_s\mu(s,u,y_u+\tilde{\varphi}_u)\big[y(u+s)+\tilde{\varphi}(u+s)\big]\right)du, & t\in[0,T]. \end{cases} \tag{3.19}$$

The next theorem guarantees the existence of solutions of IVP (3.10)-(3.11) for a fixed parameter $\gamma^0\in\Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$ and in a small neighborhood of this parameter.

Theorem 3.8 *Given $\gamma^0\in\Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$ then there exist $\alpha=\alpha(\gamma^0)>0$ and $\delta=\delta(\gamma^0)>0$ such that if $\gamma\in\Gamma_0(T,\Omega_1,\Omega_2,\Omega_3)$ and $\|\gamma-\gamma^0\|_{\Gamma_0}<\delta$ then $\gamma\in\Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$, and IVP (3.10)-(3.11) corresponding to γ has a solution, $y(t;\gamma)$, on $[-r,\alpha]$.*

Proof Let $\gamma^0=(\varphi^0,\mu^0,f^0)\in\Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$. Then by the definition of $\Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$ we have that

$$u^0\equiv\varphi^0(0)\in\Omega_1, \quad v^0\equiv\int_{-r}^0 d_s\mu^0(s,0,\varphi^0)\varphi^0(s)\in\Omega_2, \quad \text{and} \quad \varphi^0\in\Omega_3.$$

Using that $\Omega_1,\Omega_2,\Omega_3$ are open subsets of $\mathbb{R}^n$ and C, respectively, we have that there exists $\delta_1>0$ such that if $|u-u^0|<\delta_1$, $|v-v^0|<\delta_1$ and $|\varphi-\varphi^0|_C<\delta_1$ then $x\in\Omega_1$, $y\in\Omega_2$ and $\varphi\in\Omega_3$.

Pick $M>0$ and $\delta_2>0$ such that $|f^0(0,u^0,v^0)|<M-\delta_2$. The function f^0 is continuous, therefore there exist $\alpha^*>0$ and $\delta_3>0$ such that $\delta_3\le\delta_1$, and if $t\in[0,\alpha^*]$, $|u-u^0|<\delta_3$ and $|v-v^0|<\delta_3$ then $|f^0(t,u,v)|<M-\delta_2$. Therefore we have that

$$|f(t,u,v)|<M \quad \text{for} \quad t\in[0,\alpha^*],\ |u-u^0|<\delta_3,\ |v-v^0|<\delta_3 \text{ and } \|f-f^0\|<\delta_2. \tag{3.20}$$

The uniform continuity of φ^0 on $[-r,0]$ and the definition of $\omega_{\varphi^0}(h)$ imply that $\omega_{\varphi^0}(h)\to 0$ as $h\to 0$, therefore we can select $\alpha^{**}>0$ such that

$$\alpha^{**}\le\alpha^*,\quad \omega_{\varphi^0}(\alpha^{**})<\delta_3/3.$$

Then we have for $t\in[0,\alpha^{**}]$, $|\varphi-\varphi^0|_C<\delta_3/3$, and for $y\in\overline{\mathcal{G}}_{C_{\alpha^{**}}}(\delta_3/3)$ that

$$\begin{aligned} |y(t)+\tilde\varphi(t)-u^0| &= |y(t)+\varphi(0)-\varphi^0(0)|\\ &\le |y(t)|+|\varphi(0)-\varphi^0(0)|\\ &\le \delta_3/3+|\varphi-\varphi^0|_C\\ &< \delta_3/3+\delta_3/3\\ &< \delta_1, \end{aligned} \tag{3.21}$$

hence $y(t)+\tilde\varphi(t)\in\Omega_1$. Similarly, for $t\in[0,\alpha^{**}]$, $|\varphi-\varphi^0|_C<\delta_3/3$, and for $y\in\overline{\mathcal{G}}_{C_{\alpha^{**}}}(\delta_3/3)$ the definition of $\omega_{\tilde{\varphi^0}}(h)$, relations (2.10), (2.11), (3.16), the monotonicity of $\omega_\varphi(h)$ in h, and the choice of α^{**} and δ_3 imply that

$$\begin{aligned} |y_t+\tilde\varphi_t-\varphi^0|_C &\le \sup_{-r\le s\le 0}\Big(|y(t+s)|+|\tilde\varphi(t+s)-\varphi^0(s)|\Big)\\ &< \delta_3/3+\sup_{-r\le s\le 0}\Big(|\tilde\varphi(t+s)-\tilde{\varphi^0}(t+s)|+|\tilde{\varphi^0}(t+s)-\varphi^0(s)|\Big)\\ &\le \delta_3/3+|\varphi-\varphi^0|_C+\sup_{-r\le s\le 0}|\tilde{\varphi^0}(t+s)-\tilde{\varphi^0}(s)|\\ &\le \delta_3/3+|\varphi-\varphi^0|_C+\omega_{\tilde{\varphi^0}}(t)\\ &= \delta_3/3+|\varphi-\varphi^0|_C+\omega_{\varphi^0}(t)\\ &< \delta_3/3+\delta_3/3+\delta_3/3\\ &\le \delta_1. \end{aligned} \tag{3.22}$$

Inequality (3.22) yields that for $t\in[0,\alpha^{**}]$, $\|\gamma-\gamma^0\|_{\Gamma_0}<\delta_3/3$ (and hence for $|\varphi-\varphi^0|_C<\delta_3/3$), and for $y\in\overline{\mathcal{G}}_{C_{\alpha^{**}}}(\delta_3/3)$ we have that $y_t+\tilde\varphi_t\in\Omega_3$, hence $\Lambda_\mu(t,y_t+\tilde\varphi_t)$ is defined for all $\mu\in\Theta_C(T,\Omega_3)$. Using our simplifying notation defined by (2.7) we have that

$$v^0=\int_{-r}^0 d_s\mu^0(s,0,\varphi^0)\varphi^0(s)=\Lambda_{\mu^0}(0,\bar 0_0+\tilde{\varphi^0_0}),$$

where $\bar 0\in\overline{\mathcal{G}}_{C_{\alpha^*}}(\beta^*)$ is the constant zero function. By the assumption $v^0=\Lambda_{\mu^0}(0,\bar 0_0+\tilde{\varphi^0_0})\in\Omega_2$, and by the continuity of the function given in (3.12) in u, φ, μ and in y (guaranteed by Lemma 3.5) there exist constants $\alpha>0$, $\delta_4>0$ and $\beta>0$ such that

$$\alpha\le\alpha^{**},\quad \delta_4\le\delta_3/3,\quad \text{and}\quad \beta\le\delta_3/3,$$

and

$$|\Lambda_\mu(t,y_t+\tilde\varphi_t)-v^0|<\delta_3$$

for $t \in [0, \alpha]$, $\|\gamma - \gamma^0\|_{\Gamma_0} < \delta_4$, $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$ and $|y - \bar{0}|_{C_\alpha **} < \beta$. Let $\delta = \min\{\delta_2, \delta_4\}$. Then we have that

$$y(t) + \tilde{\varphi}(t) \in \Omega_1, \qquad \Lambda_\mu(t, y_t + \tilde{\varphi}_t) \in \Omega_2 \quad \text{and} \quad y_t + \tilde{\varphi}_t \in \Omega_3 \tag{3.23}$$

for t, $y \in C_\alpha$ and $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$ such that $t \in [0, \alpha]$, $y \in \overline{\mathcal{G}}_{C_\alpha}(\beta)$ and $\|\gamma - \gamma_0\|_{\Gamma_0} < \delta$. Similarly, it follows that

$$|f(t, y(t) + \tilde{\varphi}(t), \Lambda_\mu(t, y_t + \tilde{\varphi}_t))| < M, \tag{3.24}$$

for t, $y \in C_\alpha$ and $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$ such that $t \in [0, \alpha]$, $y \in \overline{\mathcal{G}}_{C_\alpha}(\beta)$ and $\|\gamma - \gamma_0\|_{\Gamma_0} < \delta$. In particular, we have that

$$\varphi(0) \in \Omega_1, \quad \varphi \in \Omega_3 \text{ and } \int_{-r}^{0} d_s\mu(s, 0, \varphi)\varphi(s) \in \Omega_2, \qquad \text{for } \|\gamma - \gamma^0\|_{\Gamma_0} < \delta,$$

i.e., $\|\gamma - \gamma^0\|_{\Gamma_0} < \delta$ implies that $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$.

Define the operator

$$S : \overline{\mathcal{G}}_{C_\alpha}(\beta) \times \Big(\mathcal{G}_{\Gamma_0(T,\Omega_1,\Omega_2,\Omega_3)}\big(\gamma^0;\, \delta\big) \cap \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)\Big) \to C_\alpha \tag{3.25}$$

by

$$S(y, \gamma)(t) \equiv \begin{cases} 0 & t \in [-r, 0] \\ \displaystyle\int_0^t f\Big(u, y(u) + \tilde{\varphi}(u), \Lambda_\mu(u, y_u + \tilde{\varphi}_u)\Big)\, du, & t \in [0, \alpha]. \end{cases} \tag{3.26}$$

Relation (3.23) and Lemma 3.6 insure that S is well-defined. Usual estimates and (3.24) imply that

$$|S(y, \gamma)(t)| \;\leq\; M\alpha \tag{3.27}$$

$$|S(y, \gamma)(t) - S(y, \gamma)(\bar{t})| \;\leq\; M|t - \bar{t}| \tag{3.28}$$

hold for all $t, \bar{t}, \in [0, \alpha]$. These inequalities yield that the function $S(\cdot, \gamma) : \overline{\mathcal{G}}_{C_\alpha}(\beta) \to C_\alpha$ is completely continuous, because it maps the bounded subset $\overline{\mathcal{G}}_{C_\alpha}(\beta)$ of the space C_α into the compact set

$$K \equiv \{w \in C([-r, \alpha]; \mathbb{R}^n) \;:\; |w(t)| \leq M\alpha,\ |w(t) - w(\bar{t})| \leq M|t - \bar{t}|,\ t, \bar{t}, \in [-r, \alpha]\}. \tag{3.29}$$

Now choose α small enough that $M\alpha \leq \beta$ is satisfied. Then $S(\cdot, \gamma) : \overline{\mathcal{G}}_{C_\alpha}(\beta) \to \overline{\mathcal{G}}_{C_\alpha}(\beta)$ holds. Using that $\overline{\mathcal{G}}_{C_\alpha}(\beta)$ is a closed bounded convex subset of C_α the Schauder fixed-point theorem (Theorem 2.21) yields that for each $\gamma \in \mathcal{G}_{\Gamma_0(T,\Omega_1,\Omega_2,\Omega_3)}(\gamma^0;\, \delta) \cap \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$ there exists a fixed point of $S(\cdot, \gamma)$ in $\overline{\mathcal{G}}_{C_\alpha}(\beta)$, i.e., there exists a solution of (3.19) on $[-r, \alpha]$. □

Note that that by using the transformation $x(t) = y(t) + \tilde{\varphi}(t)$, Theorem 3.8 provides local existence of solutions of IVP (3.1)-(3.2).

In the remaining part of this section we study the continuation of solutions of (3.1)-(3.2). Fix $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$. Then by Theorem 3.8, IVP (3.10)-(3.11) has a solution, $y(t;\, \gamma)$, on $[-r, \alpha]$, i.e., IVP (3.1)-(3.2) has a solution, $x(t;\, \gamma)$, on $[-r, \alpha]$.

We say that $\hat{x}$ is a continuation of x, if there exists $\hat{\alpha} > \alpha$ such that $\hat{x}$ is defined on $[-r, \hat{\alpha}]$ (or on $[-r, \hat{\alpha})$), coincides with x on $[-r, \alpha]$, and $\hat{x}$ satisfies

$$x(t) = \begin{cases} \varphi(t), & t \in [-r, 0] \\ \tilde{\varphi}(0) + \displaystyle\int_0^t f\Big(u, x(u), \int_{-r}^0 d_s\mu(s, u, x_u)x(u+s)\Big)du, & t \in [0, \hat{\alpha}] \quad (\text{or } t \in [0, \hat{\alpha})). \end{cases} \tag{3.30}$$

We shall show, that finding a continuation of a solution, $x(t;\gamma)$, of IVP (3.1)-(3.2) existing originally on a closed interval $[-r, \alpha]$ is equivalent to solving IVP (3.1)-(3.2) with a parameter $\gamma^1 \in \Pi_0(T - \alpha, \Omega_1, \Omega_2, \Omega_3)$, therefore, by Theorem 3.8, the solution is always continuable to a larger closed time interval.

Let $x(t)$ be a solution of (3.1)-(3.2) on $[0, \alpha]$ corresponding to parameter $\gamma = (\varphi, \mu, f)$. Then a continuation of x should satisfy for $t > \alpha$

$$\begin{aligned} x(t) &= \tilde{\varphi}(0) + \int_0^\alpha f\Big(u, x(u), \int_{-r}^0 d_s\mu(s, u, x_u)x(u+s)\Big)du \\ &\quad + \int_\alpha^t f\Big(u, x(u), \int_{-r}^0 d_s\mu(s, u, x_u)x(u+s)\Big)du \\ &= x(\alpha) + \int_\alpha^t f\Big(u, x(u), \int_{-r}^0 d_s\mu(s, u, x_u)x(u+s)\Big)du. \end{aligned}$$

Define the function $\varphi^1 : [-r, 0] \to \mathbb{R}^n$ by

$$\varphi^1(t) \equiv \begin{cases} x(t+\alpha), & t \in [-\min\{\alpha, r\}, 0] \\ \varphi(t+\alpha), & t \in [-r, -\min\{\alpha, r\}], \end{cases}$$

and let $\tilde{\varphi^1} : [-r, \infty) \to \mathbb{R}^n$ be the extension of φ^1 defined by (2.9). Introducing the new variable

$$y(t) = x(t+\alpha) - \tilde{\varphi^1}(t)$$

we transform (3.30) into

$$y(t) = \begin{cases} 0, & t \in [-r, 0], \\ \displaystyle\int_0^t f\Big(u+\alpha, y(u) + \tilde{\varphi^1}(u), & \\ \qquad \displaystyle\int_{-r}^0 d_s\mu(s, u+\alpha, y_u + (\tilde{\varphi^1})_u)\big[y(u+s) + \tilde{\varphi^1}(u+s)\big]\Big)du, & t \geq 0. \end{cases} \tag{3.31}$$

Define

$$\begin{aligned} \mu^1(s, t, \psi) &\equiv \mu(s, t+\alpha, \psi), \qquad s \in [-r, 0], \quad t \in [-r, T-\alpha], \ \psi \in C, \\ f^1(t, x, y) &\equiv f(t+\alpha, x, y), \qquad t \in [-r, T-\alpha], \ x \in \Omega_1, \ y \in \Omega_2. \end{aligned}$$

Let $\gamma^1 \equiv (\varphi^1, \mu^1, f^1)$. Then clearly $\gamma^1 \in \Gamma_0(T - \alpha, \Omega_1, \Omega_2, \Omega_3)$. Moreover, we have that $\varphi^1(0) = x(\alpha) \in \Omega_1$, $\varphi^1 = x_\alpha \in \Omega_3$, and

$$\int_{-r}^0 d_s\mu^1(s, 0, \varphi^1)\varphi^1(s) = \int_{-r}^0 d_s\mu(s, \alpha, \varphi^1)\varphi^1(s) \in \Omega_2,$$

(because $x(t)$ is defined by the assumption at $t = \alpha$), hence $\gamma^1 \in \Pi_0(T-\alpha, \Omega_1, \Omega_2, \Omega_3)$. Therefore by Theorem 3.8 there exists $\alpha^1 = \alpha(\gamma^1)$ such that $\alpha + \alpha^1 \le T$, and (3.31) has solution on $[-r, \alpha^1]$, i.e., $x(\cdot;\gamma)$ is continuable to $[-r, \alpha+\alpha^1]$. If $\alpha + \hat{\alpha} < T$, then we can repeat the process, and extend the solution to a larger interval. If we don't reach T in finitely many steps, then using Zorn-lemma we can conclude that the solution has a noncontinuable extension (i.e., a solution which has no extension) defined on an open interval. Thus, we have proved the following lemma.

Lemma 3.9 *Fix $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$. Then there exists a noncontinuable solution corresponding to γ, $x(t;\gamma)$, of IVP (3.1)-(3.2) which is defined either on $[-r, T]$ or on an interval $[-r, b)$ (where $0 < b < T$).*

3.2 Uniqueness of solutions

In this section we show that if f is locally Lipschitz-continuous in its second and third arguments, and $\mu(s, t, \psi)$ is "weakly locally Lipschitz-continuous" in its third argument, (by which we mean that the corresponding $\lambda(t, \psi, \xi)$ is locally Lipschitz-continuous in ψ), and the initial function is Lipschitz-continuous (or equivalently, $\varphi \in W^{1,\infty}$), then the solution of IVP (3.10)-(3.11) is unique.

We shall use the following assumptions:

(A4) f is locally Lipschitz-continuous in its second and third arguments, i.e., for every $\alpha > 0$, $M > 0$ there exists a constant $L_1 = L_1(\alpha, M)$ such that for all $t \in [0, \alpha]$, $x, \bar{x} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_1$, and $y, \bar{y} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_2$

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \le L_1\Big(|x - \bar{x}| + |y - \bar{y}|\Big),$$

(A5) for all $\xi \in W^{1,\infty}$ the function $\lambda(t, \psi, \xi)$ defined by (2.7) is locally Lipschitz-continuous in ψ with a Lipschitz-constant of the form $L_2|\xi|_{W^{1,\infty}}$, i.e., for every $\alpha > 0$ and $M > 0$ there exists a constant $L_2 = L_2(\alpha, M)$ such that for all $\xi \in W^{1,\infty}$, $t \in [0, \alpha]$ and $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M) \cap \Omega_3$

$$|\lambda(t, \psi, \xi) - \lambda(t, \bar{\psi}, \xi)| \le L_2|\xi|_{W^{1,\infty}}|\psi - \bar{\psi}|_C,$$

(A6) $\varphi \in W^{1,\infty}$.

First we give conditions in Examples 1.1–1.4 which imply assumption (A5). We note that the functions μ used in Examples 1.1 and 1.2 do not depend on ψ, therefore assumption (A5) holds trivially in these cases.

Example 3.10 Let μ be defined as in Examples 1.3 and 3.3. Then

$$\lambda(t, \psi, \xi) = \xi(-\tau(t, \psi)).$$

Assume that $\xi \in W^{1,\infty}$. Then Lemma 2.3 yields that

$$\begin{aligned} |\lambda(t, \psi, \xi) - \lambda(t, \bar{\psi}, \xi)| &= |\xi(-\tau(t, \psi)) - \xi(-\tau(t, \bar{\psi}))| \\ &\le |\dot{\xi}|_{L^\infty}|\tau(t, \psi) - \tau(t, \bar{\psi})| \\ &\le |\xi|_{W^{1,\infty}}|\tau(t, \psi) - \tau(t, \bar{\psi})|. \end{aligned}$$

Thus if $\tau(t,\psi)$ is locally Lipschitz-continuous in ψ, i.e., for every $\alpha>0$, $M>0$ there exists a constant $L_\tau(\alpha,M)$ such that

$$|\tau(t,\psi)-\tau(t,\bar\psi)|\le L_\tau(\alpha,M)|\psi-\bar\psi|_C,\qquad \text{for } \psi,\bar\psi\in\overline{\mathcal{G}}_C(M)\cap\Omega_3,\quad t\in[0,\alpha], \tag{3.32}$$

then μ satisfies (A5) with $L_2(\alpha,M)=L_\tau(\alpha,M)$.

Example 3.11 Consider μ defined in Examples 1.4 and 3.4. In addition to assumption (i)–(iv) of Example 3.4, we assume that each τ_k satisfies (3.32) with Lipschitz-constant L_{τ_k}. Then it is easy to see that

$$|\lambda(t,\psi,\xi)-\lambda(t,\bar\psi,\xi)|\le\left(\sum_{k=1}^m|\dot\xi|_{L^\infty}L_{\tau_k}(\alpha,M)\sup_{0\le t\le\alpha}\|A_k(t)\|+\int_{-\tau_0}^0 g(s)\,ds|\xi|_C\right)|\psi-\bar\psi|_C,$$

for $\xi\in W^{1,\infty}$, $t\in[0,\alpha]$, and $\psi,\bar\psi\in\overline{\mathcal{G}}_C(M)\cap\Omega_3$. Thus (A5) is satisfied with

$$L_2=\sum_{k=1}^m L_{\tau_k}(\alpha,M)\sup_{0\le t\le\alpha}\|A_k(t)\|+\int_{-\tau_0}^0 g(s)\,ds.$$

Lemma 3.12 *Let $\mu\in\Theta_C(T,\Omega_3)$ satisfy (A5). Then the function $\Lambda(t,\psi)$ defined by (2.8) satisfies the inequality*

$$|\Lambda(t,\psi)-\Lambda(t,\bar\psi)|\le\Big(\|\mu\|+L_2(\alpha,M)|\bar\psi|_{W^{1,\infty}}\Big)|\psi-\bar\psi|_C, \tag{3.33}$$

where $t\in[0,\alpha]$, $\psi,\bar\psi\in\overline{\mathcal{G}}_C(M)\cap\Omega_3$ and $\bar\psi\in W^{1,\infty}$.

Proof Let $\alpha,M>0$ be fixed, and $L_2(\alpha,M)$ be the corresponding constant from assumption (A5). Let ψ and $\bar\psi$ satisfy the assumptions of the lemma. Assumption (A5), inequality (2.5), and elementary estimates imply the inequalities

$$\begin{aligned}|\Lambda(t,\psi)-\Lambda(t,\bar\psi)| &\le |\lambda(t,\psi,\psi)-\lambda(t,\psi,\bar\psi)|+|\lambda(t,\psi,\bar\psi)-\lambda(t,\bar\psi,\bar\psi)|\\ &\le \left|\int_{-r}^0 d_s\mu(s,t,\psi)\Big[\psi(s)-\bar\psi(s)\Big]\right|+L_2(\alpha,\,M)|\bar\psi|_{W^{1,\infty}}|\psi-\bar\psi|_C\\ &\le \|\mu\||\psi-\bar\psi|_C+L_2(\alpha,\,M)|\bar\psi|_{W^{1,\infty}}|\psi-\bar\psi|_C.\end{aligned}$$

□

Lemma 3.13 *Assume that the parameter $\gamma=(\varphi,\mu,f)\in\Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$ satisfy (A1)-(A6). Let $x(t)$ be a solution of (3.1)-(3.2) on $[0,\alpha]$ corresponding to γ. Then*

(i) $x_t\in W^{1,\infty}$ for all $t\in[0,\alpha]$, moreover, $x_t\in C^1$ for $t\in[r,\alpha]$,

(ii) there exists a constant $M_1=M_1(\alpha,\|f\|,|\varphi|_{W^{1,\infty}})$ such that $|x_t|_{W^{1,\infty}}\le M_1$ for $t\in[0,\alpha]$.

Proof Let $\gamma = (\varphi, \mu, f)$, and $x(t)$ be a solution corresponding to γ. By (A6) the initial function is from $W^{1,\infty}$, therefore it is differentiable a.e. on $[-r, 0]$, hence Lemma 2.12, (3.1) and (3.2) imply that

$$\begin{aligned} \frac{d}{ds}x_t(s) &= \dot{x}(t+s) \\ &= \begin{cases} f(t+s, x(t+s), \Lambda(t+s, x_{t+s})), & t+s>0, \\ \dot{\varphi}(t+s), & t+s<0, \text{ for a.e. } s. \end{cases} \end{aligned} \tag{3.34}$$

Therefore x_t is differentiable for a.e. s. Moreover, if $t \geq r$ then $t+s \geq 0$ for all $s \in [-r, 0]$, therefore x_t is differentiable everywhere. Lemma 3.6 yields that the function $s \mapsto f(t+s, x(t+s), \Lambda(t+s, x_{t+s}))$ is continuous, therefore we have proved the second part of (i). To prove that $x_t \in W^{1,\infty}$, we have to show that $x_t(s)$ and $\frac{d}{ds}x_t(s)$ are bounded. It follows from (3.34) that

$$\left|\frac{d}{ds}x_t(s)\right| \leq \max\Big\{\|f\|, |\dot{\varphi}|_C\Big\}.$$

Moreover, (3.3), the definition of $\|f\|$ and $|\cdot|_C$ imply that for $t \geq 0$

$$\begin{aligned} |x(t)| &\leq |\varphi(0)| + \int_0^t \Big|f(u, x(u), \Lambda(u, x_u))\, du\Big| \\ &\leq |\varphi|_C + \|f\|t \\ &\leq |\varphi|_C + \|f\|\alpha, \end{aligned}$$

hence $|x_t|_C \leq |\varphi|_C + \|f\|\alpha$ for $t \in [0, \alpha]$. Therefore we have proved (i), and it is easy to see that (ii) is also satisfied with $M_1 = \max\Big\{\|f\|, |\dot{\varphi}|_C, |\varphi|_C + \|f\|\alpha\Big\}$. □

The next theorem shows that under assumptions (A1)–(A6), IVP (3.1)-(3.2) has a unique solution.

Theorem 3.14 *Let $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$ and assume that (A1)–(A6) are satisfied. Then there exists $\alpha > 0$ such that IVP (3.1)-(3.2) has a unique solution on $[0, \alpha]$.*

Proof Theorem 3.8 yields that there exists $\alpha > 0$ such that IVP (3.1)-(3.2) has a solution on $[0, \alpha]$. Suppose that $x(\cdot)$ and $z(\cdot)$ are two solutions of (3.1)-(3.2) on $[0, \alpha]$ corresponding to the same parameter $\gamma = (\varphi, \mu, f)$. It follows from Lemma 3.13 that $|x(t)|, |z(t)| \leq M_1$ for $t \in [0, \alpha]$. By (2.5) we have that

$$|\Lambda(t, x_t)| \leq \|\mu\||x_t|_C \leq \|\mu\|M_1,$$

and similarly $|\Lambda(t, z_t)| \leq \|\mu\|M_1$ for $t \in [0, \alpha]$. Let $M \equiv M_1 \max\{1, \|\mu\|\}$, and $L_1 = L_1(\alpha, M)$ given by (A4). The integrated form of (3.3), the Lipschitz-continuity of f, and simple estimates imply

$$\begin{aligned} |x(t) - z(t)| &\leq \int_0^t \Big|f(u, x(u), \Lambda(u, x_u)) - f(u, z(u), \Lambda(u, z_u))\Big| du \\ &\leq \int_0^t L_1\Big(|x(u) - z(u)| + |\Lambda(u, x_u) - \Lambda(u, z_u)|\Big) du, \qquad t \in [0, \alpha]. \end{aligned} \tag{3.35}$$

By the definition of M we have that $|x_t|_C, |z_t|_C \le M$, therefore if $L_2 = L_2(\alpha, M)$ is the constant from (A5), then using that by Lemma 3.13 $z_u \in W^{1,\infty}$ and $|z_u|_{W^{1,\infty}} \le M_1$, Lemma 3.12 and inequality (3.35) yield that

$$\begin{aligned} |x(t) - z(t)| &\le \int_0^t L_1\Big(|x(u) - z(u)| + (\|\mu\| + L_2|z_u|_{W^{1,\infty}})|x_u - z_u|_C\Big)\, du \\ &\le \int_0^t L_1\Big(1 + \|\mu\| + L_2 M_1\Big)|y_u - z_u|_C\, du, \qquad t \in [0, \alpha]. \end{aligned} \tag{3.36}$$

Lemma 2.14 applied to (3.36), using that $y(s) = z(s)$ for $s \in [-r, 0]$, leads to the inequality

$$\sup_{0\le s\le t} |y(s) - z(s)| \le \int_0^t L_1\Big(1 + \|\mu\| + L_2 M_1\Big) \sup_{0\le s\le u} |y(s) - z(s)|\, du, \qquad t \in [0, \alpha],$$

which by the Gronwall-Bellman inequality yields that $\sup_{0\le s\le t} |y(s)-z(s)| = 0$ for all $t \in [0, \alpha]$, and therefore the solution is unique. □

It is known, that without assumption (A4) we may loose uniqueness (take, e.g., $f(t, x, y) = \sqrt{t}$). The next example shows, that if $f(t, x, y)$ is not Lipschitz-continuous in y, then the corresponding IVP can have two solutions. The other two examples in this section show that if we violate assumptions (A5) and (A6), then we may also loose uniqueness of the solution.

Example 3.15 Consider the scalar IVP

$$\dot{x}(t) = 4\sqrt{x(t - \tau(t))}, \qquad t \ge 0, \tag{3.37}$$
$$x(t) = 0, \qquad -1 \le t \le 0, \tag{3.38}$$

where

$$\tau(t) = \min\{t/2, 1\}.$$

It is easy to see that IVP (3.37)-(3.38) has two solutions on $[0, 2]$: $x_1(t) = 0$ and $x_2(t) = t^2$.

Example 3.16 Consider the scalar IVP with state-dependent delay

$$\dot{x}(t) = x\Big(t - \tau(x(t))\Big), \qquad t \ge 0, \tag{3.39}$$
$$x(t) = -2t, \qquad -2 \le t \le 0, \tag{3.40}$$

where

$$\tau(x) \equiv 2\min\left\{\sqrt{|x|}, 1\right\}.$$

It is easy to check that this IVP has two solutions: $x_1(t) = 0$, $t \ge 0$ and $x_2(t) = t^2$ for $t \in [0, 1]$. We can rewrite (3.39)-(3.40) in the form

$$\dot{x}(t) = \int_{-2}^0 d_s\mu(s, x_t)x(t + s), \qquad t \ge 0, \tag{3.41}$$
$$x(t) = -2t, \qquad -2 \le t \le 0, \tag{3.42}$$

by defining

$$\mu(s, \psi) \equiv \chi_{[-\tau(\psi(0)),0]}(s), \qquad s \in [-2, 0].$$

We have that if $|\psi(0)| \le 1$ then

$$\lambda(\psi, \xi) = \int_{-r}^{0} d_s\mu(s,\psi)\xi(s) = \xi(-\tau(\psi(0))) = \xi\left(-2\sqrt{|\psi(0)|}\right),$$

which does not satisfy (A5). (It is enough to consider $\xi(s) = s$, and constant functions for ψ.)

Example 3.17 Consider the scalar IVP with state-dependent delay

$$\begin{aligned} \dot{x}(t) &= x\Big(t - \tau(x(t))\Big), \qquad t \ge 0 \\ x(t) &= \begin{cases} 1, & -2 \le t \le -1 \\ 1 - 2\sqrt{1+t}, & -1 \le t \le -\frac{3}{4} \\ \frac{4}{3}t + 1, & -\frac{3}{4} \le t \le 0, \end{cases} \end{aligned}$$

where $\tau(x) = \min\{|x|, 2\}$. The initial function is not Lipschitz-continuous (hence (A6) is not satisfied), therefore the uniqueness is not guaranteed by Theorem 3.14. In fact, the IVP has two solutions: $t+1$ is solution for $t \in [0,1]$ and the analytic expression on $[0, 0.5]$ for the other solution is $t + 1 - t^2$.

3.3 Continuous dependence on parameters

In this section we show that assuming uniqueness of the solution of IVP (3.10)-(3.11), the solution of IVP (3.10)-(3.11) (and therefore the solution of IVP (3.1)-(3.2)) depends continuously on the parameters, i.e., on γ.

The proof of the continuous dependence based on the following result.

Lemma 3.18 *The operator $S(\cdot,\cdot)$ defined by (3.25)-(3.26) is continuous on its domain.*

Proof Pick the sequence

$$(y^k, \gamma^k) \in \overline{\mathcal{G}}_{C_\alpha}(\beta) \times \Big(\mathcal{G}_{\Gamma_0(T,\Omega_1,\Omega_2,\Omega_3)}\Big(\gamma^0;\, \delta\Big) \cap \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)\Big)$$

such that

$$(y^k, \gamma^k) \to (\bar{y}, \bar{\gamma}) \in \overline{\mathcal{G}}_{C_\alpha}(\beta) \times \Big(\mathcal{G}_{\Gamma_0(T,\Omega_1,\Omega_2,\Omega_3)}\Big(\gamma^0;\, \delta\Big) \cap \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)\Big) \quad \text{as } k \to \infty,$$

where $\gamma^k = (\varphi^k, \mu^k, f^k)$ and $\bar{\gamma} = (\bar{\varphi}, \bar{\mu}, \bar{f})$. Using the continuity of the function $\bar{f}$, and that of the function defined by (3.12) for any fixed $u \in [0, \alpha]$, it is easy to see that for $u \in [0, \alpha]$

$$\begin{aligned} &\Big| f^k\Big(u, y^k(u) + \tilde{\varphi}^k(u), \Lambda_{\mu^k}(u, y_u + \tilde{\varphi}^k_u)\Big) - \bar{f}\Big(u, \bar{y}(u) + \tilde{\bar{\varphi}}(u), \Lambda_{\bar{\mu}}(u, \bar{y}_u + \tilde{\bar{\varphi}}_u)\Big)\Big| \\ &\quad \le \|f^k - f\| \\ &\qquad + \Big|\bar{f}\Big(u, y^k(u) + \tilde{\varphi}^k(u), \Lambda_{\mu^k}(u, y_u + \tilde{\varphi}^k_u)\Big) - \bar{f}\Big(u, \bar{y}(u) + \tilde{\bar{\varphi}}(u), \Lambda_{\bar{\mu}}(u, \bar{y}_u + \tilde{\bar{\varphi}}_u)\Big)\Big| \\ &\quad \to 0, \qquad \text{as } k \to \infty, \end{aligned}$$

hence the Lebesgue Dominated Convergence Theorem implies that

$$\int_0^t f^k\Big(u, y^k(u) + \tilde{\varphi}^k(u), \Lambda_{\mu^k}(u, y_u + \tilde{\varphi}_u^k)\Big)\, du \to \int_0^t \bar{f}\Big(u, \bar{y}(u) + \tilde{\bar{\varphi}}(u), \Lambda_{\bar{\mu}}(u, \bar{y}_u + \tilde{\bar{\varphi}}_u)\Big)\, du,$$

as $k \to \infty$, which yields that

$$S(y^k, \gamma^k)(t) \to S(\bar{y}, \bar{\gamma})(t) \qquad \text{for all } t \in [-r, \alpha]. \tag{3.43}$$

On the other hand $S(y^k, \gamma^k)$ belongs to a compact subset of $\overline{\mathcal{G}}_{C_\alpha}(\beta)$ according to the proof of Theorem 3.8, therefore an arbitrary subsequence of it contains a convergent subsubsequence, say $S(y^{k_j}, \gamma^{k_j})$, i.e.,

$$S(y^{k_j}, \gamma^{k_j}) \to y^* \in C_\alpha, \qquad \text{as } j \to \infty. \tag{3.44}$$

Combining (3.43) and (3.44) we get that $y^* = S(\bar{y}, \bar{\gamma})$. Therefore we have that an arbitrary subsequence of $S(y^k, \gamma^k)$ has a subsubsequence, which converges to $S(\bar{y}, \bar{\gamma})$, which implies that the sequence is convergent and

$$\Big| S(y^k, \gamma^k) - S(\bar{y}, \bar{\gamma}) \Big|_{C_\alpha} \to 0, \qquad \text{as } k \to \infty,$$

which proves the lemma. □

The following theorem shows that the solutions of IVP (3.10)-(3.11) depend continuously on parameters.

Theorem 3.19 *Suppose that given a parameter $\gamma^0 \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$, IVP (3.10)-(3.11) corresponding to γ^0 has a unique solution $y(t; \gamma^0)$ on $[0, \alpha]$, (where α is given by Theorem 3.8), and moreover a given sequence, $\gamma^k \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$, satisfies that $\gamma^k \to \gamma^0$ as $k \to \infty$. Then there exists $k_0 > 0$ such that if $k > k_0$, then IVP (3.10)-(3.11) corresponding to γ^k has a solution $y(t; \gamma^k)$, which exists on $[0, \alpha]$, and $y(t; \gamma^k) \to y(t; \gamma^0)$ as $k \to \infty$ uniformly on $[0, \alpha]$.*

Proof We use the notations of the proof of Theorem 3.8, i.e., let $\alpha > 0, \beta > 0$ and $\delta > 0$ such that (3.23) and (3.24) hold. Choose $k_0 > 0$ such that

$$\gamma^k \in \Big(\mathcal{G}_{\Gamma_0(T,\Omega_1,\Omega_2,\Omega_3)}\Big(\gamma^0; \delta\Big) \cap \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)\Big) \qquad \text{for } k > k_0.$$

Then by Theorem 3.8, for $k > k_0$, we have that $y(t; \gamma^k)$ exists on $[0, \alpha]$, and it is the fixed point of the operator $S(\cdot, \gamma^k)$ defined by (3.25)-(3.26), i.e.

$$y(\cdot; \gamma^k) = S(y(\cdot; \gamma^k), \gamma^k). \tag{3.45}$$

By the proof of Theorem 3.8 we know that $y(\cdot; \gamma^k) \in K$, where $K \subset \overline{\mathcal{G}}_{C_\alpha}(\beta)$ is the compact set defined by (3.29). Take an arbitrary subsequence of $\{y(\cdot; \gamma^k)\}_{k \geq k_0}$, then it contains a convergent subsubsequence. For notational convenience denote this subsubsequence again by $\{y(\cdot; \gamma^{k_j})\}$, i.e., we can assume that $y(\cdot; \gamma^{k_j}) \to y^* \in C_\alpha$, as $j \to \infty$. Then the continuity of $S(\cdot, \cdot)$ (see Lemma 3.18) and relation (3.45) imply that

$$y^* = S(y^*, \gamma^0),$$

i.e., y^* is a solution of IVP (3.10)-(3.11) corresponding to parameter γ^0. Then the assumed uniqueness of the solution at γ^0 yields that $y^* = y(\cdot;\gamma^0)$. Then using that this relation is obtained by selecting an arbitrary subsequence of $\{y(\cdot;\gamma^k)\}_{k\geq k_0}$, we get that it is a convergent sequence with limit $y(\cdot;\gamma^0)$. The proof of the theorem is complete. □

We comment that by using the transformation $x(t) = y(t) + \tilde{\varphi}(t)$, Theorem 3.19 provides continuous dependence on parameters of the solutions of IVP (3.1)-(3.2).

3.4 The state-spaces $W^{1,\infty}$ and $W^{1,p}$

Lemma 3.13 yields that the solution corresponding to $\gamma \in \Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$ always lies in $W^{1,\infty}$, hence we can use $W^{1,\infty}$ as the state-space of solutions. This is a natural choice, because uniqueness of solutions of (3.1)-(3.2) is guaranteed only for $W^{1,\infty}$ initial functions.

First we introduce a new parameter space accordingly to this new state-space of solutions:

$$\Gamma_1(T,\Omega_1,\Omega_2,\Omega_3) \equiv W^{1,\infty} \times \Theta_C(T,\Omega_3) \times BC\Big([0,T]\times\Omega_1\times\Omega_2;\ \mathbb{R}^n\Big),$$

where the norm of $\gamma = (\varphi,\mu,f) \in \Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)$ is defined by $\|\gamma\|_{\Gamma_1} \equiv |\varphi|_{W^{1,\infty}} + \|\mu\| + \|f\|$. Note, that the only difference between Γ_0 and Γ_1 is the space, and hence the norm of the φ-component.

Define the set of feasible parameters in Γ_1 by

$$\begin{aligned}\Pi_1(T,\Omega_1,\Omega_2,\Omega_3) \ \equiv\ \Big\{&(\varphi,\mu,f)\in\Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)\ :\ \varphi\in C,\ \varphi(0)\in\Omega_1,\ \varphi\in\Omega_3,\\ &\text{and} \int_{-r}^{0} d_s\mu(s,0,\varphi)\,\varphi(s)\in\Omega_2\Big\}.\end{aligned} \tag{3.46}$$

If we compare (3.46) to (3.9), we can see that

$$\Pi_1(T,\Omega_1,\Omega_2,\Omega_3) \subset \Pi_0(T,\Omega_1,\Omega_2,\Omega_3) \tag{3.47}$$

as sets, and hence Theorems 3.8 and 3.14 imply that for all $\gamma \in \Pi_1(T,\Omega_1,\Omega_2,\Omega_3)$ IVP (3.1)-(3.2) has a unique solution. Since

$$\|\gamma\|_{\Gamma_0} \leq \|\gamma\|_{\Gamma_1}, \qquad \gamma \in \Gamma_1(T,\Omega_1,\Omega_2,\Omega_3), \tag{3.48}$$

Theorem 3.19 yields, that the function

$$\Big(\Pi_1(T,\Omega_1,\Omega_2,\Omega_3) \subset \Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)\Big) \to C, \qquad \gamma \mapsto x(\cdot;\gamma)_t$$

is continuous for all $t \in [0,\alpha]$. Here and later $x(\cdot;\gamma)_t$ denotes the segment function at t of the solution corresponding to parameter γ.

The next theorem shows, that if we assume (A1)–(A6), then we have a stronger result, namely, the function

$$\Big(\Pi_1(T,\Omega_1,\Omega_2,\Omega_3) \subset \Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)\Big) \to W^{1,\infty}, \qquad \gamma \mapsto x(\cdot;\gamma)_t$$

is continuous for all $t \in [0,\alpha]$, and in fact, it is locally Lipschitz-continuous.

Theorem 3.20 *Assume that* $\bar\gamma = (\bar\varphi, \bar\mu, \bar f) \in \Pi_1(T, \Omega_1, \Omega_2, \Omega_3)$ *satisfies (A1)-(A6). Then there exist constants* $\alpha > 0$, $\delta > 0$ *and* $L_3 = L_3(\alpha, \bar\gamma, \delta)$, *such that IVP (3.1)-(3.2) has a unique solution on* $[0, \alpha]$ *for all* $\gamma \in \mathcal{G}_{\Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)}(\bar\gamma;\, \delta)$, *and*

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_{W^{1,\infty}} \le L_3\|\gamma - \bar\gamma\|_{\Gamma_1}, \quad t \in [0, \alpha].$$

Proof The existence of $\alpha > 0$ and $\delta > 0$ satisfying the first part of the statement of the theorem follows from Theorems 3.8, 3.14 and relations (3.47), (3.48).

In this proof, to indicate dependence of Λ on μ, we shall use the notation $\Lambda_\mu(t, \psi)$ for the function defined by (2.8) corresponding to μ. By using that for $\gamma = (\varphi, \mu, f) \in \mathcal{G}_{\Gamma_1}(\bar\gamma;\, \delta)$ we have

$$|\varphi|_{W^{1,\infty}} < \|\bar\gamma\|_{\Gamma_1} + \delta, \quad \|\mu\| < \|\bar\gamma\|_{\Gamma_1} + \delta, \quad \text{and} \quad \|f\| < \|\bar\gamma\|_{\Gamma_1} + \delta, \tag{3.49}$$

therefore the constant $M_1 \equiv M_1(\alpha, \|\bar\gamma\|_{\Gamma_1} + \delta, \|\bar\gamma\|_{\Gamma_1} + \delta)$ defined in Lemma 3.13 satisfies

$$|x(\cdot;\gamma)_t|_{W^{1,\infty}} \le M_1, \qquad \gamma \in \mathcal{G}_{\Gamma_1}(\bar\gamma;\, \delta), \quad t \in [0, \alpha].$$

This inequality, together with (3.49) and (2.5), implies that

$$\begin{aligned} |\Lambda_\mu(t, x(\cdot;\gamma)_t)| &\le \|\mu\||x(\cdot;\gamma)|_C \\ &< (\|\bar\gamma\|_{W^{1,\infty}} + \delta)M_1, \qquad \gamma \in \mathcal{G}_{\Gamma_1}(\bar\gamma;\, \delta), \quad t \in [0, \alpha]. \end{aligned}$$

Define $M \equiv \max\{1, \|\bar\gamma\|_{W^{1,\infty}} + \delta\}M_1$, and let $L_1 = L_1(\alpha, M)$ be the constant from (A4). Using the integrated form of (3.3), the definition of $\|f\|$, and assumption (A4) we get for $t \in [0, \alpha]$ that

$$\begin{aligned} &|x(t;\gamma) - x(t;\bar\gamma)| \\ &\le |\varphi(0) - \bar\varphi(0)| + \int_0^t \big|f(u, x(u;\gamma), \Lambda_\mu(u;(x(\cdot;\gamma)_u)) - \bar f(u, x(u;\bar\gamma), \Lambda_{\bar\mu}(u;(x(\cdot;\bar\gamma)_u))\big|\, du \\ &\le |\varphi(0) - \bar\varphi(0)| + \int_0^t \big|f(u, x(u;\gamma), \Lambda_\mu(u;(x(\cdot;\gamma)_u)) - \bar f(u, x(u;\gamma), \Lambda_\mu(u;(x(\cdot;\gamma)_u))\big|\, du \\ &\quad + \int_0^t \big|\bar f(u, x(u;\gamma), \Lambda_\mu(u;(x(\cdot;\gamma)_u)) - \bar f(u, x(u;\bar\gamma), \Lambda_{\bar\mu}(u;(x(\cdot;\bar\gamma)_u))\big|\, du \\ &\le |\varphi(0) - \bar\varphi(0)| + \alpha\|f - \bar f\| \\ &\quad + L_1 \int_0^t |x(u;\gamma) - x(u;\bar\gamma)| + |\Lambda_\mu(u;(x(\cdot;\gamma)_u) - \Lambda_{\bar\mu}(u;(x(\cdot;\bar\gamma)_u)|du. \end{aligned} \tag{3.50}$$

By applying Lemma 3.12, the definition of M and M_1, we obtain for $u \in [0, \alpha]$

$$\begin{aligned} &|\Lambda_\mu(u;(x(\cdot;\gamma)_u) - \Lambda_{\bar\mu}(u;(x(\cdot;\bar\gamma)_u)| \\ &\quad \le |\Lambda_\mu(u;(x(\cdot;\gamma)_u) - \Lambda_{\bar\mu}(u;(x(\cdot;\gamma)_u)| + |\Lambda_{\bar\mu}(u;(x(\cdot;\gamma)_u) - \Lambda_{\bar\mu}(u;(x(\cdot;\bar\gamma)_u)| \\ &\quad \le \|\mu - \bar\mu\||x(\cdot;\gamma)_u|_C + (\|\bar\mu\| + L_2|x(\cdot;\bar\gamma)_u|_{W^{1,\infty}})|x(\cdot;\gamma)_u - x(\cdot;\bar\gamma)_u|_C \\ &\quad \le \|\mu - \bar\mu\|M_1 + (\|\bar\mu\| + L_2M_1)|x(\cdot;\gamma)_u - x(\cdot;\bar\gamma)_u|_C, \end{aligned} \tag{3.51}$$

where $L_2 = L_2(\alpha, M)$ is the constant from (A5). Combining (3.50), (3.51) and the definition of $|\cdot|_{W^{1,\infty}}$, we get

$$|x(t;\gamma) - x(t;\bar\gamma)|$$

$$
\begin{aligned}
&\leq |\varphi(0)-\bar\varphi(0)| + \alpha\|f-\bar f\| + \alpha\|\mu-\bar\mu\|M_1 \\
&\qquad + L_1\int_0^t |x(u;\gamma)-x(u;\bar\gamma)| + (\|\bar\mu\|+L_2M_1)|x(\cdot;\gamma)_u - x(\cdot;\bar\gamma)_u|_C\,du \\
&\leq |\varphi-\bar\varphi|_{W^{1,\infty}} + \alpha\|f-\bar f\| + \alpha\|\mu-\bar\mu\|M_1 \\
&\qquad + L_1\int_0^t (1+\|\bar\mu\|+L_2M_1)|x(\cdot;\gamma)_u - x(\cdot;\bar\gamma)_u|_C\,du \\
&\leq \|\gamma-\bar\gamma\|_{\Gamma_1}\max\{1,\alpha,\alpha M_1\} \\
&\qquad + L_1\int_0^t (1+\|\bar\mu\|+L_2M_1)|x(\cdot;\gamma)_u - x(\cdot;\bar\gamma)_u|_C\,du. \qquad (3.52)
\end{aligned}
$$

Using Lemma 2.14, inequality (3.52) yields

$$
\begin{aligned}
&|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C \\
&\quad\leq \|\gamma-\bar\gamma\|_{\Gamma_1}\max\{1,\alpha,\alpha M_1\} + L_1\int_0^t (1+\|\bar\mu\|+L_2M_1)|x(\cdot;\gamma)_u - x(\cdot;\bar\gamma)_u|_C\,du,
\end{aligned}
$$

which, by the Gronwall-Bellman inequality, implies that

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C \leq \|\gamma-\bar\gamma\|_{\Gamma_1}\max\{1,\alpha,\alpha M_1\}\exp\left(L_1(1+\|\bar\mu\|+L_2M_1)\alpha\right).$$

Define the constant $K_1 \equiv \max\{1,\alpha,\alpha M_1\}\exp\left(L_1(1+\|\bar\mu\|+L_2M_1)\alpha\right)$, then

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C \leq K_1\|\gamma-\bar\gamma\|_{\Gamma_1}, \qquad t\in[0,\alpha]. \qquad (3.53)$$

To finish the proof we need to get a similar estimate for the difference of the derivatives of the solutions. By the estimates used in (3.50), and by (3.51) and (3.53) we get for $t\in[0,\alpha]$

$$
\begin{aligned}
|\dot x(t;\gamma)-\dot x(t;\bar\gamma)| &\leq \Big|f\Big(t,x(t;\gamma),\Lambda_\mu(t,x(\cdot;\gamma)_t\Big) - \bar f\Big(t,x(t;\bar\gamma),\Lambda_{\bar\mu}(t,x(\cdot;\bar\gamma)_t\Big)\Big| \\
&\leq \|f-\bar f\| + L_1\Big(|x(t;\gamma)-x(t;\bar\gamma)| + |\Lambda_\mu(t,x(\cdot;\gamma)_t - \Lambda_{\bar\mu}(t,x(\cdot;\bar\gamma)_t|\Big) \\
&\leq \|f-\bar f\| + L_1\Big(K_1\|\gamma-\bar\gamma\|_{\Gamma_1} + \|\mu-\bar\mu\|M_1 \\
&\qquad + (\|\bar\mu\|+L_2M_1)|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C\Big) \\
&\leq \|f-\bar f\| + L_1\Big(K_1\|\gamma-\bar\gamma\|_{\Gamma_1} + \|\mu-\bar\mu\|M_1 \\
&\qquad + (\|\bar\mu\|+L_2M_1)K_1\|\gamma-\bar\gamma\|_{\Gamma_1}\Big) \\
&\leq \Big(\max\{1,M_1L_1\} + L_1K_1(1+\|\bar\mu\|+L_2M_1)\Big)\|\gamma-\bar\gamma\|_{\Gamma_1}. \qquad (3.54)
\end{aligned}
$$

Therefore the inequality

$$|\dot x(t;\gamma)-\dot x(t;\bar\gamma)| \leq K_2\|\gamma-\bar\gamma\|_{\Gamma_1}, \qquad t\in[0,\alpha] \qquad (3.55)$$

is satisfied with the constant $K_2 \equiv \max\{1,M_1L_1\} + L_1K_1(1+\|\bar\mu\|+L_2M_1)$. On the other hand, $\varphi,\bar\varphi\in W^{1,\infty}$, hence they are almost everywhere differentiable functions, and therefore from (3.2) we get that for a.e. $t\in[-r,0]$

$$\dot x(t;\gamma)-\dot x(t;\bar\gamma) = \dot\varphi(t) - \dot{\bar\varphi}(t),$$

and therefore

$$\operatorname*{ess\,sup}_{t\in[-r,0]}|\dot{x}(t;\gamma)-\dot{x}(t;\bar{\gamma})| = \operatorname*{ess\,sup}_{t\in[-r,0]}|\dot{\varphi}(t)-\dot{\bar{\varphi}}(t)| \le |\varphi-\bar{\varphi}|_{W^{1,\infty}}. \tag{3.56}$$

Using that $K_2 \ge 1$, we get from (3.53), (3.55), (3.56), the definition of $|\cdot|_{W^{1,\infty}}$, and Lemma 2.12, that

$$|x(\cdot;\gamma)_t - x(\cdot;\bar{\gamma})_t|_{W^{1,\infty}} \le \max\{K_1, K_2\}\|\gamma-\bar{\gamma}\|_{\Gamma_1}, \qquad t\in[0,\alpha], \tag{3.57}$$

therefore the constant $L_3 = \max\{K_1, K_2\}$ satisfies the statement of the theorem. □

This state-space has an important disadvantage, namely, the solution map, i.e.,

$$[0,\alpha]\to W^{1,\infty}, \qquad t\mapsto x(\cdot;\gamma)_t \tag{3.58}$$

is not continuous, in general, for $t\in[0,r]$ (see Remark 3.22 below). The discontinuity of the map (3.58) means, that if we define the solution semigroup by

$$S(t)\varphi \equiv x(\cdot;\varphi)_t, \qquad t\ge 0, \tag{3.59}$$

then it is easy to see that $\{S(t)\}_{t\ge 0}$ is a semigroup (of nonlinear operators) on $W^{1,\infty}$, but it is not strongly continuous on $W^{1,\infty}$.

We get continuity of the map (3.58) on $[0,\alpha]$ only for sufficiently smooth initial functions. In particular, we have the following result.

For fixed f and μ define the set

$$\mathcal{M} \equiv \Big\{\varphi\in C^1 \ :\ \dot{\varphi}(0-) = f(0,\varphi(0),\Lambda(0,\varphi))\Big\}. \tag{3.60}$$

Lemma 3.21 *Let $\gamma = (\varphi,\mu,f)$ satisfy (A1)-(A6), and $x(\cdot)$ be the corresponding solution of IVP (3.1)-(3.2) on $[-r,\alpha]$. Then*

(i) the function $[r,\alpha]\to W^{1,\infty}$, $\quad t\mapsto x_t$ is continuous.

(ii) if $\varphi\in\mathcal{M}$ then the function $[0,\alpha]\to W^{1,\infty}$, $\quad t\mapsto x_t$ is continuous.

Proof Lemma 3.13 yields that $x_t\in W^{1,\infty}$ for $t\in[0,\alpha]$. By the definition of the norm $|\cdot|_{W^{1,\infty}}$, using that $x(\cdot)$ is continuous, we have

$$|x_t - x_{\bar{t}}|_{W^{1,\infty}} = \sup_{-r\le s\le 0}|x(t+s)-x(\bar{t}+s)| + \operatorname*{ess\,sup}_{-r\le s\le 0}|\dot{x}(t+s)-\dot{x}(\bar{t}+s)|.$$

Using that the function $[0,\alpha]\times[-r,0]\to\mathbb{R}^n$, $\ (t,s)\mapsto x(t+s)$ is continuous, and hence uniformly continuous, it follows that $\sup_{-r\le s\le 0}|x(t+s)-x(\bar{t}+s)|\to 0$ as $t\to\bar{t}$ for $t,\bar{t}\in[0,\alpha]$. By Lemma 3.6 the function $t\mapsto\dot{x}(t) = f(t,x(t),\Lambda(t,x_t))$ is continuous on $[0,\alpha]$, hence we can repeat the previous argument for $\dot{x}(t+s)$, and we get that $\sup_{-r\le s\le 0}|\dot{x}(t+s)-\dot{x}(\bar{t}+s)|\to 0$, as $t\to\bar{t}$, for $t,\bar{t}\in[r,0]$, therefore (i) is proved. For (ii) we note that by the definition of $\mathcal{M}$, if $\varphi\in\mathcal{M}$, then the function

$$\dot{x}(t) = \begin{cases} f(t,x(t),\Lambda(t,x_t)), & t\in[0,\alpha],\\ \dot{\varphi}(t), & t\in[-r,0)\end{cases}$$

is defined, and continuous on $[-r,\alpha]$, hence we can prove the continuity of the map $t\mapsto x_t$ for $t\in[0,\alpha]$ by repeating the previous argument. □

If $\varphi\notin\mathcal{M}$, then we have the following negative result.

Remark 3.22 *If $\dot\varphi$ has a jump at $s^0 \in (-r, 0)$, i.e., $\varepsilon \equiv |\dot\varphi(s^0+) - \dot\varphi(s^0-)|$ exists and $\varepsilon > 0$, then the function $[0, \alpha] \to W^{1,\infty}, \quad t \mapsto x_t$ is not continuous on $(0, r + s^0)$.*

Proof Fix $\bar t$ such that $0 < \bar t < r + s^0$. Then $-r < s^0 - r < 0$, so we can select a sequence $s^k \in [-r, 0]$ such that $\{s^k\}$ monotone decreasingly converges to $s^0 - \bar t$. Define the sequence $t^k \equiv 2s^0 - 2s^k - \bar t$. The it is easy to see that $t^k \to \bar t$ as $k \to \infty$, and we have

$$\bar t - r < t^k + s^k < s^0 < \bar t + s^k < 0,$$

therefore

$$\begin{aligned}
&\lim_{k\to\infty} |x_{t^k} - x_{\bar t}|_{W^{1,\infty}} \\
&\quad = \lim_{k\to\infty} \sup_{-r\le s\le 0} |x(t^k + s) - x(\bar t + s)| + \lim_{k\to\infty} \operatorname{ess\,sup}_{-r\le s\le 0} |\dot x(t^k + s) - \dot x(\bar t + s)| \\
&\quad \ge \lim_{k\to\infty} |\dot x(t^k + s^k) - \dot x(\bar t + s^k)| \\
&\quad = \varepsilon,
\end{aligned}$$

i.e., the function is not continuous at $\bar t$. □

To overcome the problem of discontinuity of the solution map, we consider $W^{1,p}$ as the spate-space of x_t. From the elementary estimate

$$|\psi|_{W^{1,p}} \le (2r)^{1/p} |\psi|_{W^{1,\infty}} \tag{3.61}$$

it follows that $W^{1,\infty} \subset W^{1,p}$, and therefore Lemma 3.13 immediately implies the first two statements of the next lemma.

Lemma 3.23 *Assume that the parameter $\gamma = (\varphi, \mu, f) \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$ satisfies (A1)–(A6). Let $x(t)$ be the solution of (3.1)-(3.2) on $[0, \alpha]$ corresponding to γ, and let $1 \le p < \infty$. Then*

(i) $x_t \in W^{1,p}$ for all $t \in [0, \alpha]$, moreover, $x_t \in C^1$ for $t \in [r, \alpha]$,

(ii) there exists a constant $M_2 = M_2(p, \alpha, \|f\|, |\varphi|_{W^{1,\infty}})$ such that $|x_t|_{W^{1,p}} \le M_2$ for $t \in [0, \alpha]$,

(iii) the map $[0, \alpha] \to W^{1,p}$, $t \mapsto x(\cdot; \gamma)_t$ is continuous.

Proof To prove (iii), consider

$$|x_t - x_{\bar t}|^p_{W^{1,p}} = \int_{-r}^0 |x(t + s) - x(\bar t + s)|^p \, ds + \int_{-r}^0 |\dot x(t + s) - \dot x(\bar t + s)|^p \, ds.$$

Since by (i) both $x \in L^p_\alpha$ and $\dot x \in L^p_\alpha$, Lemma 2.11 implies (iii). □

This lemma has the following consequence.

Corollary 3.24 *The semigroup, defined by (3.59) is a C_0-semigroup on $W^{1,p}$.*

Estimate (3.61) and Theorem 3.20 has the following consequence.

Theorem 3.25 *Assume that $1 \le p \le \infty$, and $\bar\gamma = (\bar\varphi, \bar\mu, \bar f) \in \Pi_1(T, \Omega_1, \Omega_2, \Omega_3)$ satisfies (A1)–(A6). Then there exist constants $\alpha > 0$, $\delta > 0$ and $L_4 = L_4(p, \alpha, \bar\gamma, \delta)$, such that IVP (3.1)-(3.2) has unique solution on $[0, \alpha]$ for all $\gamma \in \mathcal{G}_{\Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)}(\bar\gamma; \delta)$, and*

$$|x(\cdot; \gamma)_t - x(\cdot; \bar\gamma)_t|_{W^{1,p}} \le L_4 \|\gamma - \bar\gamma\|_{\Gamma_1}, \quad t \in [0, \alpha].$$

3.5 Remarks

Delay systems have been studied by many authors. Without completeness, we refer to [7], [8], [19], [24], [31], [38] for discussion of general theory, applications and historical remarks.

The standard reference of well-posedness results for state-dependent delay equations is [17], where the results are presented for a system of the form

$$\dot{x}_i(t) = f_i(t, x(t), x(g_2(t, x(t))), \ldots, x(g_n(t, x(t)))), \qquad i = 1, 2, \ldots, n,$$

where $x(t) = (x_1(t), \ldots, x_n(t))$, $g_i(t, x) \leq t$ for all t, x. Our results in Sections 3.1–3.3 are straightforward generalizations of that of [17] for the class of equations described by (3.1), using the methods of [31].

We comment that the class of equations described by (3.1) includes the "usual" state-dependent delay equations,

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t, x(t)))) \tag{3.62}$$

or

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t, x_t))). \tag{3.63}$$

The new feature of (3.1), in addition to the type of representation which has not been used before for the state-dependent case, is that it includes distributed state-dependent delays (like in Example 1.4), and also infinitely many state-dependent point delays of the form:

$$\Lambda(t, \psi) = \sum_{i=1}^{\infty} A_i(t, \psi)\psi(-\tau_i(t, \psi)) + \int_{-\tau_0}^{0} G(s, t, \psi)\psi(s)\, ds.$$

Clearly, the results of Chapter 3 (and the results of the later chapters as well) can be generalized for equations of the form

$$\dot{x}(t) = f\Big(t, x(t), \Lambda_1(t, x_t), \ldots, \Lambda_m(t, x_t)\Big), \quad t \in [0, T],$$

where each delayed term, Λ_i, $i = 1, 2, \ldots, m$, has the form (1.2). We restrict the presentation for the case of one delayed term in the equation to keep the notations simple in the discussions.

In this paper we assume that $r > 0$ is finite, i.e., we consider the finite (or bounded) delay case. Note, that in Section 3.1 the only point where we used the finiteness of r is the implication that the continuity of φ on $[-r, 0]$ yields that $\omega_\varphi(h) \to 0$ as $h \to 0$. By assuming uniform continuity and boundedness of the initial function on $(-\infty, 0]$, and using the supremum norm for the norm of initial functions and solutions, we can extend the existence, uniqueness and continuous dependence results for the infinite delay case. See also e.g. [29] for this choice of state-space. The topic of delay equations with unbounded delays has a large literature, we refer to [2], [14], [32] for related works. Note, that in later chapters the boundedness of the delay will be essential.

For the function $f : [0, T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ in (3.1) we assumed (in (A1)) that it is bounded (and continuous) on its domain. If the domain is compact, or only the time domain is unbounded, then this assumption is of course redundant (if we pick a finite T in the latter case, which we can do, since we are interested in local existence). On unbounded (with respect to x and/or y) domain it does not follow in general. The boundedness assumption in the proofs is not essential, since the boundedness of f is always true on compact subsets of its domain, and that is enough to use in the arguments we had (since on finite time intervals the solution lies in a compact set). (See also Theorems 2.1–2.3 in [31].) We made this assumption mainly to have a nice normed linear state-space for the parameter f, and so be easy to talk about continuous dependence on f.

For the same reasons, we assumed (in (A2)) that the function $\lambda(t, \psi, \xi)$ is bounded on $[0, T] \times \Omega_3 \times \overline{\mathcal{G}}_C(1)$. This assumption can also be omitted, with the following argument. We used this boundedness assumption in many places, but basically we used only in two situations: first, that for a given $x \in C_\alpha$ it implies that $|\lambda(t, x_t, \xi)| \le \|\mu\| |\xi|_C$, for all $t \in [0, T]$. For a fixed $\xi \in C$, the continuity of λ and Lemma 2.10 yield for finite T that $\sup_{0 \le t \le T} |\lambda(t, x_t, \xi)| < \infty$. Since $A_{t,x_t}\xi \equiv \lambda(t, x_t, \xi)$ is a linear operator from C to $\mathbb{R}^n$, the Uniform Boundedness Theorem implies that $\|\mu\| \equiv \sup_{t \in [0,T]} \|A_{t,x_t}\| < \infty$, which yields the inequality. The second case is less trivial: in the proof of Lemma 3.12 we need the estimate $|\lambda(t, \psi, \xi)| \le \|\mu\| |\xi|_C$ for all $t \in [0, \alpha]$ and $\psi \in \overline{\mathcal{G}}_C(M) \cap \Omega_3$. Since the latter set is not compact in C, the continuity does not imply the boundedness of $|\lambda(t, \psi, \xi)|$ for fixed ξ, and in fact, the statement of Lemma 3.12 is not necessarily true (without the boundedness assumption). But we always apply Lemma 3.12 for estimating $|\Lambda(t, x_t) - \Lambda(t, \bar{x}_t)|$, in which case, by the first argument, we have the required estimate.

We mention one class of state-dependent delay equations appearing frequently (especially in biological) applications, the threshold-type of delay equations, where our representation of the delayed term, (1.2), might not have natural application. Consider the delay equation

$$\dot{x}(t) = f(x(t), x(t - \tau(t, x_t)))$$

where the delay is defined through a relation

$$\int_{t-\tau}^{t} g(t, s, x(s))\, ds = m$$

or $\tau = t$. (See e.g. [23].) We could rewrite the delayed term $x(t - \tau(t, x_t))$ as a Stieltjes-integral of the form (1.2), since it contains only a point delay, but then the threshold rule would be hidden in the definition of the function μ, and more importantly, our conditions (A2) and (A5) are not satisfied naturally in this case.

We close this chapter by recalling that Cooke and Huang in [13] studied the linearization of the autonomous state-dependent delay system of the form

$$\dot{x}(t) = f\left(x_t, \int_{-r_0}^{0} d\eta(s) g\Big(x(t + s - \tau(x_t))\Big)\right), \tag{3.64}$$

where $\tau : C \to [0, r_1]$, $r_0 > 0$, and r is such that $r \ge r_0 + r_1$. Note, that (3.64) includes also the autonomous versions of (3.62) and (3.63), but when $\eta(s)$ has finitely many jumps and an absolutely continuous part, then (3.64) gives the following type of delayed term:

$$\sum_{i=1}^{m} A_i g\Big(x(t - r_i - \tau(x_t))\Big) + \int_{-r_0}^{0} H(s) g\Big(x(t + s - \tau(x_t))\Big)\, ds,$$

which gives a different type of delay dependences than that of given by (1.2) (in the autonomous case). In Example 5.8 we show an equation which is not included in (3.64) but we can rewrite it in the form (3.1), and oppositely, there are equations can be written in the form (3.64) but not in (3.1).

Chapter 4
DIFFERENTIABILITY WRT PARAMETERS

In this chapter we study differentiability of solutions of IVP (3.1)-(3.2) with respect to (wrt) parameters of the equation. We shall consider three cases. First, we discuss differentiability wrt the initial function (Section 4.1), then we consider special cases, when the delay term in the equation, i.e., μ (and hence Λ and λ), and when the right-hand side of the equation, i.e., the function f, depend explicitly on a parameter c and d, respectively, and investigate differentiability of solutions wrt these parameters, respectively. (See Sections 4.2 and 4.3.) In Chapter 3 we considered μ and f as parameters, but here we assume that only a “part” of μ and f varies, which can be represented by vector parameters. See also the introduction to Chapter 3 where we discussed how the initial time can be considered as a parameter of f. (Note, that these parameters could be elements of an infinite dimensional space, the methods we use can be applied for that case as well, like in Section 4.1, where the parameter (the initial function) is infinite dimensional.) These assumptions simplify the discussion, and also include the practically important applications.

In order to make our presentation as clear as possible, we discuss these three cases separately, but we provide full details only in Section 4.1. The remaining two cases (Sections 4.2 and 4.3) can be treated similarly (with of course some technical modifications), and we shall omit most of the proofs, since they are essentially the same as those in Section 4.1.

Differentiability results wrt parameters, beside the obvious theoretical importance, have a natural application in the problem of identification of unknown parameters of the equation (such as the initial function, some coefficients in the equation, or for a constant delay equation, the delay itself). In this direction it is important to know if the solution is differentiable wrt the parameters in some sense, since many identification methods require the use of optimization techniques, in which the knowledge of the derivative of the solution wrt the parameter is essential.

The first problem one faces trying to obtain differentiability results is the differentiability of the delay term $\Lambda(t,\psi)$ of the equation. Clearly, to be able to prove differentiability of the solution, we need to assume some kind of smoothness of $\Lambda(t,\psi)$ wrt ψ. Since $\Lambda(t,\psi) = \lambda(t,\psi,\psi)$, then we need to assume differentiability of $\lambda(t,\psi,\xi)$ wrt ψ and ξ in some sense. The latter is relatively easy, since $\lambda(t,\psi,\xi)$ is linear in ξ, therefore, it is differentiable (in every space) with derivative $\frac{\partial\lambda}{\partial\xi}(t,\psi,\xi)h = \lambda(t,\psi,h)$. It is easy to see that in order to have continuous differentiability of λ wrt ξ, we need to consider, e.g., the space $W^{1,\infty}$, since the inequality

$$|\lambda(t,\psi,h) - \lambda(t,\bar\psi,h)| \le L_2 |h|_{W^{1,\infty}} |\psi - \bar\psi|_C,$$

(provided by (A5)), guarantees the continuous differentiability of $\lambda(t,\psi,\xi)$ wrt ξ for $\xi \in W^{1,\infty}$. This suggests the use of $W^{1,\infty}$ for the state-space of solutions, and as we have seen in Section 3.4, it is a natural choice, since the solution is unique for $W^{1,\infty}$ initial functions.

The difficulty with $W^{1,\infty}$ is that for $\psi, \xi \in W^{1,\infty}$, the function $\lambda(t, \psi, \xi)$ is naturally a composition of ξ and ψ (see e.g. Example 1.3), and therefore we need to guarantee the differentiability, or preferably, continuous differentiability of the composition of $W^{1,\infty}$-functions, which is in general impossible. But in the case when the two functions are C^1 functions, the differentiability follows immediately (in our Examples) by the Chain Rule. We have seen in Section 3.4 that the solution is C^1 only for special initial functions, if $\varphi \in C^1$, and satisfies a certain boundary condition at 0 (see Lemma 3.21.) This is a strong assumption, but assuming it, we are able to prove the differentiability of solutions wrt parameters in the state-space $W^{1,\infty}$, which is a strong property. We shall discuss this special case in Sections 4.1.1, 4.2.1 and 4.3.1. (Note that in Sections 4.1.1 and 4.3.1 we can prove our results in the state-independent case without the restrictive condition on the initial function.) The method we use is a "classical" one, used to prove differentiability of the solution wrt parameters in ODEs (see e.g. [39]).

Since in $W^{1,\infty}$ the assumption for differentiability is too strong, we shall explore different spaces for the more general case, i.e., when the solution, (and the initial function) is a $W^{1,\infty}$ function only.

In [33], Hale and Ladeira investigated differentiability of solutions of the constant delay equation

$$\dot{x}(t) = f(x(t), x(t-\tau))$$

wrt to the delay, τ. They showed using an extension of the Uniform Contraction Principle to quasi-Banach spaces (see also Theorem 2.23 in Chapter 2), and selecting $W^{1,1}$ as the state-space of solutions, that the map

$$[0, r] \to W^{1,1}_\alpha, \quad \tau \mapsto x(\cdot; \tau)$$

is differentiable. This result suggests that $W^{1,p}$ could possible be used as the state-space for solutions. Again, we recall that in Section 3.4 we have seen that $W^{1,p}$ is an "ideal" state-space candidate for state-dependent equations, in the sense that the maps $t \mapsto x(\cdot;\gamma)_t$ and $\gamma \mapsto x(\cdot;\gamma)_t$ are continuous and Lipschitz-continuous on it, respectively. The method used in [33] is the following: transform the IVP into an equivalent integral equation, introduce the new variable $y(t) = x(t) - \tilde{\varphi}(t)$, and then reformulate the problem as finding the fixed point of an operator, and obtain differentiability of the fixed point wrt parameters. (Note, that we followed this method in Section 3.1 to prove existence results.) The transformed integral equation is (3.10)-(3.11), and the operator $S(y, \gamma)$ is defined by (3.25) and (3.26). The Uniform Contraction Principle says that if $S(y, \gamma)$ is a contraction in y uniformly in γ, and it is continuously differentiable wrt y and γ, then its unique fixed point, as a function of γ, is differentiable wrt γ. If we select $W^{1,p}_\alpha$ for the state-space for y, then we need the continuous differentiability of $S(y, \gamma)$ wrt y in $W^{1,p}_\alpha$. This requires the differentiability of $\Lambda(t, \psi)$ in "an L^p-type of norm".

In [10], Brokate and Colonius studied linearization of the equation

$$\dot{x}(t) = f\Big(t, x(t - \tau(t, x(t)))\Big), \qquad t \in [0, \alpha].$$

In particular, they investigated differentiability of the composition operator

$$A : \left(\bar{X} \subset W^{1,\infty}_\alpha\right) \to L^p([0, \alpha]; \mathbb{R}^n), \quad (Ax)(t) \equiv x(t - \tau(t, x(t))).$$

They obtained differentiability of this map by selecting an appropriate domain $\bar{X}$. (See more details in Section 4.1.4.)

To obtain continuous differentiability of the operator $S(y,\gamma)$ in $W^{1,p}$ for this point delay equation we would need the continuous differentiability of this composition map, but using the $|\cdot|_{W^{1,p}_\alpha}$-norm on the domain of the operator. It turns out, that the right choice for our purposes is "in between the $|\cdot|_{W^{1,\infty}_\alpha}$-norm and $|\cdot|_{W^{1,p}_\alpha}$-norm". We shall introduce a "product norm" in Section 4.1.2. Let $x \in W^{1,\infty}_\alpha$ (since all solutions are $W^{1,\infty}_\alpha$ functions, this should be the space of the solutions), and decompose x as $x = y + \tilde{\varphi}$, (where $\varphi(t) = x(t)$ for $t \in [-r, 0]$), and define the norm of x by

$$|x|_{\mathbb{X}^p_\alpha} \equiv \left(\int_0^\alpha |\dot{y}(u)|^p du\right)^{1/p} + |\varphi|_{W^{1,\infty}},$$

and consider the normed linear space $\mathbb{X}^p_\alpha \equiv (W^{1,\infty}_\alpha, |\cdot|_{\mathbb{X}^p_\alpha})$. Then this is a norm, which is weaker than the $|\cdot|_{W^{1,\infty}_\alpha}$-norm, but stronger than the $|\cdot|_{W^{1,p}_\alpha}$ norm (see Lemma 4.18). This norm is still "strong enough" that the method used in [10] go through and provide differentiability of the composition map

$$B \;:\; \left(\mathcal{K} \subset \mathbb{X}^p_\alpha\right) \to L^p([0,\alpha]; \mathbb{R}^n), \quad (Bx)(t) \equiv x(t - \tau(t, x(t))).$$

On the other hand, $|\cdot|_{\mathbb{X}^p_\alpha}$ is "weak enough" that using the differentiability of the operator B above, we can obtain obtain differentiability of the operator $S(y,\gamma) \;:\; \mathbb{X}^p_\alpha \times \Gamma \to \mathbb{X}^p_\alpha$, and be able to use a variation of the Uniform Contraction Principle (Theorem 4.14) to get differentiability of the fixed point (the solution of the IVP) wrt the parameter γ in the $|\cdot|_{\mathbb{X}^p_\alpha}$-norm. Since this product norm is stronger than the $|\cdot|_{W^{1,p}_\alpha}$-norm, the result implies the differentiability of solutions in the latter norm as well. We shall follow this method in Section 4.1.3 with detailed discussion, and in Sections 4.2.2 and 4.3.2 without detailed proofs. We provide the necessary technical preliminaries in Section 4.1.2, and modifying the method of [10] for our case, show differentiability of the composition operator associated with the delay terms of Examples 1.3 and 1.4 in Section 4.1.4.

We close this introduction by noting that differentiability of solutions of delay equations of the form

$$\dot{x}(t) = f(t, x_t)$$

wrt parameters has been studied e.g., in [31], where it was shown differentiability of solution wrt initial function and f, using C as the state-space of the solution, and the Uniform Contraction Principle. Differentiability of solutions of state-dependent delay equations wrt parameters (to the best knowledge of this author) has not been studied in the literature yet.

4.1 Differentiability of solutions wrt initial function

In this section we study differentiability of solutions of IVP

$$\dot{x}(t;\varphi) = f\Big(t,\, x(t;\varphi),\, \Lambda(t, x(\cdot;\varphi)_t)\Big), \quad t \in [0,T], \tag{4.1}$$

$$x(t;\varphi) = \varphi(t), \qquad t \in [-r, 0] \tag{4.2}$$

wrt initial function. We assume in this section, that f and μ are fixed, satisfying assumptions (A1)–(A5), and we consider the solution depending only on φ. To emphasize the dependence of the solution on the initial function, we use the notations $x(t;\varphi)$ and $x(\cdot;\varphi)_t$ for the value of the solution and for the solution segment function at t, respectively, corresponding to initial function φ. Note that by Theorems 3.8 and 3.19, assumptions (A1)–(A6) guarantee existence, uniqueness of solutions on an interval $[0,\alpha]$, and continuous dependence of solutions on φ for $\varphi \in \Phi$, where (see also (3.46))

$$\Phi \equiv \left\{ \varphi \in W^{1,\infty} \cap C \;:\; \varphi(0) \in \Omega_1, \quad \text{and} \quad \int_{-r}^{0} d_s\mu(s,0,\varphi)\varphi(s) \in \Omega_2 \right\}. \tag{4.3}$$

4.1.1 Special case, differentiability in $W^{1,\infty}$

In this subsection we shall assume that either the equation is state-independent, i.e., $\mu(s,t,\psi)$, or equivalently, $\lambda(t,\psi,\xi)$ is independent of ψ; or in the state-dependent case the initial function $\varphi \in \mathcal{M}$. In both cases we can assume that $\Lambda(t,\psi)$ is continuously differentiable wrt ψ (in the $|\cdot|_{W^{1,\infty}}$ norm) along the solution of the equation, (i.e., for each $\psi = x(\cdot;\varphi)_t$, $t \in [0,\alpha]$). This is obvious in the state-independent case, i.e., when $\lambda(t,\psi,\xi)$ does not depend on ψ (see Corollary 4.5). In the second case $\varphi \in \mathcal{M}$ guarantees that the corresponding solution is continuously differentiable for $t \in [-r,\alpha]$ (see Lemma 3.21), and therefore the corresponding solution segment functions are C^1 functions, hence we need the differentiability of $\Lambda(t,\psi)$ wrt ψ for $\psi \in C^1$. We shall show in Examples 4.1–4.3 that this is a reasonable assumption. In both cases we can argue the differentiability of solutions in the $|\cdot|_{W^{1,\infty}}$ norm wrt initial functions.

In fact, we shall need the following assumptions:

(A7) $f(t,x,y)$ has continuous partial derivatives wrt x and y on $t \in [0,T]$, $x \in \Omega_1$ and $y \in \Omega_2$,

(A8a) (i) $\lambda(t,\psi,\xi)$ is locally Lipschitz-continuous in t as well, i.e., for every $\alpha > 0$ and $M > 0$ there exists a constant $L_2 = L_2(\alpha, M)$ such that for all $\xi \in W^{1,\infty}$, $t,\bar{t} \in [0,\alpha]$ and $\psi,\bar{\psi} \in \overline{\mathcal{G}}_C(M) \cap \Omega_3$

$$|\lambda(t,\psi,\xi) - \lambda(\bar{t},\bar{\psi},\xi)| \le L_2|\xi|_{W^{1,\infty}}\Big(|t-\bar{t}| + |\psi - \bar{\psi}|_C\Big).$$

(ii) For all $t \in [0,T]$, $\psi \in W^{1,\infty} \cap \Omega_3$ and $\xi \in C^1$ the function $\lambda(t,\psi,\xi)$ is continuously differentiable wrt ψ, i.e., for each $\xi \in C^1$ the partial derivative $\frac{\partial\lambda}{\partial\psi}(\cdot,\cdot,\xi) \;:\; \Big([0,T] \times \Omega_3 \subset [0,T] \times W^{1,\infty}\Big) \to \mathcal{L}(W^{1,\infty}, \mathbb{R}^n)$ is continuous.

First we give conditions in our particular examples which yield (A8a). The functions $\lambda(t,\psi,\xi)$ used in Examples 1.1 and 1.2 are independent of ψ, therefore (A8a) holds automatically in these Examples.

Example 4.1 Let

$$\lambda(t,\psi,\xi) = \xi(-\tau(t,\psi)),$$

as in Examples 1.3, 3.3 and 3.10. If we assume that

(i) $\tau(\cdot,\cdot) \; : \; \Big([0,T] \times \Omega_3 \subset [0,T] \times C\Big) \to \mathbb{R}$ is continuous,

(ii) $\tau(t,\psi)$ is locally Lipschitz-continuous in t and ψ, i.e., for every $\alpha > 0$, $M > 0$ there exists a constant $L_\tau(\alpha, M)$ such that

$$|\tau(t,\psi) - \tau(\bar{t},\bar{\psi})| \le L_\tau(\alpha, M)\Big(|t-\bar{t}| + |\psi - \bar{\psi}|_C\Big), \text{ for } \psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M) \cap \Omega_3, \; t, \bar{t} \in [0,\alpha],$$

(iii) $\tau(t,\cdot) \; : \; \Big(W^{1,\infty} \cap \Omega_3 \subset W^{1,\infty}\Big) \to \mathbb{R}$ is differentiable for all $t \in [0,T]$,

(iv) $\frac{\partial \tau}{\partial \psi}(\cdot,\cdot) \; : \; \Big([0,T] \times (W^{1,\infty} \cap \Omega_3) \subset [0,T] \times W^{1,\infty}\Big) \to \mathcal{L}(W^{1,\infty}, \mathbb{R})$ is continuous,

then it follows from the chain rule that (A8a) (ii) holds, i.e., the function $\lambda(t,\cdot,\xi) \; : \; \Big(W^{1,\infty} \cap \Omega_3 \subset W^{1,\infty}\Big) \to \mathbb{R}^n$ is continuously differentiable for all $t \in [0,T]$, $\xi \in C^1$, and

$$\frac{\partial \lambda}{\partial \psi}(t,\psi,\xi)h = -\dot{\xi}(-\tau(t,\psi))\frac{\partial \tau}{\partial \psi}(t,\psi)h, \qquad h \in W^{1,\infty}.$$

(A8b) (i) follows from the Mean Value Theorem (Theorem 2.3) and (ii).

Example 4.2 Consider a special case of Example 4.1, when $\tau(t,\psi)$ is defined through a function, $\bar{\tau}(t,x)$, as follows: $\tau(t,\psi) \equiv \bar{\tau}(t,\psi(0))$, i.e., we consider delayed terms of the form

$$\lambda(t,\psi,\xi) = \xi(-\bar{\tau}(t,\psi(0))).$$

Then, clearly, the conditions

(i) $\bar{\tau}(\cdot,\cdot) \; : \; [0,T] \times \Omega^* \to \mathbb{R}^n$ is continuous, where $\Omega^* \subset \mathbb{R}^n$ is an open set,

(ii) $\bar{\tau}(t,x)$ is locally Lipschitz-continuous in t and x, i.e., for every $\alpha > 0$, $M > 0$ there exists a constant $L_{\bar{\tau}}(\alpha, M)$ such that

$$|\bar{\tau}(t,x) - \bar{\tau}(\bar{t},\bar{x})| \le L_{\bar{\tau}}(\alpha, M)\Big(|t-\bar{t}| + |x - \bar{x}|\Big), \quad \text{for } x, \bar{x} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_3, \; t, \bar{t} \in [0,\alpha],$$

(iii) $\bar{\tau}(t,x)$ is continuously differentiable wrt x on $t \in [0,T]$, $x \in \Omega_4$

imply conditions (i)–(iv) of Example 4.1, and hence (A2), (A5) and (A8a) as well. (Here we used that the function $g \; : \; W^{1,\infty} \to \mathbb{R}^n$, $g(\psi) \equiv \psi(0)$ is continuously differentiable with derivative $g'(\psi)h = h(0)$.)

Example 4.3 Let

$$\lambda(t,\psi,\xi) = \sum_{k=1}^{m} A_k(t)\xi(-\tau_k(t,\psi)) + \int_{-\tau_0}^{0} G(s,t,\psi)\xi(s)\,ds,$$

as in Examples 1.4, 3.4 and 3.11. Assume that for $k = 1, 2, \ldots, m$

(i) $\tau_k(\cdot,\cdot) \; : \; \Big([0,T] \times \Omega_3 \subset [0,T] \times C\Big) \to \mathbb{R}$ is continuous,

(ii) $\tau_k(t,\psi)$ is locally Lipschitz-continuous in t and ψ, i.e., for every $\alpha > 0$, $M > 0$ there exists a constant $L_{\tau_k}(\alpha, M)$ such that

$$|\tau_k(t,\psi) - \tau_k(\bar{t},\bar{\psi})| \le L_{\tau_k}(\alpha, M)\Big(|t-\bar{t}| + |\psi - \bar{\psi}|_C\Big), \text{ for } \psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M),\ t, \bar{t} \in [0,\alpha],$$

(iii) $\tau_k(t,\cdot)\ :\ \Big(W^{1,\infty} \cap \Omega_3 \subset W^{1,\infty}\Big) \to \mathbb{R}$ is differentiable for all $t \in [0,T]$,

(iv) $\frac{\partial \tau_k}{\partial \psi}(\cdot,\cdot)\ :\ [0,T] \times W^{1,\infty} \to \mathcal{L}(W^{1,\infty}, \mathbb{R})$ is continuous,

(v) the function G satisfies a Lipschitz-condition of the form

$$\|G(s,t,\psi) - G(s,\bar{t},\bar{\psi})\| \le g(s)\Big(|t-\bar{t}| + |\psi - \bar{\psi}|_C\Big),$$

for $s \in [-\tau_0, 0]$, $t, \bar{t} \in [0,T]$, and $\psi, \bar{\psi} \in \Omega_3$, where $g \in L^1([-\tau_0,0];\mathbb{R})$,

(vi) $G(s,t,\psi)\ :\ \Big([-\tau_0,0] \times [0,T] \times \Omega_3 \subset [-\tau_0,0] \times [0,T] \times W^{1,\infty}\Big) \to \mathbb{R}^{n\times n}$ has continuous partial derivative wrt ψ,

(vii) $A_k(t)$ is continuous on $[0,T]$.

Then it is easy to see that for $\xi \in C^1$ the function $\lambda(t,\psi,\xi)$ is differentiable wrt ψ, and

$$\frac{\partial \lambda}{\partial \psi}(t,\psi,\xi)h = -\sum_{k=1}^{m} A_k(t)\dot{\xi}(-\tau_k(t,\psi))\frac{\partial \tau_k}{\partial \psi}(t,\psi)h + \int_{-\tau_0}^{0} \Big(\frac{\partial G}{\partial \psi}(s,t,\psi)h\Big)\xi(s)\,ds,$$

therefore (A8a) (ii) is satisfied. (A8a) (i) easily follows from (ii) and (v).

We show, that (A8a) implies that $\Lambda(t,\psi)$ is differentiable wrt ψ for $\psi \in C^1$. The function $\Lambda(t,\psi)$ is defined as $\Lambda(t,\psi) = \lambda(t,\psi,\psi)$, therefore we have to investigate differentiability of $\lambda(t,\psi,\xi)$ wrt ψ and ξ. The latter is easy, since $\lambda(t,\psi,\xi)$ is linear in ξ. In particular, we have the following result:

Lemma 4.4 *Let $t \in [0,T]$ and $\psi \in W^{1,\infty} \cap \Omega_3$ be fixed. Assume (A2) and (A8a) (i). Then*

(i) the function $\lambda(t,\psi,\cdot)\colon W^{1,\infty} \to \mathbb{R}^n$ is differentiable, and for all $\xi \in W^{1,\infty}$

$$\frac{\partial \lambda}{\partial \xi}(t,\psi,\xi)h = \lambda(t,\psi,h), \qquad h \in W^{1,\infty},$$

and moreover,

(ii) for all $\xi, h \in W^{1,\infty}$

$$\lambda(t,\psi,\xi+h) - \lambda(t,\psi,\xi) = \frac{\partial \lambda}{\partial \xi}(t,\psi,\xi)h,$$

and

(iii) the derivative, $\frac{\partial \lambda}{\partial \xi}(t,\psi,\xi)$ is continuous in all its variables (i.e., continuous as a function $\frac{\partial \lambda}{\partial \xi}\ :\ \Big([0,T] \times \Omega_3 \times W^{1,\infty} \subset [0,T] \times W^{1,\infty} \times W^{1,\infty}\Big) \to \mathcal{L}(W^{1,\infty}, \mathbb{R}^n)$).

Proof The identity

$$\begin{aligned}\lambda(t,\psi,\xi+h)-\lambda(t,\psi,\xi) &= \int_{-r}^{0} d_s\mu(s,t,\psi)\Big(\xi(s)+h(s)\Big)-\int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)\\ &= \int_{-r}^{0} d_s\mu(s,t,\psi)h(s)\end{aligned}$$

proves the first two statements of the lemma. To prove (iii), we first comment, that by part (i) the function $\frac{\partial\lambda}{\partial\xi}(t,\psi,\xi)$ is independent of ξ. Let $\xi,h\in W^{1,\infty}$, $t,\bar t\in[0,\alpha]$, $\psi,\bar\psi\in\overline{\mathcal{G}}_{W^{1,\infty}}(M)$ for some $\alpha>0$ and $M>0$, and let $L_2=L_2(\alpha,M)$ be the constant from (A8a) (i). Then part (i) of this lemma and (A8a) (i) imply that

$$\begin{aligned}\left|\frac{\partial\lambda}{\partial\xi}(t,\psi,\xi)h-\frac{\partial\lambda}{\partial\xi}(\bar t,\bar\psi,\xi)h\right| &= |\lambda(t,\psi,h)-\lambda(\bar t,\bar\psi,h)|\\ &\le L_2|h|_{W^{1,\infty}}\Big(|t-\bar t|+|\psi-\bar\psi|_C\Big),\end{aligned}$$

and hence it follows that

$$\left\|\frac{\partial\lambda}{\partial\xi}(t,\psi,\xi)-\frac{\partial\lambda}{\partial\xi}(\bar t,\bar\psi,\xi)\right\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}\le L_2\Big(|t-\bar t|+|\psi-\bar\psi|_{W^{1,\infty}}\Big),$$

which proves (iii). □

Corollary 4.5 *If $\lambda(t,\psi,\xi)$ is independent of ψ, then $\frac{\partial\Lambda}{\partial\psi}(t,\psi)$ exists and continuous on $t\in[0,T]$ and $\psi\in W^{1,\infty}$, and $\frac{\partial\Lambda}{\partial\psi}(t,\psi)=\frac{\partial\lambda}{\partial\xi}(t,\psi,\psi)=\lambda(t,\psi,\cdot)$.*

Remark 4.6 *Note, that if in Example 4.3 there are no point delays, i.e., $A_k(t)=0$ for all $k=1,\ldots,m$, then assumption (vi) on G implies that the corresponding $\lambda(t,\psi,\xi)$ is continuously differentiable wrt ψ for $t\in[0,T]$, $\psi\in W^{1,\infty}\cap\Omega_3$ and $\xi\in W^{1,\infty}$.*

Lemma 4.4 and (A8a) together with Lemmas 2.15 and 2.17 imply the differentiability of $\Lambda(t,\psi)$ wrt $\psi\in C^1$.

Lemma 4.7 *Assume (A2), (A5) and (A8a). Then*

(i) the function $\Lambda(t,\psi)$ is differentiable wrt ψ for any $t\in[0,T]$, $\psi\in C^1\cap\Omega_3$,

(ii) for $h\in W^{1,\infty}$

$$\frac{\partial\Lambda}{\partial\psi}(t,\psi)h=\frac{\partial\lambda}{\partial\xi}(t,\psi,\psi)h+\frac{\partial\lambda}{\partial\psi}(t,\psi,\psi)h,$$

and

(iii) the function $\frac{\partial\Lambda}{\partial\psi}(\cdot,\cdot)\ :\ \Big([0,T]\times\Omega_3\subset[0,T]\times W^{1,\infty}\Big)\to\mathcal{L}(W^{1,\infty},\mathbb{R}^n)$ is continuous.

Assume that either the equation is state-independent and $\varphi \in \Phi$, or in the state-dependent case, $\varphi \in \Phi \cap \mathcal{M}$. Then in both cases $\frac{\partial \Lambda}{\partial \psi}(t, x(\cdot;\varphi)_t)$ is well-defined (by Corollary 4.5 and Lemmas 3.21 and 4.7), moreover, it is a continuous function of t. For $h \in W^{1,\infty}$ we consider the linear time-dependent IVP

$$\begin{aligned} \dot{z}(t;h) &= \frac{\partial f}{\partial x}\Big(t, x(t;\varphi), \Lambda(t, x(\cdot;\varphi)_t)\Big) z(t;h) \\ &\quad + \frac{\partial f}{\partial y}\Big(t, x(t;\varphi), \Lambda(t, x(\cdot;\varphi)_t)\Big) \frac{\partial \Lambda}{\partial \psi}(t, x(\cdot;\varphi)_t) z(\cdot;h)_t, \qquad t \in [0,T], \quad (4.4) \\ z(t;h) &= h(t), \qquad t \in [-r, 0], \quad (4.5) \end{aligned}$$

then (assuming (A7) and (A8a)) the solution, $z(\cdot;h)$, of IVP (4.4)-(4.5) exists and unique on $[0,T]$, and it is linear in h.

The next theorem shows that assumptions (A1)–(A8a) imply that the function $x(t;\cdot) : W^{1,\infty} \to \mathbb{R}^n$ is differentiable for all $t \in [0,T]$.

Theorem 4.8 *Let $(\varphi, \mu, f) \in \Pi_1(T, \Omega_1, \Omega_2, \Omega_3)$ satisfy (A1)–(A8a). Assume moreover that either*

(1) equation (4.1) is state-independent, i.e., $\mu(s,t,\psi)$, or equivalently, $\lambda(t,\psi,\xi)$ does not depend on ψ,

or

(2) in the state-dependent case $\varphi \in \mathcal{M}$.

Then

(i) the solution $x(t;\varphi)$ of IVP (4.1)-(4.2) is differentiable wrt φ for all $t \in [0,\alpha]$ and $\varphi \in W^{1,\infty}$,

(ii) $\dfrac{x(t;\varphi+h) - x(t;\varphi)}{|h|_{W^{1,\infty}}}$ converges uniformly to $\frac{\partial x}{\partial \varphi}(t;\varphi)$ on $t \in [0,\alpha]$,

(iii) the derivative is $\frac{\partial x}{\partial \varphi}(t;\varphi)h = z(t;h)$, where $z(t;h)$ is the solution of the linear IVP (4.4)-(4.5).

Proof Fix φ, μ and f satisfying the conditions of the theorem, let $\alpha > 0$ and $\delta > 0$ be the corresponding constants from Theorem 3.20, let $h \in W^{1,\infty}$, $|h|_{W^{1,\infty}} < \delta$, and let $z(t;h)$ be the corresponding solution of IVP (4.4)-(4.5). Consider

$$\begin{aligned} &|x(t;\varphi+h) - x(t;\varphi) - z(t;h)| \\ &= \left| h(0) + \int_0^t f\Big(u, x(u;\varphi+h), \Lambda(u, x(\cdot;\varphi+h)_u)\Big) - f\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big)\, du \right. \\ &\qquad\qquad \left. - z(t;h) \right| \\ &\le \int_0^t \left| f\Big(u, x(u;\varphi+h), \Lambda(u, x(\cdot;\varphi+h)_u)\Big) - f\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big) \right. \\ &\qquad - \frac{\partial f}{\partial x}\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big) z(u;h) \\ &\qquad \left. - \frac{\partial f}{\partial y}\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big) \frac{\partial \Lambda}{\partial \psi}(u, x(\cdot;\varphi)_u) z(\cdot;h)_u \right| du. \qquad (4.6) \end{aligned}$$

By assumption (A7) and Lemma 2.17, the function $(x,y)\mapsto f(u,x,y)$ is continuously differentiable for each fixed $u\in[0,T]$, therefore the function,

$$\omega^1(u,\bar x,\bar y;x,y)\equiv f(u,x,y)-f(u,\bar x,\bar y)-\frac{\partial f}{\partial x}(u,\bar x,\bar y)(x-\bar x)-\frac{\partial f}{\partial y}(u,\bar x,\bar y)(y-\bar y), \tag{4.7}$$

which is defined for $u\in[0,T]$, $x,\bar x\in\Omega_1$ and $y,\bar y\in\Omega_2$, satisfies for all $u\in[0,T]$ that

$$\frac{|\omega^1(u,\bar x,\bar y;x,y)|}{|x-\bar x|+|y-\bar y|}\to 0,\qquad \text{as } x\to\bar x,\quad\text{and}\quad y\to\bar y. \tag{4.8}$$

By assumptions (A2), (A5) and (A8a) the function

$$\omega^2(u,\bar\psi;\psi)\equiv\Lambda(u,\psi)-\Lambda(u,\bar\psi)-\frac{\partial\Lambda}{\partial\psi}(u,\bar\psi)(\psi-\bar\psi) \tag{4.9}$$

is defined for $u\in[0,\alpha]$, and for $\psi,\bar\psi\in W^{1,\infty}\cap\Omega_3$ or for $\psi,\bar\psi\in C^1\cap\Omega_3$ in case (1) and (2) of the theorem, respectively; and for $u\in[0,\alpha]$ it satisfies

$$\frac{|\omega^2(u,\bar\psi;\psi)|}{|\psi-\bar\psi|_{W^{1,\infty}}}\to 0,\qquad \text{as } |\psi-\bar\psi|_{W^{1,\infty}}\to 0. \tag{4.10}$$

Using these notations, and applying standard estimates we get from (4.6), that

$$\begin{aligned}
&|x(t;\varphi+h)-x(t;\varphi)-z(t;h)|\\
&\le \int_0^t\Big|\frac{\partial f}{\partial x}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\Big[x(u;\varphi+h)-x(u;\varphi)-z(u;h)\Big]\\
&\qquad+\frac{\partial f}{\partial y}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\Big[\Lambda(u,x(\cdot;\varphi+h)_u)\\
&\qquad\qquad-\Lambda(u,x(\cdot;\varphi)_u)-\frac{\partial\Lambda}{\partial\psi}(u,x(\cdot;\varphi)_u)z(\cdot;h)_u\Big]\\
&\qquad+\omega^1\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u);x(u;\varphi+h),\Lambda(u,x(\cdot;\varphi+h)_u)\Big)\Big|du\\
&\le \int_0^t\Big|\frac{\partial f}{\partial x}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\Big[x(u;\varphi+h)-x(u;\varphi)-z(u;h)\Big]\\
&\qquad+\frac{\partial f}{\partial y}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\frac{\partial\Lambda}{\partial\psi}(u,x(\cdot;\varphi)_u)\Big[x(\cdot;\varphi+h)_u-x(\cdot;\varphi)_u-z(\cdot;h)_u\Big]\\
&\qquad+\frac{\partial f}{\partial y}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\omega^2\Big(u,x(\cdot;\varphi)_u;x(\cdot;\varphi+h)_u\Big)\\
&\qquad+\omega^1\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u);x(u;\varphi+h),\Lambda(u,x(\cdot;\varphi+h)_u)\Big)\Big|du\\
&\le \int_0^t\Big\|\frac{\partial f}{\partial x}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\Big\|\,\Big|x(u;\varphi+h)-x(u;\varphi)-z(u;h)\Big|\\
&\qquad+\Big\|\frac{\partial f}{\partial y}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\Big\|\,\Big\|\frac{\partial\Lambda}{\partial\psi}(u,x(\cdot;\varphi)_u)\Big\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}\\
&\qquad\qquad\cdot\Big|x(\cdot;\varphi+h)_u-x(\cdot;\varphi)_u-z(\cdot;h)_u\Big|_C\\
&\qquad+\Big\|\frac{\partial f}{\partial y}\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u)\Big)\Big\|\,\Big|\omega^2\Big(u,x(\cdot;\varphi)_u;x(\cdot;\varphi+h)_u\Big)\Big|\\
&\qquad+\Big|\omega^1\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u);x(u;\varphi+h),\Lambda(u,x(\cdot;\varphi+h)_u)\Big)\Big|du.
\end{aligned}$$

Introduce the scalar functions

$$\begin{aligned}
\Psi(u;h) &\equiv \frac{1}{|h|_{W^{1,\infty}}} \max_{0\le v\le u} \Big| x(v;\varphi+h) - x(v;\varphi) - z(v;h) \Big|, \\
F(u) &\equiv \Big\| \frac{\partial f}{\partial x}\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big) \Big\| \\
&\quad + \Big\| \frac{\partial f}{\partial y}\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big) \Big\| \, \Big\| \frac{\partial \Lambda}{\partial \psi}(u, x(\cdot;\varphi)_u) \Big\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}, \\
G(u;h) &= \Big\| \frac{\partial f}{\partial y}\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big) \Big\| \frac{1}{|h|_{W^{1,\infty}}} \Big| \omega^2\Big(u, x(\cdot;\varphi)_u; x(\cdot;\varphi+h)_u\Big) \Big| \\
&\quad + \frac{1}{|h|_{W^{1,\infty}}} \Big| \omega^1\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u); x(u;\varphi+h), \Lambda(u, x(\cdot;\varphi+h)_u)\Big) \Big|.
\end{aligned}$$

Then Lemma 2.14 implies that Ψ satisfies the inequality

$$\begin{aligned}
\Psi(t;h) &\le \int_0^t F(u)\Psi(u;h) + G(u;h)\,du \\
&\le \int_0^\alpha G(u;h)\,du + \int_0^t F(u)\Psi(u;h)\,du,
\end{aligned}$$

which, by the Gronwall-Bellman inequality, yields that

$$\Psi(t;h) \le \int_0^\alpha G(u;h)\,du \exp\Big(\int_0^\alpha F(u)\,du\Big), \qquad t \in [0,\alpha]. \tag{4.11}$$

We shall show that

$$\int_0^\alpha G(u;h)\,ds \to 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0, \tag{4.12}$$

which in turn, combined with (4.11) yields that

$$\Psi(t;h) \to 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0, \text{ uniformly in } t \in [0,\alpha], \tag{4.13}$$

i.e., statements (i)–(iii) of the theorem are satisfied. To prove (4.12), it is enough to show (by the Lebesgue Dominant Convergence Theorem) that (i) $G(u;h) \to 0$ as $|h|_{W^{1,\infty}} \to 0$ for all $u \in [0,\alpha]$, and (ii) $G(u;h)$ is bounded on $[0,\alpha]$ for small h.

(i) By Lemma 3.20 it follows that there exists a constant M_1 such that

$$|x(\cdot;\varphi+h)_t|_{W^{1,\infty}} \le M_1, \qquad t \in [0,\alpha], \quad |h|_{W^{1,\infty}} < \delta. \tag{4.14}$$

Then (4.14) and (2.5) imply that

$$|\Lambda(t, x(\cdot;\varphi)_t)| \le \|\mu\| M_1, \qquad t \in [0,\alpha].$$

By assumption (A7) the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on the set $A \equiv [0,\alpha] \times \overline{\mathcal{G}}_{\mathbb{R}^n}(M_1) \times \overline{\mathcal{G}}_{\mathbb{R}^n}(\|\mu\| M_1)$, therefore the constant

$$M_2 \equiv \max\Big\{ \sup_{(u,x,y)\in A} \Big\| \frac{\partial f}{\partial x}(u,x,y) \Big\|, \sup_{(u,x,y)\in A} \Big\| \frac{\partial f}{\partial y}(u,x,y) \Big\| \Big\}$$

is well-defined, and satisfies

$$\left\| \frac{\partial f}{\partial y}\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u)\Big)\right\| \le M_2, \qquad u\in[0,\alpha]. \tag{4.15}$$

Therefore, in view of the definition of $G(u;h)$, it is enough to show that for $u\in[0,\alpha]$

$$\frac{1}{|h|_{W^{1,\infty}}}\Big|\omega^2\Big(u, x(\cdot;\varphi)_u; x(\cdot;\varphi+h)_u\Big)\Big| \to 0, \qquad \text{as } |h|_{W^{1,\infty}}\to 0, \tag{4.16}$$

and

$$\frac{1}{|h|_{W^{1,\infty}}}\Big|\omega^1\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u); x(u;\varphi+h), \Lambda(u, x(\cdot;\varphi+h)_u)\Big)\Big| \to 0, \text{ as } |h|_{W^{1,\infty}}\to 0. \tag{4.17}$$

By Theorem 3.20 there exists a constant L_3 such that

$$|x(\cdot;\varphi+h)_t - x(\cdot;\varphi)_t|_{W^{1,\infty}} \le L_3|h|_{W^{1,\infty}}, \qquad t\in[0,\alpha], \quad |h|_{W^{1,\infty}}<\delta. \tag{4.18}$$

To prove (4.16), consider the following estimate (where we use simple manipulations and (4.18)). Let $u\in[0,\alpha]$, $|h|_{W^{1,\infty}}<\delta$, then

$$\begin{aligned}
&\frac{\Big|\omega^2\Big(u, x(\cdot;\varphi)_u; x(\cdot;\varphi+h)_u\Big)\Big|}{|h|_{W^{1,\infty}}}\\
&\quad= \frac{|x(\cdot;\varphi+h)_u - x(\cdot;\varphi)_u|_{W^{1,\infty}}}{|h|_{W^{1,\infty}}}\cdot\frac{\Big|\omega^2\Big(u, x(\cdot;\varphi)_u; x(\cdot;\varphi+h)_u\Big)\Big|}{|x(\cdot;\varphi+h)_u - x(\cdot;\varphi)_u|_{W^{1,\infty}}}\\
&\quad\le L_3\frac{\Big|\omega^2\Big(u, x(\cdot;\varphi)_u; x(\cdot;\varphi+h)_u\Big)\Big|}{|x(\cdot;\varphi+h)_u - x(\cdot;\varphi)_u|_{W^{1,\infty}}}.
\end{aligned} \tag{4.19}$$

Note, that of course, the previous calculation is valid when the denominators, $|h|_{W^{1,\infty}}$ and $|x(\cdot;\varphi+h)_u - x(\cdot;\varphi)_u|_{W^{1,\infty}}$, are not equal to zero, but in the opposite case the definition of ω^2 immediately implies that $\omega^2\Big(u, x(\cdot;\varphi)_u; x(\cdot;\varphi+h)_u\Big) = 0$. From (4.19), using that by (4.18) for all $u\in[0,\alpha]$

$$|x(\cdot;\varphi+h)_u - x(\cdot;\varphi)_u|_{W^{1,\infty}} \to 0, \qquad \text{as } |h|_{W^{1,\infty}}\to 0, \tag{4.20}$$

relation (4.10) implies (4.16). Next we prove (4.17). Let $L_2 = L_2(\alpha, M_1)$ be the constant from (A5). Then estimates (4.14), (4.18) and Lemma 3.12 yield that for $u\in[0,\alpha]$ and $|h|_{W^{1,\infty}}<\delta$

$$\begin{aligned}
&\frac{1}{|h|_{W^{1,\infty}}}\Big|\omega^1\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u); x(u;\varphi+h), \Lambda(u, x(\cdot;\varphi+h)_u)\Big)\Big|\\
&\quad= \frac{|x(u;\varphi+h) - x(u;\varphi)| + |\Lambda(u, x(\cdot;\varphi+h)_u) - \Lambda(u, x(\cdot;\varphi)_u)|}{|h|_{W^{1,\infty}}}\\
&\qquad\cdot\frac{\Big|\omega^1\Big(u, x(u;\varphi), \Lambda(u, x(\cdot;\varphi)_u); x(u;\varphi+h), \Lambda(u, x(\cdot;\varphi+h)_u)\Big)\Big|}{|x(u;\varphi+h) - x(u;\varphi)| + |\Lambda(u, x(\cdot;\varphi+h)_u) - \Lambda(u, x(\cdot;\varphi)_u)|}\\
&\quad\le \left(L_3 + \frac{(\|\mu\| + L_2|x(\cdot;\varphi)_u|_{W^{1,\infty}})\Big|x(\cdot;\varphi+h)_u) - x(\cdot;\varphi)_u)\Big|_C}{|h|_{W^{1,\infty}}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \frac{\left|\omega^1\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u);x(u;\varphi+h),\Lambda(u,x(\cdot;\varphi+h)_u)\Big)\right|}{|x(u;\varphi+h)-x(u;\varphi)|+|\Lambda(u,x(\cdot;\varphi+h)_u)-\Lambda(u,x(\cdot;\varphi)_u)|}\\
\le\ & L_3(1+\|\mu\|+L_2M_1)\\
&\cdot \frac{\left|\omega^1\Big(u,x(u;\varphi),\Lambda(u,x(\cdot;\varphi)_u);x(u;\varphi+h),\Lambda(u,x(\cdot;\varphi+h)_u)\Big)\right|}{|x(u;\varphi+h)-x(u;\varphi)|+|\Lambda(u,x(\cdot;\varphi+h)_u)-\Lambda(u,x(\cdot;\varphi)_u)|}.
\end{aligned}
\tag{4.21}
$$

(As above, we could and did assume that either $|x(u;\varphi+h)-x(u;\varphi)|\neq 0$ or $|\Lambda(u,x(\cdot;\varphi+h)_u)-\Lambda(u,x(\cdot;\varphi)_u)|\neq 0$.) From this, using (4.18), the continuity of Λ, and relations (4.8) we get (4.17), and hence that $G(u;h)\to 0$ as $|h|_{W^{1,\infty}}\to 0$.

(ii) In view of inequalities (4.15), (4.21), and (4.19), we get the boundedness of $G(u,h)$ on $u\in[0,\alpha]$ for small h, if we show that the functions

$$
\frac{|\omega^1(u,\bar x,\bar y;x,y)|}{|x-\bar x|+|y-\bar y|}\qquad\text{and}\qquad\frac{|\omega^2(u,\bar\psi;\psi)|}{|\psi-\bar\psi|_{W^{1,\infty}}}
$$

are bounded for $u\in[0,\alpha]$, $x,\bar x\in\overline{\mathcal G}_{\mathbb R^n}(M_1)$, $y,\bar y\in\overline{\mathcal G}_{\mathbb R^n}(\|\mu\|M_1)$, and $\psi,\bar\psi\in\overline{\mathcal G}_{W^{1,\infty}}(M_1)$, respectively. By the Mean Value Theorem, the definition of M_2 and ω^1 yield the inequality

$$
\begin{aligned}
|\omega^1(u,\bar x,\bar y;x,y)| &\le |f(u,x,y)-f(u,\bar x,\bar y)|+\left|\frac{\partial f}{\partial x}(t,\bar x,\bar y)(x-\bar x)\right|+\left|\frac{\partial f}{\partial y}(t,\bar x,\bar y)(y-\bar y)\right|\\
&\le M_2\Big(|x-\bar x|+|y-\bar y|\Big)+M_2|x-\bar x|+M_2|y-\bar y|\\
&= 2M_2\Big(|x-\bar x|+|y-\bar y|\Big),
\end{aligned}
\tag{4.22}
$$

which proves that the first expression is bounded. Let $L_2=L_2(\alpha,M_1)$ be the constant from (A5), then (A5), Lemma 3.12, the continuity of $\frac{\partial\Lambda}{\partial\psi}(\cdot,\bar\psi)$ guaranteed by Lemma 4.4 (iii) (or Corollary 4.5 in case (1) of the theorem), and inequality (4.14) imply

$$
\begin{aligned}
|\omega^2(u,\bar\psi;\psi)| &\le |\Lambda(u,\psi)-\Lambda(u,\bar\psi)|+\left|\frac{\partial\Lambda}{\partial\psi}(u,\bar\psi)(\psi-\bar\psi)\right|\\
&\le (\|\mu\|+L_2M_1)|\psi-\bar\psi|_C+\left\|\frac{\partial\Lambda}{\partial\psi}(u,\bar\psi)\right\|_{\mathcal L(W^{1,\infty},\mathbb R^n)}|\psi-\bar\psi|_{W^{1,\infty}}\\
&\le \left(\|\mu\|+L_2M_1+\max_{u\in[0,\alpha]}\left\|\frac{\partial\Lambda}{\partial\psi}(u,\bar\psi)\right\|_{\mathcal L(W^{1,\infty},\mathbb R^n)}\right)|\psi-\bar\psi|_{W^{1,\infty}},
\end{aligned}
$$

which finishes the proof of the theorem. □

We recall that Φ is defined by (4.3). The theorem has the following corollary.

Corollary 4.9 *Assuming the conditions of Theorem 4.8, the function* $\Big(\Phi\subset W^{1,\infty}\Big)\to C$, $\varphi\mapsto x(\cdot,\varphi)_t$ *is differentiable for all* $t\in[0,\alpha]$.

Using the relation $|\psi|_{L^p}\le r^{1/p}|\psi|_C$, this result implies immediately the next corollary.

Corollary 4.10 *Assuming the conditions of Theorem 4.8, the function* $\Big(\Phi \subset W^{1,\infty}\Big) \to L^p$, $\varphi \mapsto x(\cdot,\varphi)_t$ *is differentiable for all* $t \in [0,\alpha]$, $1 \le p \le \infty$.

In the next theorem we study the differentiability of the map $\Big(\Phi \subset W^{1,\infty}\Big) \to W^{1,\infty}$, $\varphi \mapsto x(\cdot,\varphi)_t$. In the state-independent case we can show that this map is differentiable, but in the state-dependent case we can show only a weaker result, namely, for all $t \in [0,\alpha]$ the differentiability of the map $\Big((\Phi \cap \mathcal{M}) \subset W^{1,\infty}\Big) \to W^{1,\infty}$, $\varphi \mapsto x(\cdot;\varphi)_t$, when we consider the relative topology on its domain. (I.e., the derivative $\frac{\partial}{\partial\varphi}x(\cdot;\varphi)_t \in \mathcal{L}(\Phi \cap \mathcal{M}, W^{1,\infty})$).

Theorem 4.11 *Assume that the conditions of Theorem 4.8 are satisfied. Then the function*

$$\Big(\Phi \subset W^{1,\infty}\Big) \to W^{1,\infty}, \qquad \varphi \mapsto x(\cdot;\varphi)_t$$

or the function

$$\Big((\Phi \cap \mathcal{M}) \subset W^{1,\infty}\Big) \to W^{1,\infty}, \qquad \varphi \mapsto x(\cdot;\varphi)_t$$

is differentiable for all $t \in [0,\alpha]$, *in case (1) (state-independent equation) and (2) (state-dependent equation) of the theorem, respectively.*

Proof Let $\varphi \in \Phi$ or $\varphi \in \Phi \cap \mathcal{M}$ in case (1) or (2) of the theorem, respectively. In view of Corollary 4.9, to obtain differentiability in $W^{1,\infty}$, we need to show that

$$\frac{\operatorname*{ess\,sup}_{s\in[t-r,t]}\Big|\dot{x}(s;\varphi+h) - \dot{x}(s;\varphi) - \dot{z}(s;h)\Big|}{|h|_{W^{1,\infty}}} \to 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0, \tag{4.23}$$

where $t \in [0,\alpha]$, and $h \in W^{1,\infty}$ if equation (4.1) is state-independent, and $h \in C^1$ such that $\varphi + h \in \Phi \cap \mathcal{M}$ if the equation is state-dependent.

First note, that by initial conditions (4.2) and (4.5) we have that

$$x(t;\varphi+h) - x(t;\varphi) - z(t;\varphi) = 0, \qquad \text{for } t \in [-r,0],$$

and each function, $x(t;\varphi+h)$, $x(t;\varphi)$ and $z(t;h)$ is a.e. differentiable for $t \in [-r,0]$, hence $\dot{x}(t;\varphi+h) - \dot{x}(t;\varphi) - \dot{z}(t;\varphi) = 0$ for a.e. $t \in [-r,0]$. For $t \in [0,\alpha]$ each function is differentiable, and by (4.1) and (4.4) it follows that

$$\begin{aligned}
&|\dot{x}(t;\varphi+h) - \dot{x}(t;\varphi) - \dot{z}(t;h)| \\
&\quad = \bigg| f\Big(t,\, x(t;\varphi+h),\, \Lambda(t, x(\cdot;\varphi+h)_t)\Big) - f\Big(t,\, x(t;\varphi),\, \Lambda(t, x(\cdot;\varphi)_t)\Big) \\
&\qquad - \frac{\partial f}{\partial x}\Big(t, x(t;\varphi), \Lambda(t, x(\cdot;\varphi)_t)\Big) z(t;h) \\
&\qquad - \frac{\partial f}{\partial y}\Big(t, x(t;\varphi), \Lambda(t, x(\cdot;\varphi)_t)\Big) \frac{\partial \Lambda}{\partial \psi}(t, x(\cdot;\varphi)_t) z(\cdot;h)_t \bigg|.
\end{aligned}$$

Using the notations of the proof of Theorem 4.8, and repeating the estimates we used in the proof we get that

$$\frac{\Big|\dot{x}(t;\varphi+h) - \dot{x}(t;\varphi) - \dot{z}(t;h)\Big|}{|h|_{W^{1,\infty}}} \le F(t)\Psi(t;h) + G(t;h), \qquad t \in [0,\alpha]. \tag{4.24}$$

In view of (4.13), it is left to prove that: (i) the function $F(t)$ is bounded on $[0,\alpha]$, and (ii) $G(t;h) \to 0, \quad \text{as } |h|_{W^{1,\infty}} \to 0, \quad \text{uniformly on } t \in [0,\alpha]$.

To show (i), from the definition of $F(t)$ and M_2, and by Lemma 4.4 (ii) we can obtain the following estimates:

$$\begin{aligned} F(t) &\leq M_2 + M_2 \left\| \frac{\partial \Lambda}{\partial \psi}(t, x(\cdot;\varphi)_t) \right\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)} \\ &\leq M_2 + M_2 \left\| \frac{\partial \lambda}{\partial \xi}(t, x(\cdot;\varphi)_t, x(\cdot;\varphi)_t) \right\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)} \\ &\quad + M_2 \left\| \frac{\partial \lambda}{\partial \psi}(t, x(\cdot;\varphi)_t, x(\cdot;\varphi)_t) \right\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}. \end{aligned} \tag{4.25}$$

By Lemma 4.4 and (2.5) we have that

$$\begin{aligned} \left| \frac{\partial \lambda}{\partial \xi}(t,\psi,\xi)h \right| &= |\lambda(t,\psi,h)| \\ &\leq \|\mu\| |h|_{W^{1,\infty}}, \end{aligned}$$

therefore $\left\| \frac{\partial \lambda}{\partial \xi} \right\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}$ is bounded by $\|\mu\|$, and hence (4.25) implies

$$F(t) \leq M_2 + M_2\|\mu\| + M_2 \left\| \frac{\partial \lambda}{\partial \psi}(t, x(\cdot;\varphi)_t, x(\cdot;\varphi)_t) \right\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}. \tag{4.26}$$

In case (1) of the theorem (i.e., when equation (4.1) is state-independent) we have that $\frac{\partial \lambda}{\partial \psi}$ is identically zero. In case (2) using that $t \to x(\cdot;\varphi)_t$ is continuous in the $|\cdot|_{W^{1,\infty}}$ norm and $x(\cdot;\varphi)_t \in C^1$ by Lemma 3.21, and $\frac{\partial \lambda}{\partial \psi}(t,\psi,\xi)$ is continuous for $\psi, \xi \in C^1$, we get that the last term in the right hand side of inequality (4.26) is bounded for $t \in [0,\alpha]$, therefore we have shown (i).

To get (ii), by the definition of $G(t;h)$ and estimate (4.15), it is enough to show that

$$\frac{\left|\omega^1\Big(t, x(t;\varphi), \Lambda(t, x(\cdot;\varphi)_t); x(t;\varphi+h), \Lambda(t, x(\cdot;\varphi+h)_t)\Big)\right|}{|x(t;\varphi+h) - x(t;\varphi)| + |\Lambda(t, x(\cdot;\varphi+h)_t) - \Lambda(t, x(\cdot;\varphi)_t)|} \to 0, \ \text{ uniformly on } t \in [0,\alpha], \tag{4.27}$$

and

$$\frac{\left|\omega^2\Big(t, x(\cdot;\varphi)_t; x(\cdot;\varphi+h)_t\Big)\right|}{|x(\cdot;\varphi+h)_t - x(\cdot;\varphi)_t|_{W^{1,\infty}}} \to 0, \qquad \text{uniformly on } t \in [0,\alpha], \tag{4.28}$$

as $|h|_{W^{1,\infty}} \to 0$.

By applying assumption (A7), Lemmas 2.16 and 2.17, the defining relation (4.7) implies that

$$\begin{aligned} \frac{|\omega^1(t,\bar{x},\bar{y};x,y)|}{|x-\bar{x}|+|y-\bar{y}|} \leq \max\Bigg\{ &\sup_{0<\nu<1} \left\| \frac{\partial f}{\partial x}\Big(t, \bar{x}+\nu(x-\bar{x}), \bar{y}+\nu(y-\bar{y})\Big) - \frac{\partial f}{\partial x}\Big(t,\bar{x},\bar{y}\Big) \right\|, \\ &\sup_{0<\nu<1} \left\| \frac{\partial f}{\partial y}\Big(t, \bar{x}+\nu(x-\bar{x}), \bar{y}+\nu(y-\bar{y})\Big) - \frac{\partial f}{\partial y}\Big(t,\bar{x},\bar{y}\Big) \right\| \Bigg\}. \end{aligned} \tag{4.29}$$

Let $V_1 \equiv \{x(t;\varphi) \ : \ t \in [0,\alpha]\}$, and $V_2 \equiv \{\Lambda(t, x(\cdot;\varphi)_t) \ : \ t \in [0,\alpha]\}$. Then $V_1 \subset \Omega_1$ and $V_2 \subset \Omega_2$ are compact subsets of $\mathbb{R}^n$, since they are continuous images of the compact set $[0,\alpha]$.

(For V_2 we used that, by Lemma 2.10, $t \mapsto x(\cdot;\varphi)_t$ is continuous as a map $[0,\alpha] \to C$, and $\Lambda\ :\Big([0,\alpha]\times\Omega_3 \subset [0,\alpha]\times C\Big) \to \mathbb{R}^n$ is continuous by (A2).) For $i = 1,2$, let

$$U_i \text{ be an open set, such that } \bar{U}_i \text{ compact, and } V_i \subset U_i \subset \bar{U}_i \subset \Omega_i. \tag{4.30}$$

Note, that such U_1 and U_2 clearly exist. Let

$$\rho \equiv \min\Big\{\operatorname{dist}(V_1, \mathbb{R}^n \setminus U_1),\ \operatorname{dist}(V_2, \mathbb{R}^n \setminus U_2)\Big\},$$

i.e., the smallest of the distances between V_i and the complement of U_i, $i = 1,2$. Then $\rho > 0$, and if $|x-\bar{x}|+|y-\bar{y}| < \rho$ then $\bar{x}+v(x-\bar{x}) \in U_1$ and $\bar{y}+v(y-\bar{y}) \in U_2$ for all $0 < v < 1$. Then (4.29) and the continuity, and hence the uniform continuity of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ on the compact set $[0,\alpha]\times\bar{U}_1\times\bar{U}_2$ implies that for all $\bar{x} \in V_1$ and $\bar{y} \in V_2$

$$\frac{|\omega^1(t,\bar{x},\bar{y};x,y)|}{|x-\bar{x}|+|y-\bar{y}|} \to 0 \quad \text{as } x \to \bar{x},\ y \to \bar{y}, \quad \text{uniformly in } t \in [0,\alpha], \bar{x} \in V_1 \text{ and } \bar{y} \in V_2. \tag{4.31}$$

Estimate (4.18) and Lemma 3.12 imply that

$$\begin{aligned} |x(t;\varphi+h)-x(t;\varphi)| + |\Lambda(t,x(\cdot;\varphi+h)_t)-\Lambda(t,x(\cdot;\varphi)_t)| \\ \leq\ & L_3|h|_{W^{1,\infty}} + (\|\mu\| + L_2(\alpha,M_1)M_1)L_3|h|_{W^{1,\infty}} \\ <\ & \rho \end{aligned} \tag{4.32}$$

for

$$|h|_{W^{1,\infty}} < \frac{\rho}{(1+\|\mu\|+L_2(\alpha,M_1)M_1)L_3},$$

and hence for such h and $t \in [0,\alpha]$ we have that $x(t;\varphi+h) \in U_1$ and $\Lambda(t,x(\cdot;\varphi+h)_t) \in U_2$. Estimate (4.32) yields that

$$|x(t;\varphi+h)-x(t;\varphi)| + |\Lambda(t,x(\cdot;\varphi+h)_t)-\Lambda(t,x(\cdot;\varphi)_t)| \to 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0,$$

therefore (4.31) yields (4.27).

Next we concentrate on proving (4.28). The linearity of $\lambda(t,\psi,\xi)$ in ξ, and Lemmas 4.4 and 4.7 imply that

$$\begin{aligned} \omega^2(t,\bar{\psi};\psi) &= \Lambda(t,\psi)-\Lambda(t,\bar{\psi}) - \frac{\partial\Lambda}{\partial\psi}(t,\bar{\psi})(\psi-\bar{\psi}) \\ &= \lambda(t,\psi,\psi)-\lambda(t,\bar{\psi},\bar{\psi}) - \frac{\partial\lambda}{\partial\xi}(t,\bar{\psi},\bar{\psi})(\psi-\bar{\psi}) - \frac{\partial\lambda}{\partial\psi}(t,\bar{\psi},\bar{\psi})(\psi-\bar{\psi}) \\ &= \lambda(t,\psi,\psi)-\lambda(t,\bar{\psi},\bar{\psi}) - \lambda(t,\bar{\psi},\psi-\bar{\psi}) - \frac{\partial\lambda}{\partial\psi}(t,\bar{\psi},\bar{\psi})(\psi-\bar{\psi}) \\ &= \lambda(t,\psi,\psi-\bar{\psi})-\lambda(t,\bar{\psi},\psi-\bar{\psi}) \\ &\quad + \lambda(t,\psi,\bar{\psi})-\lambda(t,\bar{\psi},\bar{\psi}) - \frac{\partial\lambda}{\partial\psi}(t,\bar{\psi},\bar{\psi})(\psi-\bar{\psi}). \end{aligned} \tag{4.33}$$

Since by (A5) it follows that for $t \in [0,\alpha]$ and $\psi,\bar{\psi} \in \overline{\mathcal{G}}_{W^{1,\infty}}(M_1)$

$$|\lambda(t,\psi,\psi-\bar{\psi})-\lambda(t,\bar{\psi},\psi-\bar{\psi})| \leq L_2(\alpha,M_1)|\psi-\bar{\psi}|_{W^{1,\infty}}|\psi-\bar{\psi}|_C,$$

we have that

$$\frac{|\lambda(t,\psi,\psi-\bar\psi)-\lambda(t,\bar\psi,\psi-\bar\psi)|}{|\psi-\bar\psi|_{W^{1,\infty}}}\to 0,\quad \text{as } \psi\to\bar\psi, \tag{4.34}$$

uniformly on $t\in[0,\alpha]$, $\bar\psi\in\overline{\mathcal{G}}_{W^{1,\infty}}(M_1)$. Define the function

$$\omega^3(t,\bar\psi;\psi)\equiv\lambda(t,\psi,\bar\psi)-\lambda(t,\bar\psi,\bar\psi)-\frac{\partial\lambda}{\partial\psi}(t,\bar\psi,\bar\psi)(\psi-\bar\psi), \tag{4.35}$$

for $t\in[0,\alpha]$, $\psi,\bar\psi\in W^{1,\infty}\cap\Omega_3$. To prove (4.28), in view of (4.33) and (4.34), we need to insure that

$$\frac{|\omega^3(t,x(\cdot;\varphi)_t;x(\cdot;\varphi+h)_t)|}{|x(\cdot;\varphi+h)_t-x(\cdot;\varphi)_t|_{W^{1,\infty}}}\to 0,\quad \text{as } h\to 0,\qquad \text{uniformly on } t\in[0,\alpha]. \tag{4.36}$$

In case (1) of this theorem (i.e., if the delay is state-independent), (4.36) is automatically satisfied, therefore we can assume in the remaining part of the proof case (2) of the assumptions, i.e., that the delay is state-dependent, and we restrict the initial functions to $\Phi\cap\mathcal{M}$.

By Lemma 2.16 we get that

$$\frac{|\omega^3(t,\bar\psi;\psi)|}{|\psi-\bar\psi|_{W^{1,\infty}}}\le\sup_{0<\nu<1}\left\|\frac{\partial\lambda}{\partial\psi}(t,\bar\psi+\nu(\psi-\bar\psi),\bar\psi)-\frac{\partial\lambda}{\partial\psi}(t,\bar\psi,\bar\psi)\right\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}. \tag{4.37}$$

Unfortunately, for a compact set V in $W^{1,\infty}$, there is no U satisfying (4.30), hence the argument that was used to prove (4.27) does not work in this case. But our final goal is to prove (4.23), therefore instead of (4.36), in fact, it is enough to show that for $h^k\in W^{1,\infty}$ such that $h^k\to 0$ as $k\to\infty$ it follows that

$$\frac{|\omega^3(t,x(\cdot;\varphi)_t;x(\cdot;\varphi+h^k)_t)|}{|x(\cdot;\varphi+h^k)_t-x(\cdot;\varphi)_t|_{W^{1,\infty}}}\to 0,\quad \text{as } k\to\infty,\qquad \text{uniformly in } t\in I. \tag{4.38}$$

Fix a sequence $h^k\in W^{1,\infty}$ such that $|h^k|_{W^{1,\infty}}<\delta$ and $h^k\to 0$ as $k\to\infty$.

Define the set

$$V\equiv\Big\{x(\cdot;\varphi)_t+\nu\Big(x(\cdot;\varphi+h^k)_t-x(\cdot;\varphi)_t\Big)\ :\ t\in I,\ \nu\in[0,1],\ \text{and } k\in\mathbb{N}\Big\}. \tag{4.39}$$

We show that V is compact subset of $W^{1,\infty}$. Clearly, $V\subset W^{1,\infty}$. Pick an arbitrary sequence, $\{\psi^j\}$, from V. Then for each ψ^j there correspond $t^j\in I$, $\nu^j\in[0,1]$ and $k^j\in\mathbb{N}$, such that

$$\psi^j=x(\cdot;\varphi)_{t^j}+\nu^j\Big(x(\cdot;\varphi+h^{k^j})_{t^j}-x(\cdot;\varphi)_{t^j}\Big).$$

We need to show, that it has a convergent subsequence with limit in V. Clearly, we can and therefore do assume, in order to keep the notations simple, that $t^j\to\bar t\in I$ and $\nu^j\to\bar\nu\in[0,1]$ as $j\to\infty$. The following two cases can happen: either k^j has a subsequence converging to ∞, or k^j has a constant subsequence. Therefore, again, we can and do assume that $h^{k^j}\to\bar h$, where either $\bar h=0$ or $\bar h=h^k$ for some $k\in\mathbb{N}$. We claim that

$$\psi^j\to\bar\psi\equiv x(\cdot;\varphi)_{\bar t}+\bar\nu\Big(x(\cdot;\varphi+\bar h)_{\bar t}-x(\cdot;\varphi)_{\bar t}\Big),\qquad \text{as } j\to\infty.$$

Note first, that $\bar{\psi} \in V$. Consider

$$\begin{aligned}
|\psi^j - \bar{\psi}|_{W^{1,\infty}} &\le |x(\cdot;\varphi)_{t^j} - x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}} + |\nu^j - \bar{\nu}||x(\cdot;\varphi+\bar{h})_{\bar{t}} - x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}} \\
&\quad + |\nu^j|\Big(|x(\cdot;\varphi+h^{k^j})_{t^j} - x(\cdot;\varphi)_{t^j} - x(\cdot;\varphi+\bar{h})_{\bar{t}} + x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}}\Big) \\
&\le |x(\cdot;\varphi)_{t^j} - x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}} + |\nu^j - \bar{\nu}||x(\cdot;\varphi+\bar{h})_{\bar{t}} - x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}} \\
&\quad + |x(\cdot;\varphi+h^{k^j})_{t^j} - x(\cdot;\varphi+\bar{h})_{t^j}|_{W^{1,\infty}} \\
&\quad + |x(\cdot;\varphi+\bar{h})_{t^j} - x(\cdot;\varphi+\bar{h})_{\bar{t}}|_{W^{1,\infty}} + |x(\cdot;\varphi)_{t^j} - x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}} \\
&\le 2|x(\cdot;\varphi)_{t^j} - x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}} + |\nu^j - \bar{\nu}||x(\cdot;\varphi+\bar{h})_{\bar{t}} - x(\cdot;\varphi)_{\bar{t}}|_{W^{1,\infty}} \\
&\quad + L_3|h^{k^j} - \bar{h}|_{W^{1,\infty}} + |x(\cdot;\varphi+\bar{h})_{t^j} - x(\cdot;\varphi+\bar{h})_{\bar{t}}|_{W^{1,\infty}} \\
&\to 0, \qquad \text{as } j \to \infty,
\end{aligned}$$

where we used $|\nu^j| \le 1$, (4.18), Lemma 3.21 and our assumptions. This completes the proof of compactness of V.

Since $\frac{\partial\lambda}{\partial\psi}(t,\psi,\xi)$ is continuous, and hence uniformly continuous on the compact set $I\times V\times V$, relations (4.18) and (4.37) imply (4.38).

We have completed the proof of the theorem. □

Since $|\psi|_{W^{1,p}} \le (2r)^{1/p}|\psi|_{W^{1,\infty}}$, the theorem implies immediately:

Corollary 4.12 *Assuming the conditions of Theorem 4.8, the function*

$$\Big(\Phi \subset W^{1,\infty}\Big) \to W^{1,p}, \qquad \varphi \mapsto x(\cdot,\varphi)_t,$$

or

$$\Big((\Phi \cap \mathcal{M}) \subset W^{1,\infty}\Big) \to W^{1,p}, \qquad \varphi \mapsto x(\cdot,\varphi)_t,$$

is differentiable for all $t \in [0,\alpha]$, $1 \le p \le \infty$ for case (1) or (2) of the theorem, respectively.

Remark 4.13 *Note, that if in Example 4.3 there are no point delays, i.e., $A_k(t) = 0$ for all $k = 1,\dots,m$, then it follows from Remark 4.6 and the proofs of Theorems 4.8 and 4.11, that the corresponding solution, $x(\cdot;\varphi)_t$, is differentiable wrt φ in $W^{1,\infty}$ for all $\varphi \in \Phi$, i.e., the assumption $\varphi \in \mathcal{M}$ is not needed.*

4.1.2 Preliminaries

In this subsection we formulate a weaker version of Theorem 2.23, and introduce some new spaces which will be essential to obtain our results in the next section.

Theorem 4.14 *Let Z be a normed space, $(Y, |\cdot|)$ is a quasi-Banach space wrt the norm $\|\cdot\|$. Let W be a closed, convex subset of Y with non-empty interior, and V be an open subset of Z, and assume that $S : W \times V \to W$ satisfies:*

(i) S *is a uniform* $|\cdot|$ *and* $\|\cdot\|$ *contraction, i.e., there exists* $0 \le \theta < 1$ *such that*

$$|S(y,z) - S(\bar{y},z)| \le \theta |y - \bar{y}|, \qquad \text{for } y, \bar{y} \in W,\ z \in V,$$

and

$$\|S(y,z) - S(\bar{y},z)\| \le \theta \|y - \bar{y}\|, \qquad \text{for } y, \bar{y} \in W,\ z \in V.$$

(ii) For each $\rho > 0$ *there exists* $R > 0$ *such that*

$$S\Big((\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap W) \times (\mathcal{G}_Z(\rho) \cap V)\Big) \subset (\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap W).$$

(iii) For all $y \in W$ *the function* $S(y,\cdot)\ :\ \Big(V \subset Z\Big) \to Y$ *is continuous.*

Then for each $z \in V$*, there exists a unique fixed point* $g(z)$ *of* $S(\cdot, z)$ *in* W*, which depends continuously on* z*. Moreover, if in addition*

(iv) $S\ :\ \Big(W \times V \subset (W \cap Y) \times Z\Big) \to Y$ *is continuously differentiable on* $W \times V$ *(i.e., on the domain of* S *the relative topology generated by* W *is used when we talk about differentiability, but by a derivative* $\frac{\partial S}{\partial y}(y,z)$ *we mean a bounded linear operator from* $Y \to Y$*),*

then the map $g\ :\ \Big(V \subset Z\Big) \to Y$ *is continuously differentiable.*

Proof The proof is essentially the same as that of Theorem 2.23 (see in [33]), and therefore only the main steps are presented here, and we point out the difference in the respective arguments due to the fact that here differentiability is required in a weaker sense (in the relative topology on $Y \cap W$).

For a fixed $z \in V$, assumption (ii) implies that there exists an $R > 0$ such that

$$S(\cdot, z)\ :\ (\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap W) \to (\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R) \cap W),$$

and since $\overline{\mathcal{G}}_{(Y,\|\cdot\|)}(R)$ is a complete subset of Y, the existence of a unique fixed point of $S(\cdot,z)$, $g(z)$, follows. A standard argument (using (i) and (iii)) shows that $g(\cdot)\ :\ V \to Y$ is continuous.

Assumption (i) yields that $\left|\frac{\partial S}{\partial y}(y,z)\right|_{\mathcal{L}(Y,Y)} \le \theta$ and $\left\|\frac{\partial S}{\partial y}(y,z)\right\|_{\mathcal{L}\big((Y,\|\cdot\|),(Y,\|\cdot\|)\big)} \le \theta$ for all $(y,z) \in W \times V$, and therefore (by using a series of Lemmas in [33]), $\Big(I - \frac{\partial S}{\partial y}(y,z)\Big)^{-1} \in \tilde{\mathcal{L}}(Y)$ exists and continuous in (y,z). Define

$$M(z) \equiv \Big(I - \frac{\partial S}{\partial y}(g(z),z)\Big)^{-1} \frac{\partial S}{\partial z}(g(z),z).$$

We shall show that $g'(z) = M(z)$. Let $\gamma = \gamma(h) \equiv g(z+h) - g(z)$. Then it is easy to see that

$$\gamma = \frac{\partial S}{\partial y}(g(z),z)\gamma + \frac{\partial S}{\partial z}(g(z),z)h + \Delta,$$

where

$$\Delta \equiv S(g(z)+\gamma, z+h) - S(g(z),z) - \frac{\partial S}{\partial y}(g(z),z)\gamma - \frac{\partial S}{\partial z}(g(z),z)h.$$

By Lemmas 2.16 and 2.17, we have the following estimate for $|\Delta|$

$$\begin{aligned} |\Delta| \;\le\; & \sup_{0<\nu<1}\left|\frac{\partial S}{\partial y}(g(z)+\nu\gamma, z+\nu h) - \frac{\partial S}{\partial y}(g(z),z)\right|_{\mathcal{L}(Y,Y)} |\gamma| \\ & + \sup_{0<\nu<1}\left|\frac{\partial S}{\partial z}(g(z)+\nu\gamma, z+\nu h) - \frac{\partial S}{\partial z}(g(z),z)\right|_{\mathcal{L}(Z,Y)} |h|_Z. \end{aligned}$$

We first comment that $g(z)\in W$, and $g(z)+\gamma = g(z+h)\in W$, and since W is convex, $g(z)+\nu\gamma\in W$ for all $0\le\nu\le 1$. On the domain of S we use the relative topology defined by W, in which W itself is an open set, and hence Lemmas 2.16 and 2.17 are applicable for this case. Then the assumed continuity of the partial derivatives on $W\times V$ yields the estimate $|\Delta|\le\varepsilon(|\gamma|+|h|_Z)$, for $\varepsilon>0$ and for sufficiently small γ and h. The remaining part of the proof is identical that of Theorem 2.23. In particular, it is possible to obtain an estimate of the form

$$|g(z+h)-g(z)-M(z)h| < \frac{\varepsilon(1+k)}{1-\theta}|h|_Z,$$

which proves the statement. The details are omitted. □

We define the space

$$\mathbb{Y}^p_\alpha \equiv \Big\{ y\in W^{1,\infty}_\alpha \;:\; y(t)=0 \text{ on } [-r,0]\Big\},$$

with corresponding norms

$$|y|_{\mathbb{Y}^p_\alpha} \equiv \left(\int_0^\alpha |\dot y(s)|^p\,ds\right)^{1/p}, \qquad \text{for } 1\le p<\infty,$$

and

$$|y|_{\mathbb{Y}^\infty_\alpha} \equiv \operatorname*{ess\,sup}_{s\in[0,\alpha]} |\dot y(s)|, \qquad \text{for } p=\infty,$$

respectively. Note, that $\mathbb{Y}^p_\alpha$ is the same set for all p, but it is equipped with different norms. Clearly, $\mathbb{Y}^p_\alpha$ is a normed linear space, and $\mathbb{Y}^\infty_\alpha$ is a Banach-space, (since it is a closed subspace of $W^{1,\infty}_\alpha$).

The following lemma contains some basic properties of these norms.

Lemma 4.15 *Let $y\in\mathbb{Y}^p_\alpha$, $1\le p<\infty$, and q is the conjugate to p, i.e., $1/p+1/q=1$. Then the following estimates hold:*

(i) $|y(t)|\le\alpha^{1/q}|y|_{\mathbb{Y}^p_\alpha}$, *for* $t\in[-r,\alpha]$, $1\le p<\infty$,

(ii) $|y(t)|\le\alpha|y|_{\mathbb{Y}^\infty_\alpha}$, *for* $t\in[-r,\alpha]$,

(iii) $|y_t|_C\le\alpha^{1/q}|y|_{\mathbb{Y}^p_\alpha}$, *for* $t\in[0,\alpha]$, $1\le p<\infty$,

(iv) $|y_t|_C\le\alpha|y|_{\mathbb{Y}^\infty_\alpha}$, *for* $t\in[0,\alpha]$,

(v) $|y|_{\mathbb{Y}^p_\alpha}\le\alpha^{1/p}|y|_{\mathbb{Y}^\infty_\alpha}$, *for* $1\le p<\infty$,

(vi) $|y|_{\mathbb{Y}^p_\alpha}\le|y|_{W^{1,p}_\alpha}\le(\alpha^p+1)^{1/p}|y|_{\mathbb{Y}^p_\alpha}$, *i.e.,* $|\cdot|_{\mathbb{Y}^p_\alpha}$ *is equivalent to the norm* $|\cdot|_{W^{1,p}_\alpha}$ *on* $\mathbb{Y}^p_\alpha$, *for* $1\le p<\infty$,

(vii) $|y|_{\Upsilon_\alpha^\infty} \le |y|_{W_\alpha^{1,\infty}} \le \max\{\alpha, 1\}|y|_{\Upsilon_\alpha^\infty}$*, i.e.,* $|\cdot|_{\Upsilon_\alpha^\infty}$ *is equivalent to the norm* $|\cdot|_{W_\alpha^{1,\infty}}$ *on* Υ_α^∞*,*

Proof By the absolute continuity of $y \in \Upsilon_\alpha^p$ and $y(0) = 0$ it follows that

$$y(t) = \int_0^t \dot{y}(s)\, ds, \qquad t \in [0, \alpha],$$

and therefore the inequality

$$|y(t)| \le \int_0^\alpha |\dot{y}(s)|\, ds$$

implies (ii), and together with Hölder's inequality, implies (i). Clearly, (i) implies (iii), and (ii) yields (iv). (v) follows directly from the definition of the norms, and (vi) and (vii) easily follow from (i) and (ii). □

For $1 \le p < \infty$, Υ_α^p is not a Banach-space, but it is a quasi-Banach space wrt the $|\cdot|_{W_\alpha^{1,\infty}}$ norm. (See Chapter 2 for the definition of quasi-Banach spaces.) Hale and Ladeira applied the extension of the Uniform Contraction Theorem (Theorem 2.23) for this space (with $p = 1$) in [33] to obtain their results.

Lemma 4.16 *Let* $1 \le p < \infty$*,* $0 < \alpha < \infty$*. Then the space* Υ_α^p *is a quasi-Banach space wrt the* $|\cdot|_{\Upsilon_\alpha^\infty}$*-norm.*

Proof The lemma follows from the next result, using $\bar{y} = 0$ and that the $|\cdot|_{\Upsilon_\alpha^\infty}$ and $|\cdot|_{W_\alpha^{1,\infty}}$ norms are equivalent by Lemma 4.15 (vii). □

Lemma 4.17 *Let* $\bar{y} \in W_\alpha^{1,\infty}$*,* $\delta > 0$*,* $1 \le p < \infty$*. Then the set* $\overline{\mathcal{G}}_{W_\alpha^{1,\infty}}(\bar{y}; \delta) \cap \Upsilon_\alpha^p$ *is a closed, complete and convex subset of* Υ_α^p*.*

Proof Obviously, $\overline{\mathcal{G}}_{W_\alpha^{1,\infty}}(\bar{y}; \delta) \cap \Upsilon_\alpha^p$ is convex. Let $y^k \in \overline{\mathcal{G}}_{W_\alpha^{1,\infty}}(\bar{y}; \delta) \cap \Upsilon_\alpha^p$ be a Cauchy-sequence in the $|\cdot|_{\Upsilon_\alpha^p}$-norm. By Lemma 4.15 (vi) the $|\cdot|_{\Upsilon_\alpha^p}$ and $|\cdot|_{W_\alpha^{1,p}}$ norms are equivalent, therefore $\{y^k\}$ is a Cauchy-sequence in $W_\alpha^{1,p}$ as well. Since $W_\alpha^{1,p}$ is a Banach-space, there exists a function $y \in W_\alpha^{1,p}$ such that $|y^k - y|_{W_\alpha^{1,p}} \to 0$ as $k \to \infty$, and therefore $|y^k - y|_{\Upsilon_\alpha^p} \to 0$ as $k \to \infty$. Lemma 4.15 (i) yields that

$$\begin{aligned} |y^k(t) - y^l(t)| &\le \alpha^{1/q}|y^k - y^l|_{\Upsilon_\alpha^p}, \\ &\to 0, \qquad \text{as } k, l \to \infty, \end{aligned}$$

so $\{y^k(t)\}$ is a Cauchy-sequence for all $t \in [0, \alpha]$, and hence $\{y^k(t)\}$ is pointwise convergent to $y(t)$. We need to show that $y \in \overline{\mathcal{G}}_{W_\alpha^{1,\infty}}(\bar{y}; \delta)$. Since $|y^k - \bar{y}|_{W_\alpha^{1,\infty}} \le \delta$, it follows that $|y^k - \bar{y}|_C \le \delta$, and therefore by the pointwise convergence of y^k to y we get that $|y - \bar{y}|_C \le \delta$.

Suppose that $y \notin \overline{\mathcal{G}}_{W_\alpha^{1,\infty}}(\bar{y}; \delta)$, i.e., $|y - \bar{y}|_{W_\alpha^{1,\infty}} > \delta$. Then the previous comment implies that $\operatorname*{ess\,sup}_{0 \le u \le \alpha} |\dot{y}(u) - \dot{\bar{y}}(u)| > \delta + \varepsilon$ for some $\varepsilon > 0$, and therefore the set $A \equiv \{u \,:\, |\dot{y}(u) - \dot{\bar{y}}(u)| > \delta + \varepsilon\}$ has positive measure. Since $\operatorname*{ess\,sup}_{0 \le u \le \alpha} |\dot{y}^k(u) - \dot{\bar{y}}(u)| \le \delta$ for all $k \in \mathbb{N}$, and hence $meas(\{u \,:\, |\dot{y}^k(u) - \dot{\bar{y}}(u)| > \delta\}) = 0$, we have that the set

$$B \equiv [0, \alpha] \setminus \bigcup_{k=1}^\infty \Big\{u \,:\, |\dot{y}^k(u) - \dot{\bar{y}}(u)| > \delta\Big\} = \Big\{u \,:\, |\dot{y}^k(u) - \dot{\bar{y}}(u)| \le \delta, \ k \in \mathbb{N}\Big\}.$$

has measure α. We show that $A \cap B$ has positive measure. Suppose that $meas(A \cap B) = 0$. Then we have that

$$\begin{aligned} meas(A) &= meas(A \setminus B) + meas(A \cap B) \\ &= meas(A \setminus B) \\ &\leq meas([0,\alpha] \setminus B) \\ &= 0, \end{aligned}$$

which is a contradiction, hence $meas(A \cap B) > 0$. Then elementary estimates imply that

$$\begin{aligned} |y - y^k|_{\mathbb{Y}^p_\alpha} &\geq \left(\int\limits_{A\cap B} |\dot y(u) - \dot y^k(u)|^p \, du \right)^{1/p} \\ &\geq \left(\int\limits_{A\cap B} \Big(|\dot y(u) - \dot{\bar y}(u)| - |\dot{\bar y}(u) - \dot y^k(u)| \Big)^p \, du \right)^{1/p} \\ &\geq \varepsilon \Big(meas(A \cap B) \Big)^{1/p} \\ &> 0, \end{aligned}$$

which is a contradiction. Therefore $y \in \overline{\mathcal{G}}_{W^{1,\infty}_\alpha}(\bar y;\, \delta)$, i.e., $\overline{\mathcal{G}}_{W^{1,\infty}_\alpha}(\bar y;\, \delta) \cap \mathbb{Y}^p_\alpha$ is complete, and hence also closed in $\mathbb{Y}^p_\alpha$. □

Next we introduce a new norm on $W^{1,\infty}_\alpha$. Let $x \in W^{1,\infty}_\alpha$. Then let

$$\varphi(s) \equiv x(s), \qquad -r \leq s \leq 0, \tag{4.40}$$

and

$$y(u) \equiv \begin{cases} 0, & -r \leq u \leq 0, \\ x(u) - x(0), & 0 \leq u \leq \alpha. \end{cases} \tag{4.41}$$

Then we have that $x = y + \tilde\varphi$, and $y \in \mathbb{Y}^p_\alpha$, $\varphi \in W^{1,\infty}$, i.e., we can decompose $W^{1,\infty}_\alpha$ as a direct sum of $\mathbb{Y}^p_\alpha$ and $W^{1,\infty}$. Define the projection operators according to (4.40) and (4.41) by

$$\mathrm{Pr}_\varphi \;:\; W^{1,\infty}_\alpha \to W^{1,\infty}, \qquad (\mathrm{Pr}_\varphi\, x)(s) \equiv x(s), \quad s \in [-r, 0], \tag{4.42}$$

and

$$\mathrm{Pr}_y \;:\; W^{1,\infty}_\alpha \to \mathbb{Y}^p_\alpha, \qquad (\mathrm{Pr}_y\, x)(u) \equiv \begin{cases} 0, & -r \leq u \leq 0, \\ x(u) - x(0), & 0 \leq u \leq \alpha. \end{cases} \tag{4.43}$$

Define a "product norm" on $W^{1,\infty}_\alpha$ by

$$|x|_{\mathbb{X}^p_\alpha} \equiv |\mathrm{Pr}_y\, x|_{\mathbb{Y}^p_\alpha} + |\mathrm{Pr}_\varphi\, x|_{W^{1,\infty}}, \tag{4.44}$$

and denote the corresponding normed linear space by

$$\mathbb{X}^p_\alpha \equiv \Big(W^{1,\infty}_\alpha,\, |\cdot|_{\mathbb{X}^p_\alpha} \Big).$$

Part (i) and (ii) of the following lemma shows that this "product" norm is stronger than the $|\cdot|_{W^{1,p}_\alpha}$-norm, and weaker than the $|\cdot|_{W^{1,\infty}_\alpha}$-norm on $W^{1,\infty}_\alpha$. The estimates (iii) and (iv) will be used later.

Lemma 4.18 *Let $1 \le p < \infty$. There exist constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ and $c_4 > 0$ such that for all $x \in W^{1,\infty}_\alpha$*

(i) $|x|_{W^{1,p}_\alpha} \le c_1 |x|_{\mathbb{X}^p_\alpha}$,

(ii) $|x|_{\mathbb{X}^p_\alpha} \le c_2 |x|_{W^{1,\infty}_\alpha}$,

(iii) $|x|_{C_\alpha} \le c_3 |x|_{\mathbb{X}^p_\alpha}$.

(iv) $|\dot{x}|_{L^p_\alpha} \le c_4 |x|_{\mathbb{X}^p_\alpha}$.

Proof Let $x = y + \tilde{\varphi}$ be the direct sum decomposition of x defined by (4.40) and (4.41). Using the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ and Lemma 4.15 (i) we get

$$\begin{aligned}
|x|^p_{W^{1,p}_\alpha} &= \int_{-r}^0 |\varphi(s)|^p + |\dot{\varphi}(s)|^p \, ds + \int_0^\alpha |y(u) + \varphi(0)|^p + |\dot{y}(u)|^p \, du \\
&\le 2r|\varphi|^p_{W^{1,\infty}} + 2^{p-1} \int_0^\alpha |y(u)|^p \, du + \alpha 2^{p-1} |\varphi(0)|^p + \int_0^\alpha |\dot{y}(u)|^p \, du \\
&\le (2^{p-1}\alpha + 2r)|\varphi|^p_{W^{1,\infty}} + 2^{p-1}\alpha^{p/q+1} |y|^p_{\mathbb{Y}^p_\alpha} + |y|^p_{\mathbb{Y}^p_\alpha} \\
&\le \max\Big\{2^{p-1}\alpha + 2r, 2^{p-1}\alpha^p + 1\Big\}\Big(|y|^p_{\mathbb{Y}^p_\alpha} + |\varphi|^p_{W^{1,\infty}}\Big) \\
&\le \max\Big\{2^{p-1}\alpha + 2r, 2^{p-1}\alpha^p + 1\Big\} 2|x|^p_{\mathbb{X}^p_\alpha},
\end{aligned}$$

which proves the first statement of the lemma with $c_1 = \max\{(2^p\alpha + 4r)^{1/p}, (2^p\alpha^p + 2)^{1/p}\}$.

To show the second inequality, consider the elementary estimates

$$\begin{aligned}
|x|_{\mathbb{X}^p_\alpha} &= \left(\int_0^\alpha |\dot{y}(u)|^p \, du\right)^{1/p} + |\varphi|_{W^{1,\infty}} \\
&\le \alpha^{1/p} |\dot{y}|_{L^\infty_\alpha} + |\varphi|_{W^{1,\infty}} \\
&\le (\alpha^{1/p} + 1)|x|_{W^{1,\infty}_\alpha},
\end{aligned}$$

therefore $c_2 = (\alpha^{1/p} + 1)$ in (ii).

Consider (iii). Then by Lemma 4.15 (i) and (2.10) we get

$$\begin{aligned}
|x|_{C_\alpha} &\le |y|_{C_\alpha} + |\tilde{\varphi}|_{C_\alpha} \\
&\le \alpha^{1/q} |y|_{\mathbb{Y}^p_\alpha} + |\varphi|_{W^{1,\infty}} \\
&\le \max\{\alpha^{1/q}, 1\} |x|_{\mathbb{X}^p_\alpha},
\end{aligned}$$

therefore (iii) is satisfied with $c_3 = \max\{\alpha^{1/q}, 1\}$.

To prove (iv), consider

$$\begin{aligned}
|\dot{x}|_{L^p_\alpha} &= \left(\int_{-r}^\alpha |\dot{y}(u) + \dot{\tilde{\varphi}}(u)|^p \, du\right)^{1/p} \\
&\le \left(\int_0^\alpha |\dot{y}(u)|^p \, du\right)^{1/p} + \left(\int_{-r}^0 |\dot{\varphi}(u)|^p \, du\right)^{1/p} \\
&\le |y|_{\mathbb{Y}^p_\alpha} + r^{1/p} |\varphi|_{W^{1,\infty}} \\
&\le \max\{r^{1/p}, 1\} |x|_{\mathbb{X}^p_\alpha}. \qquad (4.45)
\end{aligned}$$

Therefore (iv) holds with $c_4 = \max\{r^{1/p}, 1\}$.

This completes the proof of the lemma. □

4.1.3 General case, differentiability in $W^{1,p}$

In this subsection we study the general case of differentiability of solutions of IVP (4.1)-(4.2), i.e., when (4.1) is state-dependent, and there is no restriction on the initial functions, they are arbitrary $W^{1,\infty}$-functions.

By Lemma 3.7, IVP (4.1)-(4.2) is equivalent to the integral equation

$$y(t)=\begin{cases} 0, & t\in[-r,0] \\ \displaystyle\int_0^t f\Big(u,\, y(u)+\tilde\varphi(u),\, \Lambda(u,y_u+\tilde\varphi_u)\Big)\,du, & t\in[0,T], \end{cases} \tag{4.46}$$

where we have used the transformation $y(t)\equiv x(t)-\tilde\varphi(t)$.

It follows from the proof of Theorem 3.8 that the solution of IVP (4.1)-(4.2) is the fixed point of the operator

$$S(y,\varphi)(t)=\begin{cases} 0, & t\in[-r,0] \\ \displaystyle\int_0^t f\Big(u,\, y(u)+\tilde\varphi(u),\, \Lambda(u,y_u+\tilde\varphi_u)\Big)\,du, & t\in[0,T]. \end{cases} \tag{4.47}$$

In the proof of Theorem 3.8, S was considered as an operator

$$S\ :\ \Big((\overline{\mathcal{G}}_{C_\alpha}(\beta)\times\mathcal{G}_C(\bar\varphi;\,\delta)\subset C_\alpha\times C\Big)\to C_\alpha$$

(with appropriate $\alpha>0$, $\beta>0$ and $\delta>0$), where

$$\mathcal{G}_C(\bar\varphi;\,\delta)\subset\Big\{\varphi\in C\ :\ \varphi(0)\in\Omega_1,\quad\text{and}\quad\int_{-r}^0 d_s\mu(s,0,\varphi)\varphi(s)\in\Omega_2\Big\}.$$

(See (3.9) and (3.25).) It follows from Theorem 3.19 that the solution of IVP (4.1)-(4.2) is unique if $\varphi\in W^{1,\infty}$, and Lemma 3.21 implies that the solution is in fact a $W^{1,\infty}_\alpha$ function. Therefore in this section we shall consider S as an operator

$$S\ :\ \Big(\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta)\times\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)\subset\Upsilon^p_{\bar\alpha}\times W^{1,\infty}\Big)\to\Upsilon^p_{\bar\alpha}, \tag{4.48}$$

where $\bar\varphi\in\Phi$ (see (4.3)) is fixed, and the constants $\delta,\ \bar\alpha,\ \bar\beta>0$ are specified by the following lemma.

Note, that in the proof of the next lemma, and in some later occasions in the manuscript, we shall need to restrict a function originally defined on an interval $[-r,\alpha^*]$, e.g., $y\in W^{1,\infty}_{\alpha^*}$, to a smaller interval, $[-r,\bar\alpha]$, $\bar\alpha<\alpha^*$. Then, of course, the restriction of the function y to $[-r,\bar\alpha]$ belongs to $W^{1,\infty}_{\bar\alpha}$, and to keep the notations simple, we shall simply write $y\in W^{1,\infty}_{\bar\alpha}$ or $|y|_{W^{1,\infty}_{\bar\alpha}}$ instead of introducing a new notation for the restriction of y to $[-r,\bar\alpha]$.

Lemma 4.19 *Let $1\le p<\infty$, $\bar\varphi\in\Phi$ and $R>0$. Then there exist $\delta>0$, $\bar\alpha>0$ and $\bar\beta>0$ such that $\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)\subset\Phi$, and the operator S defined by (4.47) satisfies*

(i) $S\ :\ \overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta)\times\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)\to\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta)$,

(ii) S is a uniform contraction on $\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta)\cap\overline{\mathcal{G}}_{W^{1,\infty}_{\bar\alpha}}(R)$ both in $|\cdot|_{\Upsilon^p_{\bar\alpha}}$ and $|\cdot|_{\Upsilon^\infty_{\bar\alpha}}$ norms, i.e., there exists $0\le\theta<1$ such that for all $\varphi\in\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)$, $y,\bar y\in\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta)\cap\overline{\mathcal{G}}_{W^{1,\infty}_{\bar\alpha}}(R)$

$$|S(y,\varphi)-S(\bar y,\varphi)|_{\Upsilon^\infty_{\bar\alpha}}\le\theta|y-\bar y|_{\Upsilon^\infty_{\bar\alpha}},$$

and

$$|S(y,\varphi)-S(\bar y,\varphi)|_{\Upsilon^p_{\bar\alpha}}\le\theta|y-\bar y|_{\Upsilon^p_{\bar\alpha}}.$$

Proof (i) Let $\alpha>0$, $\beta>0$ and $\delta>0$ be the constants from the proof of Theorem 3.8, i.e., such that $\mathcal{G}_C(\bar\varphi;\,\delta)\subset\Phi_0$, (and hence $\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)\subset\Phi$ as well), and if $\varphi\in\mathcal{G}_C(\bar\varphi;\,\delta)$, $y\in\overline{\mathcal{G}}_C(\beta)$ then $S(y,\varphi)$ is well-defined. Let

$$\alpha^*\equiv\min\left\{\alpha,\frac{\beta}{\|f\|}\right\}\quad\text{and}\quad\bar\beta\equiv(\alpha^*)^{1/p}\|f\|.$$

We shall show that $S(y,\varphi)$ is well-defined for $\varphi\in\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)$ and $y\in\overline{\mathcal{G}}_{\Upsilon^p_{\alpha^*}}(\bar\beta)$. Let $y\in\overline{\mathcal{G}}_{\Upsilon^p_{\alpha^*}}(\bar\beta)$ and suppose that there exists $\bar t\in[0,\bar\alpha]$ such that $|y(\bar t)|>\beta$. Then by Lemma 4.15 (i) and (v), the following inequalities hold:

$$\beta<|y(\bar t)|\le(\alpha^*)^{1/q}|y|_{\Upsilon^p_{\alpha^*}}\le(\alpha^*)^{1/q+1/p}\|f\|\le\alpha^*\|f\|\le\beta,$$

which is a contradiction, therefore $y\in\overline{\mathcal{G}}_{C_{\bar\alpha}}(\beta)$, and hence $S(y,\varphi)$ is well-defined on $\overline{\mathcal{G}}_{\Upsilon^p_{\alpha^*}}(\bar\beta)\times\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)$, and therefore so is on $\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta)\times\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)$ for all $0<\bar\alpha\le\alpha^*$. Finally, the inequality

$$|S(y,\varphi)|_{\Upsilon^p_{\alpha^*}}\le(\alpha^*)^{1/p}\|f\|=\bar\beta$$

completes the proof of (i).

(ii) We shall select $0<\bar\alpha\le\alpha^*$ such that (ii) is satisfied. Let $y,\bar y\in\overline{\mathcal{G}}_{\Upsilon^p_{\alpha^*}}(\bar\beta)\cap\overline{\mathcal{G}}_{W^{1,\infty}_{\alpha^*}}(\bar\beta)$. Then for $\varphi\in\mathcal{G}_{W^{1,\infty}}(\bar\varphi;\,\delta)$ it follows that $|y_t+\tilde\varphi_t|_{W^{1,\infty}}\le|y_t|_{W^{1,\infty}}+|\varphi|_{W^{1,\infty}}\le R+|\bar\varphi|_{W^{1,\infty}}+\delta$ for $t\in[0,\alpha^*]$, and therefore by (2.5) we have that $|\Lambda(t,y_t+\tilde\varphi_t)|\le\|\mu\|_\Theta(R+|\bar\varphi|_{W^{1,\infty}}+\delta)$. Let $M\equiv\max\{1,\|\mu\|\}(R+|\bar\varphi|_{W^{1,\infty}}+\delta)$, $L_1=L_1(\alpha^*,M)$ be the constant from (A4), $L_2=L_2(\alpha^*,M)$ be the constant from (A5). Then (A4), Lemma 3.12 and Lemma 4.15 (ii) and (iv) yield that for $0<\bar\alpha\le\alpha^*$

$$\begin{aligned}
&|S(y,\varphi)-S(\bar y,\varphi)|_{\Upsilon^\infty_{\bar\alpha}}\\
&\quad=\operatorname*{ess\,sup}_{0\le u\le\bar\alpha}\Big|f(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u))-f(u,\bar y(u)+\tilde\varphi(u),\Lambda(u,\bar y_u+\tilde\varphi_u))\Big|\\
&\quad\le L_1\operatorname*{ess\,sup}_{0\le u\le\bar\alpha}\Big(|y(u)-\bar y(u)|+|\Lambda(u,y_u+\tilde\varphi_u)-\Lambda(u,\bar y_u+\tilde\varphi_u)|\Big)\\
&\quad\le L_1\bar\alpha|y-\bar y|_{\Upsilon^\infty_{\bar\alpha}}+L_1(\|\mu\|+L_2M)\sup_{0\le u\le\alpha}|y_u-\bar y_u|_C\\
&\quad\le L_1\bar\alpha(1+\|\mu\|+L_2M)|y-\bar y|_{\Upsilon^\infty_{\bar\alpha}}.
\end{aligned}$$

Similarly, in $\Upsilon^p_{\bar\alpha}$ we have that

$$\begin{aligned}
&|S(y,\varphi)-S(\bar y,\varphi)|^p_{\Upsilon^p_{\bar\alpha}}\\
&\quad=\int_0^{\bar\alpha}\Big|f(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u))-f(u,\bar y(u)+\tilde\varphi(u),\Lambda(u,\bar y_u+\tilde\varphi_u))\Big|^p\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le L_1^p \int_0^{\bar\alpha} \Big(|y(u)-\bar y(u)| + |\Lambda(u, y_u+\tilde\varphi_u) - \Lambda(u,\bar y_u+\tilde\varphi_u)| \Big)^p \, ds \\
&\le L_1^p \int_0^{\bar\alpha} \Big((\bar\alpha)^{1/q} |y-\bar y|_{\Upsilon^p_{\bar\alpha}} + (\|\mu\| + L_2 M) \sup_{0\le u\le\bar\alpha} |y_u - \bar y_u|_C \Big)^p \, ds \\
&\le L_1^p (\bar\alpha)^p (1+\|\mu\|+L_2M)^p |y-\bar y|^p_{\Upsilon^p_{\bar\alpha}}.
\end{aligned}
$$

Therefore, select $0<\bar\alpha\le\alpha^*$ such that $\bar\alpha < 1/(L_1(1+\|\mu\|+L_2M))$, then (ii) is satisfied. □

Remark 4.20 *Note, that $\bar\alpha$ depends only on α, β, $\|f\|$, $\|\mu\|$, R and p, but does not depend on the initial function. The constant δ is the same as in the proof of Theorem 3.8, it does not depend on R and p.*

Lemma 4.19 provides the framework for applying Theorem 4.14 to discuss differentiability of the fixed point of $S(\cdot,\varphi)$, i.e., solutions of IVP (4.1)-(4.2) wrt the initial function. This theorem assumes that S has continuous partial derivatives on its domain, for which as one can see, it is necessary to have some kind of continuous differentiability of $\Lambda(t,\psi)$ wrt ψ. It turns out, that we need to have the differentiability of the following composition operator.

Fix $1\le p<\infty$ and let $\mathcal{K}$ be an open subset of $W^{1,\infty}_\alpha$. Define the following composition operator corresponding to $\Lambda(\cdot,\cdot)$.

$$
B_\Lambda \;:\; \Big(\mathcal{K}\subset \mathbb{X}^p_\alpha\Big) \to L^p([0,\alpha];\mathbb{R}^n), \qquad B_\Lambda(x)(t) \equiv \Lambda(t,x_t), \qquad t\in[0,\alpha]. \tag{4.49}
$$

We replace assumption (A8a) of Section 4.1.1 by the following hypothesis:

(A8b) the operator B_Λ defined by (4.49) is continuously differentiable on $\mathcal{K}$.

Remark 4.21 *For $<\bar\alpha<\alpha$ we introduce $\mathcal{K}_{\bar\alpha}$ as the set of restriction of the functions $x\in\mathcal{K}$ to $[-r,\bar\alpha]$, and consider the composition operator*

$$
B_{\Lambda,\bar\alpha} \;:\; \Big(\mathcal{K}_{\bar\alpha}\subset \mathbb{X}^p_{\bar\alpha}\Big) \to L^p([0,\bar\alpha];\mathbb{R}^n), \qquad B_{\Lambda,\bar\alpha}(x)(t) \equiv \Lambda(t,x_t), \qquad t\in[0,\bar\alpha].
$$

Then, clearly, assumption (A8b) implies that $B_{\Lambda,\bar\alpha}$ is continuously differentiable on its domain. Later, to keep the notation simple, we freely use B_Λ and $\mathcal{K}$ instead of $B_{\Lambda,\bar\alpha}$ and $\mathcal{K}_{\bar\alpha}$, respectively, so if a function x is defined on $[-r,\bar\alpha]$, and we write $x\in\mathcal{K}$, then we mean that $x\in\mathcal{K}_{\bar\alpha}$.

Note, that in Section 4.1.4 we shall present conditions implying (A8b) for the composition map corresponding to Examples 1.3 and 1.4.

The following lemma shows that assumption (A8b) yields the existence of continuous partial derivatives of $S(y,\varphi)$ if we restrict y to a certain subset of its domain, and the derivative is taken in the restricted space (in relative topology).

Lemma 4.22 *Let $\bar{\varphi} \in \Phi$, $1 \le p < \infty$ be fixed, and $R > 0$ given, and assume (A1)–(A7) and (A8b). Let δ, $\bar{\alpha}$, $\bar{\beta}$ be the constants from Lemma 4.19, i.e., such that the operator S defined by (4.47) satisfies*

$$S \;:\; \overline{\mathcal{G}}_{\Upsilon^p_{\bar{\alpha}}}(\bar{\beta}) \times \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\,\delta) \to \overline{\mathcal{G}}_{\Upsilon^p_{\bar{\alpha}}}(\bar{\beta}),$$

and it is a uniform contraction on $\overline{\mathcal{G}}_{\Upsilon^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{W^{1,\infty}_{\bar{\alpha}}}(R)$. Assume that there exists $\mathcal{W} \subset \Upsilon^p_{\bar{\alpha}}$ such that

(i) $\mathcal{W} \subset (\overline{\mathcal{G}}_{\Upsilon^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{W^{1,\infty}_{\bar{\alpha}}}(R))$,

(ii) for $y \in \mathcal{W}$ and $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\,\delta)$ it follows that $y + \tilde{\varphi} \in \mathcal{K}$.

Then the operator

$$S(y,\varphi) \;:\; \Big(\mathcal{W} \times \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\,\delta) \subset (\mathcal{W} \cap \Upsilon^p_{\bar{\alpha}}) \times W^{1,\infty}\Big) \to \Upsilon^p_{\bar{\alpha}}$$

has continuous partial derivatives wrt y and φ on its domain, and for $y \in \mathcal{W}$, $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\,\delta)$, $h \in \Upsilon^p_{\bar{\alpha}}$ we have that

$$\left(\frac{\partial S}{\partial y}(y,\varphi)h\right)(t) = \begin{cases} 0, & t \in [-r,0], \\ \displaystyle\int_0^t \frac{\partial f}{\partial x}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big)h(u) & \\ \displaystyle\quad + \frac{\partial f}{\partial y}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big)\left(\frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi})h\right)(u)\,du, & t \in [0,\bar{\alpha}], \end{cases} \tag{4.50}$$

and for $y \in \mathcal{W}$, $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\,\delta)$, $h \in W^{1,\infty}$ it follows that

$$\left(\frac{\partial S}{\partial \varphi}(y,\varphi)h\right)(t) = \begin{cases} 0, & t \in [-r,0], \\ \displaystyle\int_0^t \frac{\partial f}{\partial x}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big)h(0) & \\ \displaystyle\quad + \frac{\partial f}{\partial y}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big)\left(\frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi})\tilde{h}\right)(u)\,du. & t \in [0,\bar{\alpha}]. \end{cases} \tag{4.51}$$

Note that we consider differentiability of $S(y,\varphi)$ when y is restricted to $\mathcal{W}$, i.e., we use a relative topology on $\Upsilon^p_{\bar{\alpha}}$ defined by $\mathcal{W}$, and $\frac{\partial S}{\partial y}(y,\varphi)h$ is defined for all $h \in \Upsilon^p_{\bar{\alpha}}$.

Proof Let $y \in \mathcal{W}$, $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\,\delta)$, and $h \in \Upsilon^p_{\bar{\alpha}}$. We show, that the operator $\frac{\partial S}{\partial y}$ defined by (4.50) is, in fact, the partial derivative of S wrt y. Clearly, $\frac{\partial S}{\partial y}$ is a linear operator. We need to show, that it is bounded.

Since $y \in \Upsilon^p_{\bar{\alpha}}$, it follows that $y \in W^{1,\infty}_{\bar{\alpha}}$ as well, and hence if we define $M_3 = M_3(y,\varphi) \equiv |y|_{W^{1,\infty}_{\bar{\alpha}}} + |\varphi|_{W^{1,\infty}}$ then

$$|y_t + \tilde{\varphi}_t|_{W^{1,\infty}} \le M_3, \qquad \text{for } t \in [0,\bar{\alpha}], \tag{4.52}$$

and thus by (2.5) it follows that

$$|\Lambda(t, y_t+\tilde{\varphi}_t)| \le \|\mu\| M_3, \qquad t \in [0,\bar{\alpha}]. \tag{4.53}$$

Therefore if we define the compact set

$$A \equiv [0,\bar{\alpha}] \times \overline{\mathcal{G}}_{\mathbb{R}^n}(M_3) \times \overline{\mathcal{G}}_{\mathbb{R}^n}(\|\mu\| M_3) \tag{4.54}$$

and the constant

$$M_4 \equiv \max\left\{ \sup_{(t,x,y)\in A} \left\| \frac{\partial f}{\partial x}(t,x,y) \right\|, \sup_{(t,x,y)\in A} \left\| \frac{\partial f}{\partial y}(t,x,y) \right\| \right\}, \tag{4.55}$$

then $M_4 < \infty$ by (A7), and it satisfies

$$\left\| \frac{\partial f}{\partial x}\Big(t, y(t)+\tilde{\varphi}(t), \Lambda(t, y_t+\tilde{\varphi}_t)\Big) \right\| \le M_4, \qquad t\in[0,\bar{\alpha}], \tag{4.56}$$

and

$$\left\| \frac{\partial f}{\partial y}\Big(t, y(t)+\tilde{\varphi}(t), \Lambda(t, y_t+\tilde{\varphi}_t)\Big) \right\| \le M_4, \qquad t\in[0,\bar{\alpha}]. \tag{4.57}$$

Lemma 4.15 (i), and relation $1/p+1/q=1$ imply that

$$\begin{aligned} \left(\int_0^{\bar{\alpha}} |h(u)|^p\,du\right)^{1/p} &\le \left(\int_0^{\bar{\alpha}} (\bar{\alpha})^{p/q} |h|^p_{\mathbb{Y}^p_{\bar{\alpha}}}\,du\right)^{1/p} \\ &\le \bar{\alpha}|h|_{\mathbb{Y}^p_{\bar{\alpha}}}. \end{aligned} \tag{4.58}$$

Estimates (4.56), (4.57), (4.58), the definition of $\frac{\partial S}{\partial y}$, assumption (A8b), and triangle inequality yield

$$\begin{aligned} &\left| \frac{\partial S}{\partial y}(y,\varphi)h \right|_{\mathbb{Y}^p_{\bar{\alpha}}} \\ &\quad\le \left(\int_0^{\bar{\alpha}} \left| \frac{\partial f}{\partial x}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big) h(u) \right|^p du\right)^{1/p} \\ &\quad\quad + \left(\int_0^{\bar{\alpha}} \left| \frac{\partial f}{\partial y}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big) \left(\frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi})h\right)(u) \right|^p du\right)^{1/p} \\ &\quad\le M_4 \left(\int_0^{\bar{\alpha}} |h(u)|^p\,du\right)^{1/p} + M_4 \left| \frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi})h \right|_{L^p([0,\bar{\alpha}];\mathbb{R}^n)} \\ &\quad\le M_4\bar{\alpha}|h|_{\mathbb{Y}^p_{\bar{\alpha}}} + M_4 \left\| \frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi}) \right\|_{\mathcal{L}(\mathbb{X}^p_{\bar{\alpha}}, L^p([0,\bar{\alpha}];\mathbb{R}^n))} |h|_{\mathbb{Y}^p_{\bar{\alpha}}}, \end{aligned} \tag{4.59}$$

which shows the boundedness of $\frac{\partial S}{\partial y}(y,\varphi)$.

Next we show that it is the derivative of $S(y,\varphi)$ wrt y. Consider

$$\begin{aligned} &\left| S(y+h,\varphi) - S(y,\varphi) - \frac{\partial S}{\partial y}(y,\varphi)h \right|_{\mathbb{Y}^p_{\bar{\alpha}}} \\ &= \left(\int_0^{\bar{\alpha}} \Big| f\Big(u, y(u)+\tilde{\varphi}(u)+h(u), \Lambda(u, y_u+h_u+\tilde{\varphi}_u)\Big) \right. \\ &\quad - f\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big) - \frac{\partial f}{\partial x}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big)h(u) \\ &\quad \left. - \frac{\partial f}{\partial y}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big) \left(\frac{\partial B_\Lambda}{\partial x}(y+\tilde{\varphi})h\right)(u) \Big|^p du\right)^{1/p}. \end{aligned} \tag{4.60}$$

Introduce

$$\omega^4(t,\bar{x};x) \equiv \Lambda(t,x_t) - \Lambda(t,\bar{x}_t) - \Big(\frac{\partial B_\Lambda}{\partial x}(\bar{x})(x-\bar{x})\Big)(t) \tag{4.61}$$

for $t \in [0,\bar{\alpha}]$, $x, \bar{x} \in \mathbb{X}^p_{\bar{\alpha}}$, $\bar{x} \in \mathcal{K}$. Then it follows from (A8b) that

$$\frac{1}{|x-\bar{x}|_{\mathbb{X}^p_{\bar{\alpha}}}} \Big(\int_0^{\bar{\alpha}} |\omega^4(t,\bar{x};x)|^p dt\Big)^{1/p} \to 0, \qquad \text{as } |x-\bar{x}|_{\mathbb{X}^p_{\bar{\alpha}}} \to 0. \tag{4.62}$$

By our assumption, $y+\tilde{\varphi} \in \mathcal{K}$, hence the definition of ω^1 (defined by (4.7)), ω^4, relation (4.60) yields that

$$\begin{aligned}
&\Big|S(y+h,\varphi) - S(y,\varphi) - \frac{\partial S}{\partial y}(y,\varphi)h\Big|_{\mathbb{Y}^p_{\bar{\alpha}}} \\
&\le \Big(\int_0^{\bar{\alpha}} \Big|\omega^1(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u); y(u)+h(u)+\tilde{\varphi}(u), \Lambda(u, y_u+h_u+\tilde{\varphi}_u)) \\
&\qquad + \frac{\partial f}{\partial y}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big)\omega^4(u, y+\tilde{\varphi}; y+h+\tilde{\varphi})\Big|^p du\Big)^{1/p} \\
&\le \Big(\int_0^{\bar{\alpha}} \Big|\omega^1(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u); y(u)+h(u)+\tilde{\varphi}(u), \Lambda(u, y_u+h_u+\tilde{\varphi}_u))\Big|^p du\Big)^{1/p} \\
&\qquad + \Big(\int_0^{\bar{\alpha}} \Big|\frac{\partial f}{\partial y}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u)\Big)\omega^4(u, y+\tilde{\varphi}; y+h+\tilde{\varphi})\Big|^p du\Big)^{1/p}.
\end{aligned} \tag{4.63}$$

By using (4.57), estimate (4.63) implies that

$$\begin{aligned}
&\frac{1}{|h|_{\mathbb{Y}^p_{\bar{\alpha}}}}\Big|S(y+h,\varphi) - S(y,\varphi) - \frac{\partial S}{\partial y}(y,\varphi)h\Big|_{\mathbb{Y}^p_{\bar{\alpha}}} \\
&\le \Big(\int_0^{\bar{\alpha}} \Big|\frac{\omega^1(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u); y(u)+h(u)+\tilde{\varphi}(u), \Lambda(u, y_u+h_u+\tilde{\varphi}_u))}{|h|_{\mathbb{Y}^p_{\bar{\alpha}}}}\Big|^p du\Big)^{1/p} \\
&\qquad + M_4\Big(\int_0^{\bar{\alpha}} \Big|\frac{\omega^4(u, y+\tilde{\varphi}; y+h+\tilde{\varphi})}{|h|_{\mathbb{Y}^p_{\bar{\alpha}}}}\Big|^p du\Big)^{1/p}.
\end{aligned} \tag{4.64}$$

We show first that $|\omega^1(\cdot,\cdot;\cdot,\cdot)|/|h|_{\mathbb{Y}^p_{\bar{\alpha}}}$ in (4.64) converges to zero pointwise as $|h|_{\mathbb{Y}^p_{\bar{\alpha}}} \to 0$. By (4.8), it is enough to show that for all $u \in [0,\bar{\alpha}]$ it follows that $y(u)+h(u)+\tilde{\varphi}(u) \to y(u)+\tilde{\varphi}(u)$ and $\Lambda(u, y_u+h_u+\tilde{\varphi}_u) \to \Lambda(u, y_u+\tilde{\varphi}_u)$ as $|h|_{\mathbb{Y}^p_{\bar{\alpha}}} \to 0$. The first relation follows from the inequality $|h(u)| \le \bar{\alpha}^{1/q}|h|_{\mathbb{Y}^p_{\bar{\alpha}}}$ (guaranteed by Lemma 4.15 (i)). Let $R_1 \equiv R + |\bar{\varphi}|_{W^{1,\infty}} + \delta$. For the second relation, by using Lemma 3.12 with $L_2 = L_2(\alpha, M_3 + (\bar{\alpha})^{1/q})$, assumption (ii) of the theorem and Lemma 4.15 (i), we get for $|h|_{\mathbb{Y}^p_{\bar{\alpha}}} \le 1$ that

$$\begin{aligned}
|\Lambda(u, y_u+h_u+\tilde{\varphi}_u) - \Lambda(u, y_u+\tilde{\varphi}_u)| &\le (\|\mu\| + L_2|y_u+\tilde{\varphi}_u|_{W^{1,\infty}})|h_u|_C \\
&\le (\|\mu\| + L_2R_1)\bar{\alpha}^{1/q}|h|_{\mathbb{Y}^p_{\bar{\alpha}}} \\
&\to 0, \qquad \text{as } |h|_{\mathbb{Y}^p_{\bar{\alpha}}} \to 0.
\end{aligned}$$

Next we show that $|\omega^1(\cdot,\cdot;\cdot,\cdot)|/|h|_{\mathbb{Y}^p_{\bar{\alpha}}}$ is bounded on $[0,\bar{\alpha}]$. As in the proof of (4.22), it follows from the Mean Value Theorem and the definition of M_4 and the previous estimates for $|h|_{\mathbb{Y}^p_{\bar{\alpha}}} \le 1$

that

$$\begin{aligned}
&|\omega^1(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u); y(u) + h(u) + \tilde{\varphi}(u), \Lambda(u, y_u + h_u + \tilde{\varphi}_u))| \\
&\quad \leq \; 2M_4(|h(u)| + |\Lambda(u, y_u + h_u + \tilde{\varphi}_u) - \Lambda(u, y_u + \tilde{\varphi}_u)|) \\
&\quad \leq \; 2M_4\bar{\alpha}^{1/q}(1 + \|\mu\| + L_2R_1)|h|_{\mathbb{Y}^p_{\bar{\alpha}}}.
\end{aligned}$$

Therefore the Lebesgue Dominant Theorem yields that the first term in (4.64) goes to zero as $|h|_{\mathbb{Y}^p_{\bar{\alpha}}} \to 0$. So does the second term by (4.62), therefore we have proved (4.50).

Next we show that $\frac{\partial S}{\partial y}(y, \varphi)$ is continuous. Let $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi}; \delta)$ and $y \in \mathcal{W}$, and select sequences $\varphi^k \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi}; \delta)$ and $y^k \in \mathcal{W}$ such that $|\varphi^k - \varphi|_{W^{1,\infty}} \to 0$ and $|y^k - y|_{\mathbb{Y}^p_{\bar{\alpha}}} \to 0$ as $k \to \infty$. Let $h \in \mathbb{Y}^p_{\bar{\alpha}}$. By (4.29) and Lemma 4.4 (ii) we have that

$$\begin{aligned}
&\Big|\frac{\partial S}{\partial y}(y^k, \varphi^k)h - \frac{\partial S}{\partial y}(y, \varphi)h\Big|_{\mathbb{Y}^p_{\bar{\alpha}}} \\
&= \; \Big(\int_0^{\bar{\alpha}} \Big|\frac{\partial f}{\partial x}(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big)h(u) \\
&\qquad + \frac{\partial f}{\partial y}\Big(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big)\Big(\frac{\partial B_\Lambda}{\partial x}(y^k + \tilde{\varphi^k})h\Big)(u) \\
&\qquad - \frac{\partial f}{\partial x}(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big)h(u) \\
&\qquad - \frac{\partial f}{\partial y}\Big(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big)\Big(\frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi})h\Big)(u)\Big|^p\, du\Big)^{1/p} \\
&\leq \; \Big(\int_0^{\bar{\alpha}} \Big\|\frac{\partial f}{\partial x}(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big) \\
&\qquad - \frac{\partial f}{\partial x}(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big)\Big\|^p |h(u)|^p\, du\Big)^{1/p} \\
&\quad + \Big(\int_0^{\bar{\alpha}} \Big\|\frac{\partial f}{\partial y}\Big(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big) \\
&\qquad - \frac{\partial f}{\partial y}(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big)\Big\|^p \Big|\Big(\frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi})h\Big)(u)\Big|^p\, du\Big)^{1/p} \\
&\quad + \Big(\int_0^{\bar{\alpha}} \Big\|\frac{\partial f}{\partial y}(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big)\Big\|^p \\
&\qquad \cdot \Big|\Big(\frac{\partial B_\Lambda}{\partial x}(y^k + \tilde{\varphi^k})h - \frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi})h\Big)(u)\Big|^p\, du\Big)^{1/p}.
\end{aligned}$$

Therefore, using the definition of M_4, (4.58), we get

$$\begin{aligned}
&\Big\|\frac{\partial S}{\partial y}(y^k, \varphi^k) - \frac{\partial S}{\partial y}(y, \varphi)\Big\|_{\mathcal{L}(\mathbb{Y}^p_{\bar{\alpha}}, \mathbb{Y}^p_{\bar{\alpha}})} \\
&\leq \; \bar{\alpha} \sup_{0 \leq u \leq \bar{\alpha}} \Big\|\frac{\partial f}{\partial x}(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big) - \frac{\partial f}{\partial x}(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big)\Big\| \\
&\quad + \sup_{0 \leq u \leq \bar{\alpha}} \Big\|\frac{\partial f}{\partial y}\Big(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big) \\
&\qquad - \frac{\partial f}{\partial y}(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big)\Big\|\Big\|\frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi})\Big\|_{\mathcal{L}(\mathbb{X}^p_{\bar{\alpha}}, L^p([0,\bar{\alpha}];\mathbb{R}^n))}
\end{aligned}$$

$$+ M_4\left\| \frac{\partial B_\Lambda}{\partial x}(y^k + \tilde{\varphi^k}) - \frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi})\right\|_{\mathcal{L}(\mathbb{X}^p_{\bar\alpha}, L^p([0,\bar\alpha];\mathbb{R}^n))}. \tag{4.65}$$

Lemma 4.15 (i) and (2.10) imply that

$$\begin{aligned} |y^k(u) + \tilde{\varphi^k}(u) - y(u) - \tilde{\varphi}(u)| &\leq |y^k(u) - y(u)| + |\tilde{\varphi^k}(u) - \tilde{\varphi}(u)| \\ &\leq \bar\alpha^{1/q}|y^k - y|_{\mathbb{Y}^p_{\bar\alpha}} + |\varphi^k - \varphi|_{W^{1,\infty}} \\ &\to 0, \qquad \text{as } k \to \infty, \end{aligned} \tag{4.66}$$

and since $y^k \in \mathcal{W}$, and $\varphi^k \in \mathcal{G}_{W^{1,\infty}}(\bar\varphi;\, \delta)$, by the definition of M_3, assumption (ii) of the theorem if follows for $u \in [0, \bar\alpha]$ that

$$\begin{aligned} |y^k_u - y_u + \varphi^k_u - \varphi_u|_C &\leq |y + \tilde\varphi|_C + |y^k_u|_C + |\varphi|_C + |\varphi^k - \varphi|_C \\ &\leq M_3 + R + M_3 + \delta, \end{aligned}$$

and therefore Lemma 4.15 (iii) and Lemma 3.12 with $L_2 = L_2(\bar\alpha, 2M_3 + R + \delta)$ imply that

$$\begin{aligned} |\Lambda(u, y^k_u + \tilde{\varphi^k_u}) - \Lambda(u, y_u + \tilde\varphi_u)| &\leq (\|\mu\| + L_2|y_u + \tilde\varphi_u|_{W^{1,\infty}})(|y^k_u - y_u|_C + |\tilde\varphi^k_u - \tilde\varphi_u|_C\Big) \\ &\leq (\|\mu\| + L_2 M_3)\Big((\bar\alpha)^{1/q}|y^k - y|_{\mathbb{Y}^p_{\bar\alpha}} + |\varphi^k - \varphi|_{W^{1,\infty}}\Big) \\ &\to 0, \qquad \text{as } k \to \infty. \end{aligned} \tag{4.67}$$

Then (4.66), (4.67) and (4.31) (by an argument similar to the proof of (4.27)) yield that the first and second terms in the right hand side of (4.65) go to zero as $k \to \infty$. So does the third term, since by (A8b), $\frac{\partial B_\Lambda}{\partial x}$ is continuous on $\mathcal{K}$ (in the $\|\cdot\|_{\mathcal{L}(\mathbb{X}^p_{\bar\alpha}, L^p([0,\bar\alpha];\mathbb{R}^n))}$-norm). This completes the proof of continuity of $\frac{\partial S}{\partial y}$.

The proof of (4.51) is similar. Clearly, $\frac{\partial S}{\partial \varphi}$ is linear, and similarly to (4.59), we can get

$$\left|\frac{\partial S}{\partial \varphi}(y,\varphi)h\right|_{\mathbb{Y}^p_{\bar\alpha}} \leq M_4(\bar\alpha)^{1/p}|h|_{W^{1,\infty}} + M_4\left\|\frac{\partial B_\lambda}{\partial x}(y + \tilde\varphi)\right\|_{\mathcal{L}(\mathbb{X}^p_{\bar\alpha}, L^p([0,\bar\alpha];\mathbb{R}^n))}|h|_{W^{1,\infty}},$$

which implies the boundedness of $\frac{\partial S}{\partial \varphi}(y,\varphi)$.

Let $h \in W^{1,\infty}$, then using the definitions of ω^1 and ω^4, estimates (4.57) and (4.51), we get

$$\begin{aligned} &\frac{1}{|h|_{W^{1,\infty}}}\left|S(y, \varphi + h) - S(y,\varphi) - \frac{\partial S}{\partial \varphi}(y,\varphi)h\right|_{\mathbb{Y}^p_{\bar\alpha}} \\ &\leq \left(\int_0^{\bar\alpha}\left|\frac{\omega^1(u, y(u) + \tilde\varphi(u), \Lambda(u, y_u + \tilde\varphi_u); y(u) + \tilde\varphi(u) + \tilde h(u), \Lambda(u, y_u + \tilde\varphi_u + \tilde h_u))}{|h|_{W^{1,\infty}}}\right|^p du\right)^{1/p} \\ &\quad + M_4\left(\int_0^{\bar\alpha}\left|\frac{\omega^4(u, y + \tilde\varphi; y + \tilde\varphi + \tilde h)}{|h|_{W^{1,\infty}}}\right|^p du\right)^{1/p}. \end{aligned} \tag{4.68}$$

Lemma 3.12 yields that for $|h|_{W^{1,\infty}} < \delta^*$

$$\begin{aligned} |\tilde h(t)| + |\Lambda(t, y_t + \tilde\varphi_t + \tilde h_t) - \Lambda(t, y_t + \tilde\varphi_t)| &\leq |h|_C + (\|\mu\| + L_2(\bar\alpha, M_3 + \delta^*)M_3)|h|_C \\ &\to 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0, \end{aligned} \tag{4.69}$$

therefore $|\omega^1(\cdot,\cdot;\cdot,\cdot)|/|h|_{W^{1,\infty}}$ converges to zero pointwise as $|h|_{W^{1,\infty}} \to 0$, and since it is clearly bounded, the Lebesgue Dominant Convergent Theorem implies that the first term in the right hand side of (4.68) goes to zero as $|h|_{W^{1,\infty}} \to 0$. Since

$$|\tilde{h}|_{\mathbb{X}^p_{\bar{\alpha}}} = |h|_{W^{1,\infty}}, \tag{4.70}$$

(4.70) and (4.62) yield that

$$\begin{aligned} \frac{\left(\int_0^{\bar{\alpha}} |\omega^4(t, y+\tilde{\varphi}; y+\tilde{\varphi}+\tilde{h})|^p dt\right)^{1/p}}{|h|_{W^{1,\infty}}} &= \frac{\left(\int_0^{\bar{\alpha}} |\omega^4(t, y+\tilde{\varphi}; y+\tilde{\varphi}+\tilde{h})|^p dt\right)^{1/p}}{|\tilde{h}|_{\mathbb{X}^p_{\bar{\alpha}}}} \\ &\to 0, \qquad \text{as } |h|_{W^{1,\infty}} \to 0, \end{aligned} \tag{4.71}$$

therefore $\frac{\partial S}{\partial \varphi}$ defined by (4.51) is really the partial derivative of S wrt φ.

Finally, we show that $\frac{\partial S}{\partial \varphi}$ is continuous. Let $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta)$ and $y \in \mathcal{W}$, and select sequences $\varphi^k \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta)$ and $y^k \in \mathcal{W}$ such that $|\varphi^k - \varphi|_{W^{1,\infty}} \to 0$ and $|y^k - y|_{\mathbb{Y}^p_{\bar{\alpha}}} \to 0$ as $k \to \infty$. Similarly to (4.65) we can show that

$$\begin{aligned} &\left\| \frac{\partial S}{\partial \varphi}(y^k, \varphi^k) - \frac{\partial S}{\partial \varphi}(y, \varphi) \right\|_{\mathcal{L}(W^{1,\infty}, \mathbb{Y}^p_{\bar{\alpha}})} \\ &\leq \bar{\alpha}^{1/p} \sup_{0 \leq u \leq \bar{\alpha}} \left\| \frac{\partial f}{\partial x}\Big(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big) - \frac{\partial f}{\partial x}\Big(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big) \right\| \\ &\quad + \sup_{0 \leq u \leq \bar{\alpha}} \left\| \frac{\partial f}{\partial y}\Big(u, y^k(u) + \tilde{\varphi^k}(u), \Lambda(u, y^k_u + \tilde{\varphi^k_u})\Big) \right. \\ &\qquad \left. - \frac{\partial f}{\partial y}\Big(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u)\Big) \right\| \left\| \frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi}) \right\|_{\mathcal{L}(\mathbb{X}^p_{\bar{\alpha}}, L^p([0,\bar{\alpha}];\mathbb{R}^n))} \\ &\quad + M_4 \left\| \frac{\partial B_\Lambda}{\partial x}(y^k + \tilde{\varphi^k}) - \frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi}) \right\|_{\mathcal{L}(\mathbb{X}^p_{\bar{\alpha}}, L^p([0,\bar{\alpha}];\mathbb{R}^n))}. \end{aligned}$$

which implies the continuity of $\frac{\partial S}{\partial \varphi}$, since it is essentially the same as (4.65).

This completes the proof of the lemma. □

Theorem 4.23 *Assume that $\bar{\varphi}$, μ and f satisfy (A1)–(A7), (A8b). Then there exist $\alpha > 0$, $\delta > 0$ such that IVP (4.1)-(4.2) has unique solution, $x(t;\varphi)$, on $[0,\alpha]$ corresponding to any initial function $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta)$. Assume that $x(\cdot;\bar{\varphi}) \in \mathcal{K}$, then $x(t;\varphi)$ is continuously differentiable wrt φ, as a function*

$$\Big(\mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta) \subset W^{1,\infty}\Big) \to \mathbb{X}^p_\alpha, \qquad \varphi \mapsto x(\cdot;\varphi).$$

Proof The existence of $\alpha > 0$ and $\delta^1 > 0$, such that the solution of IVP (4.1)-(4.2) exists and is unique on $[0,\alpha]$ for $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta^1)$ follows from Theorems 3.8 and 3.14.

Since $x(\cdot;\bar{\varphi}) \in \mathcal{K}$, and $\mathcal{K}$ is open in $W^{1,\infty}_\alpha$, there exists $\delta^2 > 0$ such that

$$\overline{\mathcal{G}}_{W^{1,\infty}_\alpha}\Big(x(\cdot;\bar{\varphi});\, \delta^2\Big) \subset \mathcal{K}. \tag{4.72}$$

We shall use the notation $\bar{y} \equiv \Pr_y x(\cdot;\bar{\varphi})$, i.e., $x(\cdot;\bar{\varphi}) = \bar{y} + \tilde{\bar{\varphi}}$. By (4.72) we have that

$$\text{if} \quad y \in \overline{\mathcal{G}}_{W^{1,\infty}_\alpha}\Big(\bar{y};\, \delta^2/2\Big),\ \varphi \in \mathcal{G}_{W^{1,\infty}}\Big(\bar{\varphi};\, \delta^2/2\Big) \quad \text{then} \quad y + \tilde{\varphi} \in \mathcal{K}, \tag{4.73}$$

since $|y + \tilde{\varphi} - \bar{y} - \bar{\varphi}|_{W^{1,\infty}_\alpha} \le \delta^2$.

Let $R \equiv \max\Big\{|x(\cdot;\bar{\varphi})|_{W^{1,\infty}_\alpha} + \delta^2/2,\, \|f\|\Big\}$. Then by Lemma 4.19 there exist $\delta^3 > 0$, $\bar{\alpha} > 0$ and $\bar{\beta} > 0$ such that $\bar{\alpha} \le \alpha$, and the operator S defined by (4.47) satisfies

$$S\ :\ \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \times \mathcal{G}_{W^{1,\infty}}\Big(\bar{\varphi};\, \delta^3\Big) \to \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}),$$

and S is uniform contraction on $\overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{W^{1,\infty}_\alpha}(R)$. Let $\delta \equiv \min\{\delta^1, \delta^2/2, \delta^3\}$, and consider S as an operator:

$$S\ :\ \mathcal{W} \times \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta) \to \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}),$$

where

$$\mathcal{W} \equiv \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{W^{1,\infty}_{\bar{\alpha}}}\Big(\bar{y};\, \delta^2/2\Big).$$

Then $\mathcal{W} \subset (\overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{W^{1,\infty}_\alpha}(R))$, and by Lemma 4.17, $\mathcal{W}$ is convex and closed in $\mathbb{Y}^p_{\bar{\alpha}}$. It is easy to see that $|S(y,\varphi)|_{\mathbb{Y}^\infty_{\bar{\alpha}}} \le \|f\|$ for all y and ψ, and hence

$$S\Big((\overline{\mathcal{G}}_{\mathbb{Y}^\infty_{\bar{\alpha}}}(R) \cap \mathcal{W}) \times \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta)\Big) \subset (\overline{\mathcal{G}}_{\mathbb{Y}^\infty_{\bar{\alpha}}}(R) \cap \mathcal{W}),$$

and the operator

$$S\ :\ \Big(\mathcal{W} \times \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta) \subset (\mathcal{W} \cap \mathbb{Y}^p_{\bar{\alpha}}) \times W^{1,\infty}\Big) \to \mathbb{Y}^p_{\bar{\alpha}}$$

is continuously differentiable by assumption (A8b), (4.73), and Lemma 4.22. Therefore an application of Theorem 4.14 yields that the fixed point of $S(\cdot,\varphi)$, (i.e., the solution of (4.50)), called $y(\cdot;\varphi)$, is continuously differentiable as a map

$$\Big(\mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta) \subset W^{1,\infty}\Big) \to \mathbb{Y}^p_{\bar{\alpha}}, \qquad \varphi \mapsto y(\cdot;\varphi).$$

Now we show that the function, $x(t;\varphi) \equiv y(t;\varphi) + \tilde{\varphi}(t)$, i.e., the solution of IVP (4.1)-(4.2) with initial function φ, is continuously differentiable as a map

$$\Big(\mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\, \delta) \subset W^{1,\infty}\Big) \to \mathbb{X}^p_{\bar{\alpha}}, \qquad \varphi \mapsto x(\cdot;\varphi),$$

with derivative

$$\frac{\partial x}{\partial \varphi}(t;\varphi)h = \frac{\partial y}{\partial \varphi}(t;\varphi)h + \tilde{h}, \qquad h \in W^{1,\infty}.$$

To prove the claim, it is enough to consider the obvious relation

$$\left|x(\cdot;\varphi+h) - x(\cdot;\varphi) - \frac{\partial x}{\partial \varphi}(\cdot;\varphi)h\right|_{\mathbb{X}^p_{\bar{\alpha}}} = \left|y(\cdot;\varphi+h) - y(\cdot;\varphi) - \frac{\partial y}{\partial \varphi}(\cdot;\varphi)h\right|_{\mathbb{Y}^p_{\bar{\alpha}}}.$$

Suppose that $\bar{\alpha} < \alpha$. Then we need to show that $x(\cdot;\varphi)$ is continuously differentiable wrt φ in $\mathbb{X}^p_\alpha$ as well. Consider the equation

$$\dot{z}(t) = f(t+\bar{\alpha}, z(t), \Lambda(t+\bar{\alpha}, z_t)), \qquad t \in [0, \alpha - \bar{\alpha}], \tag{4.74}$$

with initial condition

$$z(t) = x(t + \bar{\alpha}; \varphi), \qquad t \in [-r, 0]. \tag{4.75}$$

Then, clearly, $z(t) = x(t + \bar{\alpha}; \varphi)$ is the unique solution of IVP (4.74)-(4.75) on $[0, \alpha - \bar{\alpha}]$. On the other hand, Lemma 4.19 and Remark 4.20 imply that if we define $\alpha^* \equiv \min\{\bar{\alpha}, \alpha - \bar{\alpha}\}$ then the operator S corresponding to (4.74)-(4.75) satisfies

$$S \ : \ \overline{\mathcal{G}}_{\mathbb{Y}^p_{\alpha^*}}(\bar{\beta}) \times \mathcal{G}_{W^{1,\infty}}(x(\bar{\alpha} + \cdot; \varphi); \delta^*) \to \overline{\mathcal{G}}_{\mathbb{Y}^p_{\alpha^*}}(\bar{\beta})$$

with some $\delta^* > 0$. But then the first part of this theorem yields that z is continuously differentiable wrt its initial function on $[0, \alpha^*]$. Therefore, since for $h \in W^{1,\infty}$ Lemma 3.20 implies that

$$|x(\cdot; \varphi + h) - x(\cdot; \varphi)|_{W^{1,\infty}_{\bar{\alpha}+\alpha^*}} \le L_3 |h|_{W^{1,\infty}},$$

it follows that $x(\cdot; \varphi)$ is continuously differentiable wrt φ on $[0, \bar{\alpha} + \alpha^*]$. By repeating this argument finitely many times, we obtain that $x(\cdot; \varphi)$ is actually continuously differentiable in $\mathbb{X}^p_\alpha$. □

Since by Lemma 4.18 (i) the $|\cdot|_{\mathbb{X}^p_\alpha}$-norm is stronger than the $|\cdot|_{W^{1,p}_\alpha}$-norm, the theorem has the following corollary.

Corollary 4.24 *Assume the conditions of Theorem 4.23. Then $x(t; \varphi)$ is continuously differentiable wrt φ, as a function*

$$\Big(\mathcal{G}_{W^{1,\infty}}(\bar{\varphi}; \delta) \subset W^{1,\infty}\Big) \to W^{1,p}_\alpha, \quad \varphi \mapsto x(\cdot; \varphi).$$

4.1.4 Differentiability of the composition map B_Λ

In this section we study differentiability the composition map B_Λ defined by (4.40), and show conditions in our examples implying assumption (A8b), i.e., the differentiability of B_Λ on a set $\mathcal{K}$.

We introduce the composition map B_λ corresponding to λ:

$$B_\lambda \ : \ \Big(\mathcal{K} \times W^{1,\infty}_\alpha \subset \mathbb{X}^p_\alpha \times \mathbb{X}^p_\alpha\Big) \to L^p([0, \alpha], \mathbb{R}^n), \qquad B_\lambda(x, z)(t) \equiv \lambda(t, x_t, z_t), \tag{4.76}$$

where $1 \le p < \infty$, $0 < \alpha \le T$ finite.

By Lemma 2.17, to obtain (A8b), we need to show, that $B_\lambda(x, z)$ has continuous partial derivatives wrt x and z on $\mathcal{K} \times W^{1,\infty}_\alpha$ for some $\mathcal{K} \subset W^{1,\infty}_\alpha$.

In [10], Brokate and Colonius studied linearization of the equation

$$\dot{x}(t) = f\Big(t, x(t - \tau(t, x(t)))\Big), \qquad t \in [0, \alpha].$$

In particular, they investigated differentiability of the composition operator

$$A \ : \ \Big(\bar{X} \subset W^{1,\infty}_\alpha\Big) \to L^p([0, \alpha]; \mathbb{R}^n), \quad (Ax)(t) \equiv x(t - \tau(t, x(t))),$$

where it was assumed that $\tau(t,x)$ is twice continuously differentiable satisfying $0 \le \tau(t,x) \le r$ for all $t \in [0,\alpha]$ and $x \in \mathbb{R}$, and

$$\bar{X} \equiv \Big\{ x \in W^{1,\infty}_\alpha \ :\ \text{there exists } \varepsilon > 0 \text{ s.t. } \frac{d}{dt}\Big(t - \tau(t, x(t))\Big) \ge \varepsilon \quad \text{a.e. } t \in [0,\alpha] \Big\}.$$

It was shown in [10], that under these assumptions, A is continuously (Frechét-)differentiable on its domain with derivative

$$((A'x)h)(t) = h(t - \tau(t,x(t))) + \dot{x}(t - \tau(t,x(t)))\frac{\partial \tau}{\partial x}(t,x(t))h(t). \tag{4.77}$$

The key idea of obtaining the result in [10] is the choice of the domain, $\bar{X}$. With minor modifications, the argument of [10] is applicable to obtain differentiability of $B_\Lambda \ :\ \Big(\mathcal{K} \subset (W^{1,\infty}_\alpha, |\cdot|_{\mathbb{X}^p_\alpha})\Big) \to L^p([0,\alpha];\mathbb{R}^n)$ in our examples as well. (The main difference between our case and that of [10] is that we need differentiability of B_Λ in the $|\cdot|_{\mathbb{X}^p_\alpha}$-norm.) We can proceed as follows.

Examples 1.1 and 1.2 are state-independent equations (and also can be considered as special cases of Example 1.4), and therefore omitted here.

Example 4.25 First we study Example 1.3, i.e., the equation

$$\dot{x}(t) = f\Big(t, x(t), x(t - \tau(t, x_t))\Big), \qquad t \in [0,T]. \tag{4.78}$$

Fix $\bar{\varphi} \in W^{1,\infty}$ and $\alpha > 0$ such that (4.78) have a unique solution on $[0,\alpha]$ corresponding to all initial function $\varphi \in \mathcal{G}_{W^{1,\infty}}(\bar{\varphi};\bar{\delta})$ for some $\bar{\delta} > 0$. Consider the function λ corresponding to (4.78):

$$\lambda(t,\psi,\xi) \equiv \xi(-\tau(t,\psi)). \tag{4.79}$$

(See also Examples 1.3, 3.3 and 3.10.) We assume that τ satisfies the following assumptions:

(i) $\tau(t,\psi)$ is locally Lipschitz-continuous in ψ, i.e., for every $M > 0$ there exists a constant $L_\tau = L_\tau(\alpha, M)$ such that

$$|\tau(t,\psi) - \tau(t,\bar{\psi})| \le L_\tau |\psi - \bar{\psi}|_C, \quad \text{for } \psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M),\ t \in [0,\alpha],$$

(ii) $\tau(t,\psi)\ :\ \Big([0,\alpha] \times \Omega_3 \subset [0,\alpha] \times C\Big) \to \mathbb{R}$ is continuously differentiable wrt t and ψ,

(iii) $\frac{\partial \tau}{\partial t}(t,\psi)$ and $\frac{\partial \tau}{\partial \psi}(t,\psi)$ are locally Lipschitz-continuous in ψ, i.e., for every $M > 0$ there exists $L^*_\tau = L^*_\tau(\alpha, M)$ such that for all $t \in [0,\alpha]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M) \cap \Omega_3$ it follows that

$$\left| \frac{\partial \tau}{\partial t}(t,\psi) - \frac{\partial \tau}{\partial t}(t,\bar{\psi}) \right| \le L^*_\tau |\psi - \bar{\psi}|_C,$$

and

$$\left\| \frac{\partial \tau}{\partial \psi}(t,\psi) - \frac{\partial \tau}{\partial \psi}(t,\bar{\psi}) \right\|_{\mathcal{L}(C,\mathbb{R})} \le L^*_\tau |\psi - \bar{\psi}|_C.$$

For each $\varepsilon > 0$ define the set

$$X_\varepsilon \equiv \Big\{ x \in W^{1,\infty}_\alpha \ : \ \frac{d}{dt}\Big(t - \tau(t, x_t)\Big) \geq \varepsilon \ \text{ a.e. } t \in [0, \alpha] \Big\}. \tag{4.80}$$

We shall need the following lemma (see also Lemma 3.1 in [10]).

Lemma 4.26 *Assume that τ satisfies (i) and (ii), and let $g \in L^p_\alpha$, $x \in X_\varepsilon$ for some $\varepsilon > 0$. Then*

$$\int_0^\alpha |g(t - \tau(t, x_t))|^p \, dt \leq \frac{1}{\varepsilon} |g|^p_{L^p_\alpha}.$$

Moreover, if $x^k \in X_\varepsilon$, $|x^k - x|_{W^{1,\infty}_\alpha} \leq \delta$ for $k \in \mathbb{N}$ with some $\delta > 0$, and $|x^k - x|_{\mathbb{X}^p_\alpha} \to 0$ as $k \to \infty$, then

$$\lim_{k\to\infty} \int_0^\alpha \Big| g(t - \tau(t, x^k_t)) - g(t - \tau(t, x_t)) \Big|^p \, dt = 0.$$

Proof Elementary manipulations give

$$\begin{aligned}
\int_0^\alpha |g(t - \tau(t, x_t))|^p \, dt &= \int_0^\alpha |g(t - \tau(t, x_t))|^p \left(\frac{d}{dt}(t - \tau(t, x_t)) \right) \frac{1}{\frac{d}{dt}(t - \tau(t, x_t))} \, dt \\
&\leq \frac{1}{\varepsilon} \int_{-\tau(0, x_0)}^{\alpha - \tau(\alpha, x_\alpha)} |g(u)|^p \, du \\
&\leq \frac{1}{\varepsilon} |g|^p_{L^p_\alpha},
\end{aligned}$$

which proves the first part of the lemma.

For the second part, first assume that $g(t) = \chi_{[a,b]}(t)$ for some $[a, b] \subset [-r, \alpha]$. Then it is easy to see (see also the proof of Lemma 3.1 in [10]) that

$$meas\Big\{ t \ : \ \chi_{[a,b]}(t - \tau(t, x^k_t)) \neq \chi_{[a,b]}(t - \tau(t, x_t)) \Big\} \leq \frac{4}{\varepsilon} \sup_{0 \leq t \leq \alpha} |t - \tau(t, x^k_t) - (t - \tau(t, x_t))|,$$

therefore by assumption (i) with $L_\tau = L_\tau(\alpha, |x|_{C_\alpha} + \delta)$, and Lemma 4.18 (iii) we get that

$$\begin{aligned}
\int_0^\alpha \Big| g(t - \tau(t, x^k_t)) - g(t - \tau(t, x_t)) \Big|^p \, dt &\leq \frac{4}{\varepsilon} L_\tau \sup_{0 \leq t \leq \alpha} |x^k_t - x_t|_C \\
&\leq \frac{4}{\varepsilon} L_\tau c_3 |x^k - x|_{\mathbb{X}^p_\alpha}, \\
&\to 0, \qquad \text{as } k \to \infty,
\end{aligned}$$

which proves the statement for this case. Clearly, we can extend this result for the case when g is a step function. Let s be a step function on $[-r, \alpha]$. Then by the first part of the lemma we have that

$$\int_0^\alpha \Big| g(t - \tau(t, x_t)) - s(t - \tau(t, x_t)) \Big|^p \, dt \leq \frac{1}{\varepsilon} |g - s|^p_{L^p_\alpha},$$

and

$$\int_0^\alpha \Big| g(t - \tau(t, x^k_t)) - s(t - \tau(t, x^k_t)) \Big|^p \, dt \leq \frac{1}{\varepsilon} |g - s|^p_{L^p_\alpha},$$

therefore the triangle inequality yields that

$$\begin{aligned}\Big(\int_0^\alpha \Big|g(t-\tau(t,x_t^k)) - g(t-\tau(t,x_t))\Big|^p\,dt\Big)^{1/p} \\ \le\ \frac{2}{\varepsilon}|g-s|_{L^p_\alpha} + \Big(\int_0^\alpha \Big|s(t-\tau(t,x_t^k)) - s(t-\tau(t,x_t))\Big|^p\,dt\Big)^{1/p},\end{aligned}$$

and the statement follows from that it is true for step functions, and that the step functions are dense in L^p_α. $\square$

Lemma 4.27 *Let $x^0 \in X_{\varepsilon^0}$ for some $\varepsilon^0 > 0$. Assume that τ satisfies (i)–(iii). Then there exists $\delta > 0$ and $\varepsilon > 0$ such that $\overline{\mathcal{G}}_{W^{1,\infty}_\alpha}(x^0;\delta) \subset X_\varepsilon$.*

Proof Since x^0 is a.e. differentiable, the assumption $x^0 \in X_{\varepsilon^0}$ is equivalent to

$$1 - \frac{\partial\tau}{\partial t}(t,x_t^0) - \frac{\partial\tau}{\partial\psi}(t,x_t^0)\dot{x}_t^0 \ge \varepsilon^0, \qquad \text{for a.e. } t\in[0,\alpha],$$

and hence it is equivalent to

$$\frac{\partial\tau}{\partial t}(t,x_t^0) + \frac{\partial\tau}{\partial\psi}(t,x_t^0)\dot{x}_t^0 \le 1-\varepsilon^0, \qquad \text{for a.e. } t\in[0,\alpha].$$

Consider

$$\begin{aligned}&\frac{\partial\tau}{\partial t}(t,x_t) + \frac{\partial\tau}{\partial\psi}(t,x_t)\dot{x}_t \\ &\quad\le\ \frac{\partial\tau}{\partial t}(t,x_t^0) + \frac{\partial\tau}{\partial\psi}(t,x_t^0)\dot{x}_t^0 + \frac{\partial\tau}{\partial t}(t,x_t) - \frac{\partial\tau}{\partial t}(t,x_t^0) \\ &\qquad+\ \Big(\frac{\partial\tau}{\partial\psi}(t,x_t) - \frac{\partial\tau}{\partial\psi}(t,x_t^0)\Big)\dot{x}_t^0 + \frac{\partial\tau}{\partial\psi}(t,x_t)(\dot{x}_t - \dot{x}_t^0) \\ &\quad\le\ 1-\varepsilon^0 + \Big|\frac{\partial\tau}{\partial t}(t,x_t) - \frac{\partial\tau}{\partial t}(t,x_t^0)\Big| + \Big\|\frac{\partial\tau}{\partial\psi}(t,x_t) - \frac{\partial\tau}{\partial\psi}(t,x_t^0)\Big\|_{\mathcal{L}(C,\mathbb{R})}|\dot{x}_t^0|_C \\ &\qquad+\ \Big\|\frac{\partial\tau}{\partial\psi}(t,x_t)\Big\|_{\mathcal{L}(C,\mathbb{R})}|\dot{x}_t - \dot{x}_t^0|_C.\end{aligned}$$

Fix $0<\bar\varepsilon<\varepsilon^0$, and Let $L_\tau^* = L_\tau^*(\alpha, |x^0|_{C_\alpha}+1)$. Then for $|x-x^0|_{C_\alpha}\le 1$ we have that

$$\begin{aligned}&\frac{\partial\tau}{\partial t}(t,x_t) + \frac{\partial\tau}{\partial\psi}(t,x_t)\dot{x}_t \\ &\quad\le\ 1-\varepsilon^0 + L_\tau^*|x_t - x_t^0|_C(1+|\dot{x}_t^0|_C) + \left(\Big\|\frac{\partial\tau}{\partial\psi}(t,x_t^0)\Big\|_{\mathcal{L}(C,\mathbb{R})} + L_\tau^*\right)|\dot{x}_t - \dot{x}_t^0|_C \\ &\quad\le\ 1-\varepsilon^0 + \left(L_\tau^*(1+|\dot{x}_t^0|_C) + \max_{0\le t\le\alpha}\Big\|\frac{\partial\tau}{\partial\psi}(t,x_t^0)\Big\|_{\mathcal{L}(C,\mathbb{R})} + L_\tau^*\right)|x-x^0|_{W^{1,\infty}_\alpha}.\end{aligned}$$

Therefore, there exists $\delta>0$ such that for $|x-x^0|_{W^{1,\infty}_\alpha}\le\delta$ it follows that

$$\frac{\partial\tau}{\partial t}(t,x_t) + \frac{\partial\tau}{\partial\psi}(t,x_t)\dot{x}_t \le 1-\varepsilon^0+\bar\varepsilon, \qquad \text{for a.e. } t\in[0,\alpha],$$

i.e., $x \in X_\varepsilon$ with $\varepsilon = \varepsilon^0 - \bar\varepsilon$. $\square$

Define

$$\mathcal{K} \equiv \mathcal{G}_{W^{1,\infty}_\alpha}\Big(x^0;\,\delta\Big), \tag{4.81}$$

where x^0, $\delta > 0$ satisfy the previous lemma, i.e., $x^0 \in X_{\varepsilon^0}$ for some $\varepsilon^0 > 0$, and $\delta > 0$ is such that $\overline{\mathcal{G}}_{W^{1,\infty}_\alpha}(x^0;\,\delta) \subset X_\varepsilon$ for some $\varepsilon > 0$. We use this notation throughout the discussion of Example 4.25.

Next we show that $B_\lambda(x,z)$ has continuous partial derivatives wrt x and z for $x \in \mathcal{K}$, $z \in W^{1,\infty}_\alpha$.

Lemma 4.28 *Assume that τ satisfies (i)–(iii), and let $\mathcal{K}$ defined by (4.81). Then the composition operator $B_\lambda(x,z)$ defined by (4.76) has continuous partial derivatives wrt x and z for $x \in \mathcal{K}$, $z \in \mathbb{X}^p_\alpha$. Moreover,*

$$\frac{\partial B_\lambda}{\partial z}(x,z)h = B_\lambda(x,h), \qquad h \in \mathbb{X}^p_\alpha, \tag{4.82}$$

and

$$\Big(\frac{\partial B_\lambda}{\partial x}(x,z)h\Big)(t) = -\dot z(t-\tau(t,x_t))\frac{\partial \tau}{\partial \psi}(t,x_t)h_t, \qquad h \in \mathbb{X}^p_\alpha, \quad t \in [0,\alpha]. \tag{4.83}$$

Proof Since the map $z \mapsto B_\lambda(x,z)$ is linear, it is obvious, that it is differentiable, and (4.82) is satisfied provided that $\frac{\partial B_\lambda}{\partial z}(x,z)$ is bounded. Let $h \in \mathbb{X}^p_\alpha$, $x \in \mathcal{K}$ and $z \in \mathbb{X}^p_\alpha$. Then since $x \in X_\varepsilon$, Lemma 4.26 and Lemma 4.18 (i) imply that

$$\begin{aligned}\Big|\frac{\partial B_\lambda}{\partial z}(x,z)h\Big|_{L^p([0,\alpha];\mathbb{R}^n)} &= \Big(\int_0^\alpha |h(t-\tau(t,x_t))|^p dt\Big)^{1/p}\\ &\le \frac{1}{\varepsilon^{1/p}}|h|_{L^p_\alpha}\\ &\le \frac{c_1}{\varepsilon^{1/p}}|h|_{\mathbb{X}^p_\alpha},\end{aligned}$$

which shows the boundedness of $\frac{\partial B_\lambda}{\partial z}(x,z)$.

Next we show the continuity of the derivative. First we comment that $\frac{\partial B_\lambda}{\partial z}(x,z)$ is independent of z. Let $x, \bar x \in \mathcal{K}$, $z, \bar z \in \mathbb{X}^p_\alpha$, and consider

$$\begin{aligned}&\Big|\frac{\partial B_\lambda}{\partial z}(x,z)h - \frac{\partial B_\lambda}{\partial z}(\bar x,\bar z)h\Big|^p_{L^p([0,\alpha];\mathbb{R}^n)}\\ &\quad= |B_\lambda(x,h) - B_\lambda(\bar x,h)|^p_{L^p([0,\alpha];\mathbb{R}^n)}\\ &\quad= \int_0^\alpha |h(t-\tau(t,x_t)) - h(t-\tau(t,\bar x_t))|^p\,dt\\ &\quad= \int_0^\alpha \Big|\int_0^1 \dot h\Big(t-\tau(t,\bar x_t)+u(\tau(t,\bar x_t)-\tau(t,x_t))\Big)\,du\Big(\tau(t,\bar x_t)-\tau(t,x_t)\Big)\Big|^p\,dt\\ &\quad\le \int_0^\alpha \Big|\int_0^1 \Big|\dot h\Big(t-\tau(t,\bar x_t)+u(\tau(t,\bar x_t)-\tau(t,x_t))\Big)\Big|\,du\Big|\tau(t,x_t)-\tau(t,\bar x_t)\Big|\Big|^p\,dt.\end{aligned}$$

The definition of $\mathcal{K}$ implies that $|x - x^0|_{W^{1,\infty}_\alpha} \le \delta$ and $|\bar{x} - x^0|_{W^{1,\infty}_\alpha} \le \delta$, hence $x, \bar{x} \in \overline{\mathcal{G}}_C\Big(|x^0|_{W^{1,\infty}_\alpha} + \delta\Big)$. Let $L_\tau = L_\tau(\alpha, |x^0|_{W^{1,\infty}_\alpha} + \delta)$, then by (i), Hölder's inequality and Fubini's theorem it follows that

$$
\begin{aligned}
&\Big|\frac{\partial B_\lambda}{\partial z}(x,z)h - \frac{\partial B_\lambda}{\partial z}(\bar{x},\bar{z})h\Big|^p_{L^p([0,\alpha];\mathbb{R}^n)} \\
&\quad \le L_\tau^p \int_0^\alpha \Big|\int_0^1 \Big|\dot{h}\Big(t - \tau(t,\bar{x}_t) + u(\tau(t,\bar{x}_t) - \tau(t,x_t))\Big)\Big|\,du \Big|x_t - \bar{x}_t\Big|_C\Big|^p\,dt \\
&\quad \le L_\tau^p |x - \bar{x}|^p_{C_\alpha} \int_0^\alpha \int_0^1 \Big|\dot{h}\Big(t - \tau(t,\bar{x}_t) + u(\tau(t,\bar{x}_t) - \tau(t,x_t))\Big)\Big|^p\,du\,dt \\
&\quad \le L_\tau^p |x - \bar{x}|^p_{C_\alpha} \int_0^1 \int_0^\alpha \Big|\dot{h}\Big(t - \tau(t,\bar{x}_t) + u(\tau(t,\bar{x}_t) - \tau(t,x_t))\Big)\Big|^p\,dt\,du.
\end{aligned}
\tag{4.84}
$$

Since $x, \bar{x} \in X_\varepsilon$, we have for $u \in [0,1]$ that

$$
\begin{aligned}
\frac{d}{dt}\Big((t - \tau(t,\bar{x}_t) + u(\tau(t,\bar{x}_t) - \tau(t,x_t))\Big) &= \frac{d}{dt}\Big(u(t - \tau(t,x_t)) + (1-u)(t - \tau(t,\bar{x}_t))\Big) \\
&= u\frac{d}{dt}\Big(t - \tau(t,x_t)\Big) + (1-u)\frac{d}{dt}\Big(t - \tau(t,\bar{x}_t)\Big) \\
&> u\varepsilon + (1-u)\varepsilon \\
&= \varepsilon,
\end{aligned}
\tag{4.85}
$$

therefore (4.84), Lemma 4.18 (iii), (iv) and Lemma 4.26 imply that

$$
\begin{aligned}
\Big|\frac{\partial B_\lambda}{\partial z}(x,z)h - \frac{\partial B_\lambda}{\partial z}(\bar{x},\bar{z})h\Big|^p_{L^p([0,\alpha];\mathbb{R}^n)} &\le \frac{1}{\varepsilon} L_\tau^p |x - \bar{x}|^p_{C_\alpha} |\dot{h}|^p_{L^p_\alpha} \\
&\le \frac{1}{\varepsilon} L_\tau^p c_3^p c_4^p |x - \bar{x}|^p_{\mathbb{X}^p_\alpha} |h|^p_{\mathbb{X}^p_\alpha},
\end{aligned}
$$

i.e.,

$$
\Big\|\frac{\partial B_\lambda}{\partial z}(x,z) - \frac{\partial B_\lambda}{\partial z}(\bar{x},\bar{z})\Big\|_{\mathcal{L}(\mathbb{X}^p_\alpha, L^p([0,\alpha];\mathbb{R}^n))} \le \frac{L_\tau c_3 c_4}{\varepsilon^{1/p}} |x - \bar{x}|_{\mathbb{X}^p_\alpha},
\tag{4.86}
$$

hence $\frac{\partial B_\lambda}{\partial z}$ is continuous on its domain.

Now we shall show that the function defined by (4.83) is, in fact, the partial derivative of B_λ wrt x. The boundedness of $\frac{\partial B_\lambda}{\partial x}(x,z)$ follows from Lemma 4.18 (iii) and from the estimates

$$
\begin{aligned}
\Big|\frac{\partial B_\lambda}{\partial x}(x,z)h\Big|_{L^p([0,\alpha];\mathbb{R}^n)} &= \Big(\int_0^\alpha \Big|\dot{z}(t - \tau(t,x_t))\frac{\partial \tau}{\partial \psi}(t,x_t)h_t\Big|^p\,dt\Big)^{1/p} \\
&\le |z|_{W^{1,\infty}_\alpha} \sup_{0\le t\le\alpha} \Big\|\frac{\partial \tau}{\partial \psi}(t,x_t)\Big\|_{\mathcal{L}(C,\mathbb{R})} |h|_{C_\alpha} \alpha^{1/p} \\
&\le |z|_{W^{1,\infty}_\alpha} \sup_{0\le t\le\alpha} \Big\|\frac{\partial \tau}{\partial \psi}(t,x_t)\Big\|_{\mathcal{L}(C,\mathbb{R})} c_3 |h|_{\mathbb{X}^p_\alpha} \alpha^{1/p}.
\end{aligned}
$$

Elementary manipulations yield that

$$
\Big|B_\lambda(x+h,z) - B_\lambda(x,z) - \frac{\partial B_\lambda}{\partial x}(x,z)h\Big|^p_{L^p([0,\alpha];\mathbb{R}^n)}
$$

$$
\begin{aligned}
&= \int_0^\alpha \left| z(t-\tau(t,x_t+h_t)) - z(t-\tau(t,x_t)) + \dot z(t-\tau(t,x_t))\frac{\partial \tau}{\partial \psi}(t,x_t)h_t \right|^p dt \\
&= \int_0^\alpha \bigg| \int_0^1 \dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big)\,du(\tau(t,x_t)-\tau(t,x_t+h_t)) \\
&\qquad + \dot z(t-\tau(t,x_t))\frac{\partial \tau}{\partial \psi}(t,x_t)h_t \bigg|^p dt \\
&= \int_0^\alpha \bigg| \int_0^1 \Big(\dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big) - \dot z(t-\tau(t,x_t))\Big)\,du \\
&\qquad \cdot (\tau(t,x_t)-\tau(t,x_t+h_t)) \\
&\qquad + \dot z(t-\tau(t,x_t))\Big(\tau(t,x_t)-\tau(t,x_t+h_t)+\frac{\partial \tau}{\partial \psi}(t,x_t)h_t\Big) \bigg|^p dt.
\end{aligned}
$$

Then by the triangle and Hölder's inequalities it follows that

$$
\begin{aligned}
&\left| B_\lambda(x+h,z) - B_\lambda(x,z) - \frac{\partial B_\lambda}{\partial x}(x,z)h \right|_{L^p([0,\alpha];\mathbb{R}^n)} \\
&\le \bigg(\int_0^\alpha \bigg| \int_0^1 \Big(\dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big) - \dot z(t-\tau(t,x_t))\Big)\,du \\
&\qquad \cdot (\tau(t,x_t)-\tau(t,x_t+h_t)) \bigg|^p dt \bigg)^{1/p} \\
&\qquad + \bigg(\int_0^1 \bigg| \dot z(t-\tau(t,x_t))\Big(\tau(t,x_t)-\tau(t,x_t+h_t)+\frac{\partial \tau}{\partial \psi}(t,x_t)h_t\Big) \bigg|^p dt \bigg)^{1/p} \\
&\le \bigg(\int_0^\alpha \int_0^1 \Big| \dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big) - \dot z(t-\tau(t,x_t)) \Big|^p du \\
&\qquad \cdot \Big| \tau(t,x_t)-\tau(t,x_t+h_t) \Big|^p dt \bigg)^{1/p} \\
&\qquad + \bigg(\int_0^1 \Big| \dot z(t-\tau(t,x_t)) \Big|^p \Big| \tau(t,x_t)-\tau(t,x_t+h_t)+\frac{\partial \tau}{\partial \psi}(t,x_t)h_t \Big|^p dt \bigg)^{1/p}. \qquad (4.87)
\end{aligned}
$$

First consider the first term of the right hand side of (4.87). Since $x+h \in \mathcal{K}$, we have that $|x+h|_C \le |x^0|_C + \delta$. Let $L_\tau = L_\tau(\alpha, |x^0|_{C_\alpha} + \delta)$. Then assumption (i), Fubini's theorem, and Lemma 4.18 (iii) imply that

$$
\begin{aligned}
&\bigg(\int_0^\alpha \int_0^1 \Big| \dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big) - \dot z(t-\tau(t,x_t)) \Big|^p du \\
&\qquad \cdot \Big| \tau(t,x_t)-\tau(t,x_t+h_t) \Big|^p dt \bigg)^{1/p} \\
&\le L_\tau |h|_{C_\alpha} \bigg(\int_0^\alpha \int_0^1 \Big| \dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big) - \dot z(t-\tau(t,x_t)) \Big|^p du\,dt \bigg)^{1/p} \\
&= L_\tau |h|_{C_\alpha} \bigg(\int_0^1 \int_0^\alpha \Big| \dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big) - \dot z(t-\tau(t,x_t)) \Big|^p dt\,du \bigg)^{1/p} \\
&\le L_\tau c_3 |h|_{\mathbb{X}^p_\alpha} \bigg(\int_0^1 \int_0^\alpha \Big| \dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big) - \dot z(t-\tau(t,x_t)) \Big|^p dt\,du \bigg)^{1/p}. \qquad (4.88)
\end{aligned}
$$

Lemma 4.26 yields that

$$\int_0^\alpha \Big|\dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big)-\dot z(t-\tau(t,x_t))\Big|^p\,dt\to 0,$$

as $|h|_{\mathbb{X}^p_\alpha}\to 0$, since

$$\begin{aligned}\Big|t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))-\Big(t-\tau(t,x_t)\Big)\Big| &= u\Big|\tau(t,x_t)-\tau(t,x_t+h_t)\Big|\\ &\to 0,\qquad \text{as } |h|_{\mathbb{X}^p_\alpha}\to 0,\end{aligned}$$

by the continuity of τ and Lemma 4.18 (iii), and because, similarly to (4.85), we can show that

$$\frac{d}{dt}\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big)\geq\varepsilon.$$

Since $z\in W^{1,\infty}_\alpha$, we get that the function

$$u\mapsto\int_0^\alpha \Big|\dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t)\Big)-\dot z(t-\tau(t,x_t))\Big|^p\,dt$$

is bounded on $[0,1]$, therefore the Lebesgue Dominant Convergence Theorem yields that

$$\int_0^1\int_0^\alpha \Big|\dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t)\Big)-\dot z(t-\tau(t,x_t))\Big|^p\,dt\,du\to 0,\ \text{as } |h|_{\mathbb{X}^p_\alpha}\to 0. \tag{4.89}$$

Consider the second term of the right hand side of (4.87). By applying Lemma 2.16, assumption (iii) with $L^*_\tau=L^*_\tau(\alpha,|x^0|_{C_\alpha}+\delta)$, Lemma 4.18 (iii) and (iv), Lemma 4.26, and that $x\in X_\varepsilon$, we get

$$\begin{aligned}&\left(\int_0^1\Big|\dot z(t-\tau(t,x_t))\Big|^p\Big|\tau(t,x_t)-\tau(t,x_t+h_t)+\frac{\partial\tau}{\partial\psi}(t,x_t)h_t\Big|^p\,dt\right)^{1/p}\\ &\quad\leq\left(\int_0^1\Big|\dot z(t-\tau(t,x_t))\Big|^p\sup_{0\leq\nu\leq1}\Big\|\frac{\partial\tau}{\partial\psi}(t,x_t+\nu h_t)-\frac{\partial\tau}{\partial\psi}(t,x_t)\Big\|^p_{\mathcal{L}(C,\mathbb{R})}|h_t|^p_C\,dt\right)^{1/p}\\ &\quad\leq L^*_\tau\left(\int_0^1\Big|\dot z(t-\tau(t,x_t))\Big|^p|h_t|^{2p}_C\,dt\right)^{1/p}\\ &\quad\leq L^*_\tau|h|^2_{C_\alpha}\left(\int_0^1\Big|\dot z(t-\tau(t,x_t))\Big|^p\,dt\right)^{1/p}\\ &\quad\leq\frac{1}{\varepsilon^{1/p}}L^*_\tau|h|^2_{C_\alpha}|\dot z|_{L^p_\alpha}\\ &\quad\leq\frac{1}{\varepsilon^{1/p}}L^*_\tau c_3^2c_4|h|^2_{\mathbb{X}^p_\alpha}|z|_{\mathbb{X}^p_\alpha}.\end{aligned}\tag{4.90}$$

Combining (4.87), (4.88), (4.89) and (4.90), we get that

$$\begin{aligned}&\frac{1}{|h|_{\mathbb{X}^p_\alpha}}\Big|B_\lambda(x+h,z)-B_\lambda(x,z)-\frac{\partial B_\lambda}{\partial x}(x,z)h\Big|^p_{L^p([0,\alpha];\mathbb{R}^n)}\\ &\quad\leq L_\tau c_3\Big(\int_0^1\int_0^\alpha\Big|\dot z\Big(t-\tau(t,x_t)+u(\tau(t,x_t)-\tau(t,x_t+h_t))\Big)\\ &\qquad\qquad-\dot z(t-\tau(t,x_t))\Big|^p\,dt\,du\Big)^{1/p}+\frac{1}{\varepsilon^{1/p}}L^*_\tau c_3^2c_4|h|_{\mathbb{X}^p_\alpha}|z|_{\mathbb{X}^p_\alpha}\\ &\quad\to 0,\qquad\text{as } |h|_{\mathbb{X}^p_\alpha}\to 0,\end{aligned}$$

which proves (4.83).

Next we show that $\frac{\partial B_\lambda}{\partial x}$ is continuous on $\mathcal{K}\times\mathbb{X}^p_\alpha$.

$$\begin{aligned}
&\left|\frac{\partial B_\lambda}{\partial x}(x,z)h-\frac{\partial B_\lambda}{\partial x}(\bar x,\bar z)h\right|_{L^p([0,\alpha];\mathbb{R}^n)}\\
&\qquad = \left(\int_0^\alpha\left|\dot z(t-\tau(t,x_t))\frac{\partial\tau}{\partial\psi}(t,x_t)h_t-\dot{\bar z}(t-\tau(t,\bar x_t))\frac{\partial\tau}{\partial\psi}(t,\bar x_t)h_t\right|^p dt\right)^{1/p}\\
&\qquad \le \left(\int_0^\alpha\left|\dot z(t-\tau(t,x_t))-\dot{\bar z}(t-\tau(t,x_t))\right|^p\left|\frac{\partial\tau}{\partial\psi}(t,x_t)h_t\right|^p dt\right)^{1/p}\\
&\qquad\quad + \left(\int_0^\alpha\left|\dot{\bar z}(t-\tau(t,\bar x_t))\right|^p\left|\frac{\partial\tau}{\partial\psi}(t,x_t)h_t-\frac{\partial\tau}{\partial\psi}(t,\bar x_t)h_t\right|^p dt\right)^{1/p}. \qquad (4.91)
\end{aligned}$$

Assumption (iii) with $L^*_\tau = L^*_\tau(\alpha,|x^0|_{W^{1,\infty}_\alpha}+\delta)$, Lemma 4.26 and Lemma 4.18 (iii) and (iv) yield

$$\begin{aligned}
&\left|\frac{\partial B_\lambda}{\partial x}(x,z)h-\frac{\partial B_\lambda}{\partial x}(\bar x,\bar z)h\right|_{L^p([0,\alpha];\mathbb{R}^n)}\\
&\le \left(\int_0^\alpha\left|\dot z(t-\tau(t,x_t))-\dot{\bar z}(t-\tau(t,x_t))\right|^p\left\|\frac{\partial\tau}{\partial\psi}(t,x_t)\right\|^p_{\mathcal{L}(C,\mathbb{R})}|h_t|_C\,dt\right)^{1/p}\\
&\quad + L^*_\tau\left(\int_0^\alpha\left|\dot{\bar z}(t-\tau(t,\bar x_t))\right|^p\left|x_t-\bar x_t\right|^p_C|h_t|^p_C\,dt\right)^{1/p}\\
&\le \left(\max_{0\le t\le\alpha}\left\|\frac{\partial\tau}{\partial\psi}(t,\bar x_t)\right\|_{\mathcal{L}(C,\mathbb{R})}+L^*_\tau 2\delta\right)|h|_{C_\alpha}\left(\int_0^\alpha\left|\dot z(t-\tau(t,x_t))-\dot{\bar z}(t-\tau(t,x_t))\right|^p dt\right)^{1/p}\\
&\quad + L^*_\tau|x-\bar x|_{C_\alpha}|h|_{C_\alpha}\alpha^{1/p}|z|_{W^{1,\infty}_\alpha}\\
&\le \frac{1}{\varepsilon^{1/p}}\left(\max_{0\le t\le\alpha}\left\|\frac{\partial\tau}{\partial\psi}(t,\bar x_t)\right\|_{\mathcal{L}(C,\mathbb{R})}+L^*_\tau 2\delta\right)|h|_{C_\alpha}|\dot z-\dot{\bar z}|_{L^p_\alpha}+L^*_\tau|x-\bar x|_{C_\alpha}|h|_{C_\alpha}\alpha^{1/p}|z|_{W^{1,\infty}_\alpha}\\
&\le \frac{1}{\varepsilon^{1/p}}\left(\max_{0\le t\le\alpha}\left\|\frac{\partial\tau}{\partial\psi}(t,\bar x_t)\right\|_{\mathcal{L}(C,\mathbb{R})}+L^*_\tau 2\delta\right)c_3c_4|h|_{\mathbb{X}^p_\alpha}|z-\bar z|_{\mathbb{X}^p_\alpha}\\
&\qquad + L^*_\tau c_3c_4|x-\bar x|_{\mathbb{X}^p_\alpha}|h|_{\mathbb{X}^p_\alpha}\alpha^{1/p}|z|_{W^{1,\infty}_\alpha},
\end{aligned}$$

which implies the continuity of $\frac{\partial B_\lambda}{\partial x}$. □

This concludes the discussion of Example 4.25.

Example 4.29 Consider a special case of Example 4.25, when $\tau(t,\psi)$ ($t\in[0,\alpha]$, $\psi\in C$) is defined through a function, $\bar\tau(t,x)$, by $\tau(t,\psi)\equiv\bar\tau(t,\psi(0))$. (See also Example 4.2.) It is easy to check, that the assumptions (i)–(iii) of Example 4.2 on $\bar\tau$ together with the continuous differentiability of $\bar\tau(t,x)$ wrt t on $[0,\alpha]\times\Omega^*$ imply conditions (i)–(iii) of Example 4.25. (Here we use that the function $G : C\to\mathbb{R}^n$, $G(\psi)\equiv\psi(0)$ is continuously differentiable with derivative $G'(\psi)h=h(0)$.)

Example 4.30 Let

$$\lambda(t,\psi,\xi)=\sum_{k=1}^m A_k(t)\xi(-\tau_k(t,\psi))+\int_{-\tau_0}^0 G(s,t,\psi)\xi(s)\,ds,$$

as in Examples 1.4, 3.4 and 3.11. Assume that for $k = 1, 2, \ldots, m$ each τ_k satisfies condition (i)–(iii) of Example 4.25, then it is easy to see that if there exists $x \in W^{1,\infty}_{\alpha}$ such that

$$\frac{d}{dt}(t - \tau_k(t, x_t)) \geq \varepsilon > 0, \qquad k = 1, 2, \ldots, m,$$

for some $\varepsilon > 0$, then the composite operator B_{λ_1} corresponding to

$$\lambda_1(t, \psi, \xi) \equiv \sum_{k=1}^{m} A_k(t)\xi(-\tau_k(t, \psi))$$

is differentiable for some $\mathcal{K}$. Define

$$\lambda_2(t, \psi, \xi) = \int_{-\tau_0}^{0} G(s, t, \psi)\xi(s)\, ds.$$

This is not a composite function of ξ and ψ, therefore it is easy to discuss differentiability of the corresponding composition map, B_{λ_2}, e.g., if we assume that

(iv) $G(s, t, \psi) : \Big([-r, 0] \times [0, \alpha] \times \Omega_3 \subset [-r, 0] \times [0, \alpha] \times C\Big) \to \mathbb{R}^{n\times n}$ has continuous partial derivative wrt ψ,

then $B_{\lambda_2}(x, z)$ (defined by (4.76)) is continuously differentiable wrt x and z. Since $B_\lambda = B_{\lambda_1} + B_{\lambda_2}$, we can get continuous differentiability of B_λ wrt x and z.

4.2 Differentiability wrt a parameter in the delay

In this section we study differentiability of solutions of IVP

$$\dot{x}(t; c) = f\Big(t, x(u; c), \Lambda(t, x(\cdot; c)_t, c)\Big), \quad t \in [0, T], \tag{4.92}$$

$$x(t; c) = \varphi(t), \qquad t \in [-r, 0] \tag{4.93}$$

wrt the parameter c of the delayed term. Here we use the notation

$$\Lambda(t, \psi, c) \equiv \int_{-r}^{0} d_s\mu(s, t, \psi, c)\psi(s), \tag{4.94}$$

where $c \in \mathbb{R}^m$, i.e., $\mu : [-r, 0] \times [0, T] \times \Omega_3 \times \Omega_4 \to \mathrm{NBV}$, and Ω_4 is an open subset of $\mathbb{R}^m$.

In this section the initial function, μ and f are considered to be fixed, and hence the solution depends only on the parameter c of μ. We shall use $x(t; c)$ and $x(\cdot; c)_t$ to denote the value of the solution and the solution segment function at t, respectively, corresponding to parameter c.

Define the function

$$\lambda(t, \psi, \xi, c) \equiv \int_{-r}^{0} d_s\mu(s, t, \psi, c)\xi(s). \tag{4.95}$$

In this section we modify assumptions (A2), (A5) (since μ depends on the variable c as well) as follows:

(A2') the function $\mu\ :\ \Big([-r,0]\times[0,T]\times\Omega_3\times\Omega_4\subset[-r,0]\times[0,T]\times C\times\mathbb{R}^m\Big)\to\text{NBV}$ is such that the function

$$[0,T]\times\Omega_3\times\Omega_4\to\mathbb{R}^n,\qquad (t,\psi,c)\mapsto\int_{-r}^0 d_s\mu(s,t,\psi,c)\xi(s)$$

is continuous and bounded on its domain for all $\xi\in\overline{\mathcal{G}}_C(1)$, (where $\mu(\cdot,t,\psi,c)$ denotes the image function corresponding to t,ψ and c),

(A5') for every $\alpha>0$, $M_1,M_2>0$ there exists a constant $L_2=L_2(\alpha,M_1,M_2)$ such that for all $\xi\in W^{1,\infty}$, $t\in[0,\alpha]$, $\psi,\bar\psi\in\overline{\mathcal{G}}_C(M_1)\cap\Omega_3$ and $c,\bar c\in\overline{\mathcal{G}}_{\mathbb{R}^m}(M_2)\cap\Omega_4$

$$|\lambda(t,\psi,\xi,c)-\lambda(t,\bar\psi,\xi,\bar c)|\le L_2|\xi|_{W^{1,\infty}}\Big(|\psi-\bar\psi|_C+|c-\bar c|_{\mathbb{R}^m}\Big),$$

For a given $c\in\Omega_4$ define the function

$$\mu^c(s,t,\psi)\equiv\mu(s,t,\psi,c).$$

Assume that c, φ, and μ are such that

$$\varphi(0)\in\Omega_1,\quad \varphi\in\Omega_3\quad\text{and}\quad\int_{-r}^0 d_s\mu(s,0,\varphi,c)\varphi(s)\,ds\in\Omega_2,\tag{4.96}$$

(see (3.9)). By (A2') and (A5') the function μ^c satisfies (A2) and (A5), thus assumption (4.96) together with Theorems 3.8 and 3.19 imply that IVP (3.1)-(3.2) corresponding to (φ,μ^c,f) has a unique solution, and consequently, IVP (4.92)-(4.93) with the fixed parameter value, c, has a unique solution on an interval $[0,\alpha]$. By assumption (A2') and the facts that Ω_2 and Ω_4 are open, it follows that the unique solution exists in a neighborhood of c as well.

Define the norm of a function μ satisfying (A2') by

$$\|\mu\|\equiv\sup\left\{\left|\int_{-r}^0 d_s\mu(s,t,\psi,c)\xi(s)\right|\ :\ t\in[0,T],\ \psi\in\Omega_3,\ c\in\Omega_4,\ \xi\in\overline{\mathcal{G}}_C(1)\right\}.\tag{4.97}$$

The following lemma is an easy consequence of the definition of $\|\cdot\|$ and (A5'). (See the proof of Lemma 3.12.)

Lemma 4.31 *Assume (A2') and (A5'). Then*

$$|\Lambda(t,\psi,c)-\Lambda(t,\bar\psi,\bar c)|\le\Big(\|\mu\|+L_2(\alpha,M_1,M_2)|\bar\psi|_{W^{1,\infty}}\Big)\Big(|\psi-\bar\psi|_C+|c-\bar c|_{\mathbb{R}^m}\Big),$$

where $t\in[0,\alpha]$, $\psi,\bar\psi\in\overline{\mathcal{G}}_C(M_1)\cap\Omega_3$, $\bar\psi\in W^{1,\infty}$ *and* $c,\bar c\in\overline{\mathcal{G}}_{\mathbb{R}^m}(M_2)\cap\Omega_4$.

The following theorem can be proved the same way we proved Theorem 3.25, using Lemma 4.31. The proof is omitted.

Theorem 4.32 *Assume that φ, μ, f and $\bar c$ satisfy (A1), (A2'), (A3), (A4), (A5'), (A6) and (4.96), and let $1\le p\le\infty$. Then there exist constants $\alpha>0$, $\delta>0$ and $L_3=L_3(p,\alpha,\bar\varphi,\bar c,\delta)$, such that IVP (4.92)-(4.93) has a unique solution on $[0,\alpha]$ for all $c\in\Omega_4$ with $|c-\bar c|_{\mathbb{R}^m}<\delta$, and*

$$|x(\cdot;c)_t-x(\cdot;\bar c)_t|_{W^{1,p}}\le L_3|c-\bar c|_{\mathbb{R}^m},\quad t\in[0,\alpha].$$

4.2.1 Special case, differentiability in $W^{1,\infty}$

In this subsection case we shall obtain differentiability of solutions wrt c in a special case (similarly to Section 4.1.1). In particular, we assume that the initial function guarantees that the solution is a C^1 function, i.e., φ satisfies that

$$\varphi \in C^1 \quad \text{and} \quad \dot{\varphi}(0-) = f(0, \varphi, \Lambda(0, \varphi, c)).$$

We make the following assumption (see (A8a) of Section 4.1.1 for comparison):

(A8a') (i) For every $\alpha > 0$, $M_1, M_2 > 0$ there exists a constant $L_2 = L_2(\alpha, M_1, M_2)$ such that for all $\xi \in W^{1,\infty}$, $t, \bar{t} \in [0, \alpha]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M_1) \cap \Omega_3$, and $c, \bar{c} \in \overline{\mathcal{G}}_{\mathbb{R}^m}(M_2) \cap \Omega_4$,

$$|\lambda(t, \psi, \xi, c) - \lambda(\bar{t}, \bar{\psi}, \xi, \bar{c})| \leq L_2 |\xi|_{W^{1,\infty}} \Big(|t - \bar{t}| + |\psi - \bar{\psi}|_C + |c - \bar{c}|_{\mathbb{R}^m} \Big),$$

(ii) For all $t \in [0, T]$, $\psi \in W^{1,\infty} \cap \Omega_3$, $\xi \in C^1$ and $c \in \Omega_4$ the function $\lambda(t, \psi, \xi, c)$ is continuously differentiable wrt ψ and c on its domain,

(iii) For all $\psi \in \Omega_3$ we have that $\Lambda(0, \psi, c)$ is independent of $c \in \Omega_4$.

We comment, that even in the state-independent case, c appears naturally inside the argument of ξ in $\lambda(t, \psi, \xi, c)$ (see Example 4.35 below), therefore differentiability of λ wrt c can not be obtained for arbitrary $\xi \in W^{1,\infty}$ functions.

The next lemma shows that (A2') and (A8a') (i) yield that $\lambda(t, \psi, \xi, c)$ is continuously differentiable wrt ξ on its domain.

Lemma 4.33 *Assume (A2') and (A8a') (i). Then the function $\lambda(t, \psi, \xi, c)$ is continuously differentiable wrt ξ, and for $t \in [0, T]$, $\psi \in W^{1,\infty} \cap \Omega_3$, $\xi \in W^{1,\infty}$ and $c \in \Omega_4$, and*

$$\frac{\partial \lambda}{\partial \xi}(t, \psi, \xi, c)h = \lambda(t, \psi, h, c), \qquad h \in W^{1,\infty},$$

Proof The differentiability of $\lambda(t, \psi, \xi, c)$ wrt ξ with the above derivative follows from the linearity of λ in ξ. The continuity of the derivative is the consequence of assumption (A8a') (i) using the inequality

$$\left| \frac{\partial \lambda}{\partial \xi}(t, \psi, \xi, c)h - \frac{\partial \lambda}{\partial \xi}(\bar{t}, \bar{\psi}, \xi, \bar{c})h \right| \leq L_2 |h|_{W^{1,\infty}} \Big(|t - \bar{t}| + |\psi - \bar{\psi}|_C + |c - \bar{c}|_{\mathbb{R}^m} \Big)$$

for $t, \bar{t} \in [0, \alpha]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_{W^{1,\infty}}(M_1)$, $c, \bar{c} \in \overline{\mathcal{G}}_{\mathbb{R}^m}(M_2)$, $L_2 = L_2(\alpha, M_1, M_2)$. □

Assumption (A8a') (i), and (ii), Lemma 4.33, Lemma 2.17 and the Chain Rule imply immediately:

Lemma 4.34 *Assume (A2'), (A5') and (A8a') (i), (ii). Then the function $\Lambda(t, \psi, c)$ is continuously differentiable wrt ψ for $t \in [0, T]$, $\psi \in W^{1,\infty} \cap \Omega_3$, $c \in \Omega_4 \subset \mathbb{R}^m$.*

Example 4.35 We illustrate conditions (A2'), (A5'), and (A8a') on the delay function of Example 1.3. Consider the equation

$$\dot{x}(t) = f\Big(t, x(t), x(t - \tau(t, x_t, c))\Big), \qquad t \in [0, T], \tag{4.98}$$

where we assume that the point delay function, $\tau(t, \psi, c)$, depends on a parameter $c \in \Omega_4$ as well. As in Example 1.3, we can see that by defining the function $\mu(s, t, \psi, c) \equiv \chi_{[-\tau(t,\psi,c),0]}(s)I$, Equation (4.98) transforms into the form of (4.92). The function λ corresponding to μ has the form

$$\lambda(t, \psi, \xi, c) = \xi(-\tau(t, \psi, c)). \tag{4.99}$$

(See also Examples 1.3, 3.3 and 3.10.) Assume that τ satisfies

(i) $\tau(\cdot, \cdot, \cdot) \; : \; \Big([0, T] \times \Omega_3 \times \Omega_4 \subset [0, T] \times C \times \mathbb{R}^m\Big) \to \mathbb{R}$ is continuous,

(ii) $\tau(t, \psi, c)$ is locally Lipschitz-continuous in ψ and c, i.e., for every $\alpha > 0$, $M_1 > 0$, $M_2 > 0$ there exists a constant $L_\tau = L_\tau(\alpha, M_1, M_2)$ such that for $t, \bar{t} \in [0, \alpha]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M_1) \cap \Omega_3$ and $c, \bar{c} \in \overline{\mathcal{G}}_{\mathbb{R}^m}(M_2) \cap \Omega_4$

$$|\tau(t, \psi, c) - \tau(\bar{t}, \bar{\psi}, \bar{c})| \le L_\tau\Big(|t - \bar{t}| + |\psi - \bar{\psi}|_{W^{1,\infty}} + |c - \bar{c}|_{\mathbb{R}^m}\Big),$$

(iii) $\tau(t, \psi, c) \; : \; \Big([0, T] \times (W^{1,\infty} \cap \Omega_3) \times \Omega_4 \subset [0, T] \times W^{1,\infty} \times \mathbb{R}^m\Big) \to \mathbb{R}$ is continuously differentiable wrt ψ and c,

(iv) $\frac{\partial \tau}{\partial c}$ is bounded on $[0, T] \times (W^{1,\infty} \cap \Omega_3) \times \Omega_4$, and

(v) $\tau(0, \psi, c)$ is independent of c, (e.g., τ has the form $\tau(t, \psi, c) = ct + \bar{\tau}(\psi)$ for some $\bar{\tau} \; : \; C \to \mathbb{R}^+$).

Then it is easy to see that these conditions imply (A2'), (A5') and (A8a').

Now we are at the position to state our results concerning the differentiability of the solution wrt a parameter in the delay.

For $c \in \Omega_4$, $h \in \mathbb{R}^m$ define the function $z(\cdot; h)$ as the solution of the linear IVP

$$\begin{aligned} \dot{z}(t; h) &= \frac{\partial f}{\partial x}\Big(t, x(t; c), \Lambda(t, x(\cdot; c)_t, c)\Big) z(t; h) \\ &\quad + \frac{\partial f}{\partial y}\Big(t, x(t; c), \Lambda(t, x(\cdot; c)_t, c)\Big)\Big(\frac{\partial \Lambda}{\partial \psi}(t, x(\cdot; c)_t, c) z(\cdot; h)_t \\ &\qquad + \frac{\partial \Lambda}{\partial c}(t, x(\cdot; c)_t, c) h\Big), \qquad t \in [0, T], \end{aligned} \tag{4.100}$$

$$z(t; h) = 0, \qquad t \in [-r, 0]. \tag{4.101}$$

We comment, that assuming (A1)–(A8a'), the solution, $z(\cdot; h)$, of this IVP exists, and depends linearly on h.

The next theorem shows that (the modified) assumptions (A1)–(A8a') imply that the function $\Big(\Omega_4 \subset \mathbb{R}^m\Big) \to \mathbb{R}^n$, $c \mapsto x(t; c)$ is differentiable for all $t \in [0, T]$.

Theorem 4.36 *Let φ, c, μ and f be fixed satisfying (A1), (A2'), (A3), (A4), (A5'), (A6), (A7), (A8a') and (4.96), and assume that $\varphi \in C^1$, $\dot{\varphi}(0-) = f(0, \varphi, \Lambda(0, \varphi, c))$. Then*

(i) the solution $x(t;c)$ of IVP (4.92)-(4.93) is differentiable wrt c for all $t \in [0, \alpha]$ and $c \in \Omega_4$,

(ii) $\dfrac{x(t;c+h) - x(t;c)}{|h|_{\mathbb{R}^m}}$ converges uniformly to $\frac{\partial x}{\partial c}(t;c)$ on $t \in [0, \alpha]$,

(iii) the derivative is $\frac{\partial x}{\partial c}(t;c)h = z(t;h)$, where $z(t;h)$ is the solution of the linear IVP (4.100)-(4.101).

Proof The proof is analogous to that of Theorem 4.8, and therefore it is omitted. □

Corollary 4.37 *Assuming the conditions of Theorem 4.36, the function*

$$\Omega_4 \to C, \quad c \mapsto x(\cdot, c)_t$$

is differentiable for all $t \in [0, \alpha]$.

Next we state the result for differentiability of the map $\Big(\Omega_4 \subset \mathbb{R}^m\Big) \to W^{1,\infty}, \quad c \mapsto x(\cdot, c)_t$ without the proof, which is analogous to that of Theorem 4.11.

Theorem 4.38 *Assume that the conditions of Theorem 4.36 are satisfied. Then the function $\Omega_4 \to W^{1,\infty}$, $c \mapsto x(\cdot; c)_t$ is differentiable for all $t \in [0, \alpha]$.*

4.2.2 General case, differentiability in $W^{1,p}$

In this section we study the general case of differentiability of solutions of IVP (4.92)-(4.93), without the strong assumption (A8a') of the previous section. We shall use the same method that was used in Section 4.1.3. We transform (4.92)-(4.93) by the new variable $y(t) \equiv x(t) - \tilde{\varphi}(t)$ into

$$y(t) = \begin{cases} 0, & t \in [-r, 0] \\ \displaystyle\int_0^t f\Big(u,\, y(u) + \tilde{\varphi}(u),\, \Lambda(u, y_u + \tilde{\varphi}_u, c)\Big)\, du, & t \in [0, T], \end{cases} \tag{4.102}$$

and introduce the operator

$$S(y,c)(t) = \begin{cases} 0, & t \in [-r, 0] \\ \displaystyle\int_0^t f\Big(u,\, y(u) + \tilde{\varphi}(u),\, \Lambda(u, y_u + \tilde{\varphi}_u, c)\Big)\, du, & t \in [0, T]. \end{cases} \tag{4.103}$$

As in Section 4.1.3, we consider $S(y,c)$ as a map

$$S \;:\; \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \times \overline{\mathcal{G}}_{\mathbb{R}^m}(\bar{c};\, \delta) \to \mathbb{Y}^p_{\bar{\alpha}}$$

for some $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\delta > 0$.

It is easy to see, by repeating the proof of Lemma 4.19, that the following result holds.

Lemma 4.39 *Let $1 \le p < \infty$, $\bar{c} \in \Omega_4$ and $R > 0$. Then there exist $\delta > 0$, $\bar{\alpha} > 0$ and $\bar{\beta} > 0$ such that $\mathcal{G}_{\mathbb{R}^m}(\bar{c};\,\delta) \subset \Omega_4$, and the operator S defined by (4.103) satisfies*

(i) $S \;:\; \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \times \mathcal{G}_{\mathbb{R}^m}(\bar{c};\,\delta) \to \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta})$,

(ii) S is a uniform contraction on $\overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{W^{1,\infty}_{\bar{\alpha}}}(R)$ both in $|\cdot|_{\mathbb{Y}^p_{\bar{\alpha}}}$ and $|\cdot|_{\mathbb{Y}^\infty_{\bar{\alpha}}}$ norms, i.e., there exists $0 \le \theta < 1$ such that for all $c \in \mathcal{G}_{\mathbb{R}^m}(\bar{c};\,\delta)$, $y, \bar{y} \in \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{W^{1,\infty}_{\bar{\alpha}}}(R)$

$$|S(y,c) - S(\bar{y},c)|_{\mathbb{Y}^\infty_{\bar{\alpha}}} \le \theta |y - \bar{y}|_{\mathbb{Y}^\infty_{\bar{\alpha}}},$$

and

$$|S(y,c) - S(\bar{y},c)|_{\mathbb{Y}^p_{\bar{\alpha}}} \le \theta |y - \bar{y}|_{\mathbb{Y}^p_{\bar{\alpha}}}.$$

Next we define the composite operator B_Λ in this section. Fix $1 \le p < \infty$ and let $\mathcal{K}$ be an open subset of $W^{1,\infty}_\alpha$. Then define

$$B_\Lambda \;:\; \Big(\mathcal{K} \times \Omega_4 \subset \mathbb{X}^p_\alpha \times \mathbb{R}^m\Big) \to L^p([0,\alpha];\mathbb{R}^n), \qquad B_\Lambda(x,c)(t) \equiv \Lambda(t, x_t, c), \qquad t \in [0,\alpha]. \tag{4.104}$$

We assume that:

(A8b') the operator B_Λ defined by (4.104) is continuously differentiable on $\mathcal{K} \times \Omega_4$ wrt x and c.

The following lemma shows that assumption (A8b) yields the existence of continuous partial derivatives of $S(y,\varphi)$ if we restrict y to a certain subset of its domain, and the derivative is taken in the restricted space (in relative topology).

Lemma 4.40 *Let $\bar{c} \in \Omega_4$, $1 \le p < \infty$ be fixed, and $R > 0$ given, and assume (A1)–(A7) and (A8b'). Let δ, $\bar{\alpha}$, $\bar{\beta}$ be the constants from Lemma 4.39, i.e., such that the operator S defined by (4.103) satisfies*

$$S \;:\; \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \times \mathcal{G}_{\mathbb{R}^m}(\bar{c};\,\delta) \to \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}),$$

and it is a uniform contraction on $\overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{\mathbb{R}^m}(R)$. Assume that there exists $\mathcal{W} \subset \mathbb{Y}^p_{\bar{\alpha}}$ such that

(i) $\mathcal{W} \subset (\overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \cap \overline{\mathcal{G}}_{\mathbb{R}^m}(R))$,

(ii) for $y \in \mathcal{W}$ it follows that $y + \tilde{\varphi} \in \mathcal{K}$.

Then the operator

$$S(y,c) \;:\; \Big(\mathcal{W} \times \mathcal{G}_{\mathbb{R}^m}(\bar{c};\,\delta) \subset (\mathcal{W} \cap \mathbb{Y}^p_{\bar{\alpha}}) \times \mathbb{R}^m\Big) \to \mathbb{Y}^p_{\bar{\alpha}}$$

has continuous partial derivatives wrt y and c on its domain, and for $y \in \mathcal{W}$, $c \in \mathcal{G}_{\mathbb{R}^m}(\bar{c};\,\delta)$, $h \in \mathbb{Y}^p_{\bar{\alpha}}$ we have that

$$\Big(\frac{\partial S}{\partial y}(y,c)h\Big)(t) = \begin{cases} 0, & t \in [-r,0], \\ \displaystyle\int_0^t \frac{\partial f}{\partial x}\Big(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u, c)\Big)h(u) & \\ \quad + \displaystyle\frac{\partial f}{\partial y}\Big(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u, c)\Big)\Big(\frac{\partial B_\Lambda}{\partial x}(y + \tilde{\varphi}, c)h\Big)(u)\,du, & t \in [0,\bar{\alpha}], \end{cases} \tag{4.105}$$

and for $y \in \mathcal{W}$, $c \in \mathcal{G}_{\mathbb{R}^m}(\bar{c};\,\delta)$, $h \in \mathbb{R}^m$ it follows that

$$\left(\frac{\partial S}{\partial c}(y,c)h\right)(t) = \begin{cases} 0, & t \in [-r,0], \\ \displaystyle\int_0^t \frac{\partial f}{\partial y}\Big(u, y(u)+\tilde{\varphi}(u), \Lambda(u, y_u+\tilde{\varphi}_u, c)\Big)\left(\frac{\partial B_\Lambda}{\partial c}(y+\tilde{\varphi},c)h\right)(u)\,du. & t \in [0,\bar{\alpha}]. \end{cases} \tag{4.106}$$

Proof Relation (4.105) is a restatement of (4.50) using μ^c in the equation. The continuity of $\frac{\partial S}{\partial y}(y,c)$ for a fixed c also follows from Lemma 4.22, but the continuity wrt y and c needs to be proved. The proof goes similarly to that in Lemma 4.22, using the assumed continuity of $\frac{\partial B_\Lambda}{\partial x}(x,c)$, and (A5'), and it is omitted.

To show the second part of the lemma, first note, that the operator $\frac{\partial S}{\partial c}(y,c)$ defined by (4.106) is clearly linear. Fix $y \in \mathcal{W}$, $c \in \Omega_4$. Then for this fixed c consider μ^c, and with it we can define the constant M_4 (which then depends on c), repeating (4.52)–(4.55). Then M_4 satisfies the estimates (4.56) and (4.57) (with using μ^c), and hence it is easy to obtain that

$$\left\|\frac{\partial S}{\partial c}(y,c)\right\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{Y}^p_{\bar{\alpha}})} \le M_4\left\|\frac{\partial B_\Lambda}{\partial c}(y+\tilde{\varphi},c)\right\|_{\mathcal{L}(\mathbb{R}^m,L^p([0,\bar{\alpha}];\mathbb{R}^n))},$$

which gives the boundedness of $\frac{\partial S}{\partial c}(y,c)$.

To show that it is the derivative of $S(y,c)$ wrt c, let $h \in \mathbb{R}^m$, and consider

$$\begin{aligned} &\left|S(y,c+h) - S(y,c) - \frac{\partial S}{\partial c}(y,c)h\right|_{\mathbb{Y}^p_{\bar{\alpha}}} \\ &= \left(\int_0^{\bar{\alpha}}\Big|f(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,c+h)) - f(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,c))\right. \\ &\qquad \left. - \frac{\partial f}{\partial y}\Big(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,c)\Big)\left(\frac{\partial B_\Lambda}{\partial c}(y+\tilde{\varphi},c)h\right)(u)\Big|^p du\right)^{1/p}. \end{aligned} \tag{4.107}$$

Define the function

$$\omega^5(t,x,c;h) \equiv \Lambda(t,x_t,c+h) - \Lambda(t,x_t,c) - \left(\frac{\partial B_\Lambda}{\partial c}(x,c)h\right)(t)$$

for $t \in [0,\bar{\alpha}]$, $x \in \mathcal{K}$, $c \in \Omega_4$ and $h \in \mathbb{R}^m$. Then (A8b') implies that

$$\frac{1}{|h|_{\mathbb{R}^m}}\left(\int_0^{\bar{\alpha}}|\omega^5(t,x,c;h)|^p\,dt\right)^{1/p} \to 0, \qquad \text{as } |h|_{\mathbb{R}^m} \to 0. \tag{4.108}$$

Using this notation and the function ω^1 defined by (4.7), (4.107) and the definition of M_4 yield

$$\begin{aligned} &\left|S(y,c+h) - S(y,c) - \frac{\partial S}{\partial c}(y,c)h\right|_{\mathbb{Y}^p_{\bar{\alpha}}} \\ &= \left(\int_0^{\bar{\alpha}}\Big|\omega^1(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,c);y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,c+h))\Big|^p du\right)^{1/p} \\ &\quad + \left(\int_0^{\bar{\alpha}}\Big|\frac{\partial f}{\partial y}\Big(u,y(u)+\tilde{\varphi}(u),\Lambda(u,y_u+\tilde{\varphi}_u,c)\Big)\omega^5(u,y+\tilde{\varphi},c;h)\Big|^p du\right)^{1/p} \end{aligned}$$

$$\leq \left(\int_0^{\bar{\alpha}} \left| \omega^1(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u, c); y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u, c + h)) \right|^p du \right)^{1/p}$$
$$+ M_4 \left(\int_0^{\bar{\alpha}} \left| \omega^5(u, y + \tilde{\varphi}, c; h) \right|^p du \right)^{1/p},$$

which by (4.108), and by that the first integral divided by $|h|_{\mathbb{R}^m}$ goes to zero, as $|h|_{\mathbb{R}^m} \to 0$ proves that $\frac{\partial S}{\partial c}(y, c)$ is, in fact, the partial derivative of S wrt c. (Here to prove the first fact, we use (4.8), and that by (A5'), $|\Lambda(u, y_u + \tilde{\varphi}_u, c + h) - \Lambda(u, y_u + \tilde{\varphi}_u, c)| \to 0$ as $|h|_{\mathbb{R}^m} \to 0$, and the Lebesgue Dominant Convergence Theorem.)

Finally, we show the continuity of $\frac{\partial S}{\partial c}(y, c)$. Let $y^k \in \mathbb{Y}^p_{\bar{\alpha}}$ and $c^k \in \Omega_4$ such that $|y^k - y|_{\mathbb{Y}^p_{\bar{\alpha}}} \to 0$ and $|c^k - c|_{\mathbb{R}^m} \to 0$ as $k \to \infty$. Let $h \in \mathbb{R}^m$, and consider

$$\left| \frac{\partial S}{\partial c}(y^k, c^k)h - \frac{\partial S}{\partial c}(y, c)h \right|_{\mathbb{Y}^p_{\bar{\alpha}}}$$
$$\leq \left(\int_0^t \left\| \frac{\partial f}{\partial y}\Big(u, y^k(u) + \tilde{\varphi}(u), \Lambda(u, y^k_u + \tilde{\varphi}_u, c^k)\Big) - \frac{\partial f}{\partial y}\Big(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u, c)\Big) \right\|^p \right.$$
$$\left. \cdot \left| \Big(\frac{\partial B_\Lambda}{\partial c}(y + \tilde{\varphi}, c)h \Big)(u) \right|^p du \right)^{1/p}$$
$$+ \left(\int_0^t \left\| \frac{\partial f}{\partial y}\Big(u, y^k(u) + \tilde{\varphi}(u), \Lambda(u, y^k_u + \tilde{\varphi}_u, c^k)\Big) \right\|^p \right.$$
$$\left. \cdot \left| \Big(\frac{\partial B_\Lambda}{\partial c}(y^k + \tilde{\varphi}, c^k)h - \frac{\partial B_\Lambda}{\partial c}(y + \tilde{\varphi}, c)h \Big)(u) \right|^p du \right)^{1/p}$$
$$\leq \sup_{0 \leq u \leq \bar{\alpha}} \left\| \frac{\partial f}{\partial y}\Big(u, y^k(u) + \tilde{\varphi}(u), \Lambda(u, y^k_u + \tilde{\varphi}_u, c^k)\Big) \right.$$
$$\left. - \frac{\partial f}{\partial y}\Big(u, y(u) + \tilde{\varphi}(u), \Lambda(u, y_u + \tilde{\varphi}_u, c)\Big) \right\| \cdot \left\| \frac{\partial B_\Lambda}{\partial c}(y + \tilde{\varphi}, c) \right\|_{\mathcal{L}(\mathbb{R}^m, L^p([0,\bar{\alpha}];\mathbb{R}^n))} |h|_{\mathbb{R}^m}$$
$$+ M_4 \left\| \frac{\partial B_\Lambda}{\partial c}(y^k + \tilde{\varphi}, c^k) - \frac{\partial B_\Lambda}{\partial c}(y + \tilde{\varphi}, c) \right\|_{\mathcal{L}(\mathbb{R}^m, L^p([0,\bar{\alpha}];\mathbb{R}^n))} |h|_{\mathbb{R}^m},$$

which yields the continuity of $\frac{\partial S}{\partial c}$, using the assumed continuity of $\frac{\partial B_\Lambda}{\partial c}$, and the continuity of $\frac{\partial f}{\partial y}$ and Λ, and that $|y^k(u) + \tilde{\varphi}(u) - (y(u) + \tilde{\varphi}(u))| \to 0$ and $|y^k_u + \tilde{\varphi}_u - (y_u + \tilde{\varphi}_u)|_C \to 0$ as $k \to \infty$, which follows from the proof of Lemma 4.22. □

Using Lemmas 4.39 and 4.40, the proof of the following theorem is essentially the same as that of Theorem 4.23, and therefore it is omitted.

Theorem 4.41 *Assume that φ, μ, f and $\bar{c}$ satisfy (A1)–(A7) and (A8b'). Then there exist $\alpha > 0$ and $\delta > 0$ such that IVP (4.1)-(4.2) has a unique solution, $x(t; c)$, on $[0, \alpha]$ corresponding to any parameter $c \in \mathcal{G}_{\mathbb{R}^m}(\bar{c}; \delta)$. Assume that $x(\cdot; \bar{c}) \in \mathcal{K}$, then $x(\cdot; c)$ is continuously differentiable wrt c, as a function*

$$\mathcal{G}_{\mathbb{R}^m}(\bar{c}; \delta) \to \mathbb{X}^p_\alpha, \qquad c \mapsto x(\cdot; c).$$

To conclude this section, we present conditions applying (A8b') in two special examples.

Example 4.42 Consider again Example 1.1, i.e.,

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k), \tag{4.109}$$

where $A_k \in \mathbb{R}^{n\times n}$. Here we can think of the delays, $c = (\tau_1, \tau_2, \ldots, \tau_m)^T$, as a vector parameter. Define

$$\Omega_4 \equiv \Big\{(c_1, \ldots, c_m)^T \in \mathbb{R}^m \ : 0 < c_i < r,\ i = 1, 2, \ldots, m\Big\}.$$

Let $c = (c_1, \ldots, c_m)^T$. As we have seen in Example 1.1, by defining

$$\mu(s, c) \equiv \sum_{k=1}^{m} A_k \chi_{[-c_k, 0]}(s),$$

we can rewrite (4.109) in the form

$$\dot{x}(t) = A_0 x(t) + \int_{-r}^{0} d_s \mu(s, c) x_t.$$

It is easy to see that the function λ corresponding to μ depends only on ξ and c, and has the form

$$\lambda(\xi, c) = \sum_{k=1}^{m} A_k \xi(-c_k). \tag{4.110}$$

Then, clearly, $|\lambda(\xi, c)| \le \sum_{k=1}^{m} \|A_k\|$ is satisfied for $\xi \in \bar{G}_C(1)$ and $c \in \Omega_4$, therefore (A2') is satisfied. The equivalence of norms on a finite dimensional space implies that there exist constants $P_1, P_2 > 0$ (depending on the norm $|\cdot|_{\mathbb{R}^m}$) such that

$$P_1 |c|_{\mathbb{R}^m} \le \sum_{k=1}^{m} |c_k| \le P_2 |c|_{\mathbb{R}^m} \tag{4.111}$$

for all $c = (c_1, \ldots, c_m)^T \in \mathbb{R}^m$. Let $\xi \in W^{1,\infty}$, $c = (c_1, \ldots, c_m)^T, \bar{c} = (\bar{c}_1, \ldots, \bar{c}_m)^T \in \Omega_4$. Then it follows from (4.110) using the Mean Value Theorem and (4.111) that

$$\begin{aligned} |\lambda(\xi, c) - \lambda(\xi, \bar{c})| &\le \sum_{k=1}^{m} \|A_k\| |\xi(-c_k) - \xi(-\bar{c}_k)| \\ &\le |\xi|_{1,\infty} \max_{k=1,\ldots,m} \|A_k\| \sum_{k=1}^{m} |c_k - \bar{c}_k| \\ &\le |\xi|_{1,\infty} P_2 \max_{k=1,\ldots,m} \|A_k\| |c - \bar{c}|_{\mathbb{R}^m}. \end{aligned}$$

This shows that (A5') is satisfied, moreover, the constant $L_2 = P_2 \max_{k=1,\ldots,m} \|A_k\|$ is global in c. The composition operator corresponding to (4.110) has the form:

$$B_\Lambda \ : \ \mathbb{X}^p_\alpha \times \Omega_4 \to L^p([0, \alpha]; \mathbb{R}^n), \qquad B_\Lambda(x, c)(t) = \sum_{k=1}^{m} A_k x(t - c_k), \qquad t \in [0, \alpha].$$

Then we show that (A8b') is satisfied with $\mathcal{K} = W^{1,\infty}_\alpha$, and for $x \in \mathbb{X}^p_\alpha$ and $h \in \mathbb{X}^p_\alpha$ we have that

$$\Big(\frac{\partial B_\Lambda}{\partial x}(x,c)h\Big)(t) = \sum_{k=1}^m A_k h(t-c_k), \quad t \in [0,\alpha], \tag{4.112}$$

and for $h = (h_1, h_2, \ldots, h_m)^T \in \mathbb{R}^m$ it follows that

$$\Big(\frac{\partial B_\Lambda}{\partial c}(x,c)h\Big)(t) = -\sum_{k=1}^m A_k \dot{x}(t-c_k)h_k, \quad t \in [0,\alpha]. \tag{4.113}$$

To prove (4.112), first note, that the formula follows trivially from the linearity of $B_\Lambda(x,c)$ in x, we need to show that this is a bounded operator, and then that $\frac{\partial B_\Lambda}{\partial x}(x,c)$ is continuous in x and c. The boundedness follows from

$$\Big|\frac{\partial B_\Lambda}{\partial x}(x,c)h\Big|_{L^p([0,\alpha];\mathbb{R}^n)} \le C_1|h|_{\mathbb{X}^p_\alpha}\sum_{k=1}^m \|A_k\|,$$

which is easy to obtain, using Lemma 4.18 (i) with constant C_1. For the continuity first note that $\frac{\partial B_\Lambda}{\partial x}(x,c)$ is independent of x. Let $|c^i - c|_{\mathbb{R}^m} \to 0$ as $i \to \infty$, $c^i = (c^i_1, \ldots, c^i_m)^T$, and consider

$$\begin{aligned}
&\Big|\frac{\partial B_\Lambda}{\partial x}(x,c^i)h - \frac{\partial B_\Lambda}{\partial x}(x,c)h\Big|_{L^p([0,\alpha];\mathbb{R}^n)}\\
&\quad\le \sum_{k=1}^m \|A_k\| \Big(\int_0^\alpha |h(t-c^i_k) - h(t-c_k)|^p dt\Big)^{1/p}\\
&\quad= \sum_{k=1}^m \|A_k\| \Big(\int_0^\alpha \Big|\int_0^1 \dot{h}(t-c_k+u(c_k-c^i_k))\,du\Big|^p dt\Big)^{1/p} |c^i_k - c_k|\\
&\quad\le \sum_{k=1}^m \|A_k\||\dot{h}|_{L^p_\alpha}|c^i_k - c_k|.
\end{aligned}$$

Then Lemma 4.18 (iv) with constant C_4 and (4.111) imply that

$$\begin{aligned}
\Big|\frac{\partial B_\Lambda}{\partial x}(x,c^i)h - \frac{\partial B_\Lambda}{\partial x}(x,c)h\Big|_{L^p([0,\alpha];\mathbb{R}^n)} &\le \max_{k=1,\ldots,m}\|A_k\|C_4|h|_{\mathbb{X}^p_\alpha}\sum_{k=1}^m |c^i_k - c_k|\\
&\le \max_{k=1,\ldots,m}\|A_k\|C_4P_2|h|_{\mathbb{X}^p_\alpha}|c^i - c|_{\mathbb{R}^m},
\end{aligned}$$

which proves the continuity of $\frac{\partial B_\Lambda}{\partial x}$.

Next we show (4.113). The boundedness of $\frac{\partial B_\Lambda}{\partial c}$ follows from

$$\begin{aligned}
\Big|\frac{\partial B_\Lambda}{\partial c}(x,c)h\Big|_{L^p([0,\alpha];\mathbb{R}^n)} &\le \sum_{k=1}^m \|A_k\||x|_{W^{1,\infty}_\alpha}|h_k|\\
&\le \max_{k=1,\ldots,m}\|A_k\||x|_{W^{1,\infty}_\alpha}P_2|h|_{\mathbb{R}^m}.
\end{aligned}$$

To show that this is the derivative, consider

$$\begin{aligned}
&\left| B_\Lambda(x,c+h) - B_\Lambda(x,c) - \frac{\partial B_\Lambda}{\partial c}(x,c)h \right|_{L^p([0,\alpha];\mathbb{R}^n)} \\
&\quad = \left(\int_0^\alpha \left| \sum_{k=1}^m A_k\Big(x(t-c_k-h_k) - x(t-c_k) + \dot{x}(t-c_k)h_k \Big) \right|^p dt \right)^{1/p} \\
&\quad \le \sum_{k=1}^m \|A_k| \left(\int_0^\alpha \Big| x(t-c_k-h_k) - x(t-c_k) + \dot{x}(t-c_k)h_k \Big|^p dt \right)^{1/p},
\end{aligned}$$

which implies (4.113), since for a.e. $t \in [0,\alpha]$

$$\frac{1}{|h|_{\mathbb{R}^m}} \Big| x(t-c_k-h_k) - x(t-c_k) + \dot{x}(t-c_k)h_k \Big| \to 0, \qquad \text{as } |h|_{\mathbb{R}^m} \to 0,$$

and therefore the Lebesgue Dominant Convergence Theorem implies the statement. To show continuity of $\frac{\partial B_\Lambda}{\partial c}(x,c)$, let $|x^i - x|_{\mathbb{X}^p_\alpha} \to 0$ and $|c^i - c|_{\mathbb{R}^m} \to 0$, then by applying similar estimates that we used above we get

$$\begin{aligned}
&\left| \frac{\partial B_\Lambda}{\partial c}(x^i,c^i)h - \frac{\partial B_\Lambda}{\partial c}(x,c)h \right|_{L^p([0,\alpha];\mathbb{R}^n)} \\
&\quad \le \sum_{k=1}^m \|A_k\| \Big(|\dot{x}^i - \dot{x}|_{L^p_\alpha} + |\dot{x}|_{L^p_\alpha} |c^i_k - c_k| \Big) |h_k| \\
&\quad \le \max_{k=1,\ldots,m} \|A_k\| \Big(C_4 |x^i - x|_{\mathbb{X}^p_\alpha} + |\dot{x}|_{L^p_\alpha} \max_{k=1,\ldots,m} |c^i_k - c_k| \Big) |h|_{\mathbb{R}^m},
\end{aligned}$$

which proves the continuity of $\frac{\partial B_\Lambda}{\partial c}(x,c)$.

Example 4.43 Consider the delay function of Example 4.35:

$$\lambda(t,\psi,\xi,c) = \xi(-\tau(t,\psi,c)).$$

Assume that τ satisfies condition (i) and (ii) of Example 4.35. Then (A2') and (A5') are satisfied. The composite function B_Λ of this example is

$$B_\Lambda(x,c)(t) = x(t - \tau(t,x_t,c)).$$

If we assume that

(iii) $\tau(t,\psi,c) : \Big([0,\alpha] \times \Omega_3 \times \Omega_4 \subset [0,\alpha] \times C \times \mathbb{R}^m \Big) \to \mathbb{R}$ is continuously differentiable wrt t, ψ and c,

(iv) $\frac{\partial \tau}{\partial t}(t,\psi,c)$, $\frac{\partial \tau}{\partial \psi}(t,\psi,c)$ and $\frac{\partial \tau}{\partial c}(t,\psi,c)$ are locally Lipschitz-continuous in ψ and c, i.e., for every $M_1 > 0$, $M_2 > 0$ there exists $L^*_\tau = L^*_\tau(\alpha, M_1, M_2)$ such that for all $t \in [0,\alpha]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M_1) \cap \Omega_3$ and $c, \bar{c} \in \overline{\mathcal{G}}_{\mathbb{R}^m}(M_2) \cap \Omega_4$ it follows that

$$\left| \frac{\partial \tau}{\partial t}(t,\psi,c) - \frac{\partial \tau}{\partial t}(t,\bar{\psi},\bar{c}) \right| \le L^*_\tau \Big(|\psi - \bar{\psi}|_C + |c - \bar{c}|_{\mathbb{R}^m} \Big),$$

$$\left\| \frac{\partial \tau}{\partial \psi}(t,\psi,c) - \frac{\partial \tau}{\partial \psi}(t,\bar{\psi},\bar{c}) \right\|_{\mathcal{L}(C,\mathbb{R})} \le L^*_\tau \Big(|\psi - \bar{\psi}|_C + |c - \bar{c}|_{\mathbb{R}^m} \Big).$$

and

$$\left\| \frac{\partial \tau}{\partial c}(t,\psi,c) - \frac{\partial \tau}{\partial c}(t,\bar{\psi},\bar{c}) \right\|_{\mathbb{R}^{1\times m}} \le L^*_\tau \Big(|\psi - \bar{\psi}|_C + |c - \bar{c}|_{\mathbb{R}^m} \Big).$$

and assuming that

$$\frac{d}{dt}\Big(t-\tau(t,x_t,c)\Big)\geq\varepsilon,\qquad \text{a.e. } t\in[0,\alpha],$$

for some $x\in W^{1,\infty}_{\alpha}$, $c\in\Omega_4$ and $\varepsilon>0$, then by repeating the proofs of Example 4.25 we can show that (A8b') is satisfied in a neighborhood of (x,c). The details are omitted.

4.3 Differentiability wrt a parameter in the equation

In this section we study differentiability of solutions of IVP

$$\dot{x}(t;d) = f\Big(t,\, x(u;d),\, \Lambda(t,x(\cdot;d)_t),d\Big),\quad t\in[0,T] \tag{4.114}$$

$$x(t;d) = \varphi(t),\qquad t\in[-r,0] \tag{4.115}$$

wrt the parameter d of the equation. We assume that $d\in\mathbb{R}^m$, i.e., $f\ :\ [0,T]\times\Omega_1\times\Omega_2\times\Omega_5\to\mathbb{R}^n$, where Ω_5 is an open subset of $\mathbb{R}^m$. In this section the initial function, f and μ are fixed, and only the parameter d of the equation varies, and therefore, to emphasize the dependence of the solution on d, we use the notations $x(t;d)$ and $x(\cdot;d)_t$ for the value of the solution and for the solution segment function at t, respectively, corresponding to parameter d.

In this section we replace assumptions (A1), (A4) and (A7) by the following ones, respectively.

(A1') the function $f\ :\ [0,T]\times\Omega_1\times\Omega_2\times\Omega_5\to\mathbb{R}^n$ is bounded and continuous on its domain,

(A4') for each d the function $f(t,x,y,d)$ is locally Lipschitz-continuous in its second and third variables, i.e., for every $d\in\Omega_5$, $\alpha>0$, $M>0$ there exists a constant $L_1=L_1(d,\alpha,M)$ such that for all $t\in[0,\alpha]$, $x,\bar{x}\in\bar{G}_{\mathbb{R}^n}(M)\cap\Omega_1$ and $y,\bar{y}\in\bar{G}_{\mathbb{R}^n}(M)\cap\Omega_2$

$$|f(t,x,y,d)-f(t,\bar{x},\bar{y},d)|\leq L_1\Big(|x-\bar{x}|+|y-\bar{y}|\Big),$$

(A7') (i) The function $f(t,x,y,d)\ :\ [0,T]\times\Omega_1\times\Omega_2\times\Omega_5\to\mathbb{R}^n$ is continuously differentiable wrt x, y and d, and

(ii) $\frac{\partial f}{\partial d}(t,x,y,d)$ is bounded on $[0,T]\times\Omega_1\times\Omega_2\times\Omega_5$.

By (A1'), the following definition is meaningful.

$$\|f\|\equiv\sup\Big\{|f(t,x,y,d)|\ :\ t\in[0,T],\ x\in\Omega_1,\ y\in\Omega_2\quad\text{and}\quad d\in\Omega_5\Big\}. \tag{4.116}$$

For given $d\in\Omega_5$ define the function

$$f^d(t,x,y)\equiv f(t,x,y,d).$$

Then by (A1') and (A4') the function f^d satisfies (A1) and (A4), and then by applying Theorems 3.8 and 3.19, we get that IVP (3.1)-(3.2) corresponding to the function f^d has a unique solution, and consequently, IVP (4.114)-(4.115) has a unique solution on an interval $[0,\alpha]$ for the fixed parameter value d.

By assumption (A7'), the constant

$$N \equiv \sup\left\{ \left\| \frac{\partial f}{\partial d}(t,x,y,d) \right\|_{\mathbb{R}^{n\times m}} \ : \ t \in [0,T], x \in \Omega_1, y \in \Omega_2, d \in \Omega_5 \right\} \tag{4.117}$$

is well-defined and satisfies

$$\|f^d - f^{\bar d}\| \le N|d - \bar d|_{\mathbb{R}^m}, \qquad \text{for } d, \bar d \in \Omega_5, \tag{4.118}$$

therefore Theorems 3.20 and 3.25 imply the following result.

Theorem 4.44 *Assume (A1'), (A2), (A3), (A4'), (A5)-(A6) and (A7'), and let* $1 \le p \le \infty$. *For a given* $\bar d \in \Omega_5$ *there exist constants* $\alpha > 0$, $\delta > 0$ *and* $L_3 = L_3(p, \alpha, \bar d, \delta)$, *such that IVP (4.114)-(4.115) has a unique solution on* $[0,\alpha]$ *for all* $d \in \Omega_5$ *with* $|d - \bar d|_{\mathbb{R}^m} < \delta$, *and*

$$|x(\cdot; d)_t - x(\cdot; \bar d)_t|_{W^{1,p}} \le L_3 |d - \bar d|_{\mathbb{R}^m}, \quad t \in [0, \alpha].$$

We comment, that condition (A7') (ii) is assumed only for simplicity of the discussion. (Theorem 4.44 could be proved without this assumption.) Note also that, of course, (A7') implies (A4').

4.3.1 Special case, differentiability in $W^{1,\infty}$

In this subsection we study the special case corresponding to that in Section 4.1.1, i.e., we shall assume that either the equation is state-independent, (i.e., $\mu(s,t,\psi)$, or equivalently, $\lambda(t,\psi,\xi)$ is independent of ψ) or in the state-dependent case the initial function $\varphi \in C^1$ such that $\dot\varphi(0-) = f(0, \varphi(0), \Lambda(0,\varphi), d)$, and we assume that $f(0,x,y,d)$ is independent of d for $x \in \Omega_1$, $y \in \Omega_2$. As in Section 4.1.1, we shall use assumption (A8a) on the delay function to obtain our results.

For $d \in \Omega_5$, $h \in \mathbb{R}^m$ let $z(\cdot; h)$ be the solution of the linear IVP

$$\begin{aligned} \dot z(t;h) &= \frac{\partial f}{\partial x}\Big(t, x(t;d), \Lambda(t, x(\cdot;d)_t), d\Big) z(t;h) \\ &\quad + \frac{\partial f}{\partial y}\Big(t, x(t;d), \Lambda(t, x(\cdot;d)_t), d\Big) \frac{\partial \Lambda}{\partial \psi}(t, x(\cdot;d)_t) z(\cdot;h)_t \\ &\quad + \frac{\partial f}{\partial d}\Big(t, x(t;d), \Lambda(t, x(\cdot;d)_t), d\Big) h, \qquad t \in [0,T], \qquad (4.119) \\ z(t;h) &= 0, \qquad t \in [-r, 0]. \qquad (4.120) \end{aligned}$$

We comment, that the solution, $z(\cdot;h)$ of this IVP exists, and depends linearly on h.

Next we state the results corresponding to Theorems 4.8 and 4.11. The proofs are omitted, since they are analogous to those in Section 4.1.1.

Theorem 4.45 *Let* φ, d, μ *and* f *be fixed satisfying (A1'), (A2), (A3), (A4'), (A5), (A6), (A7') and (A8a), and assume that either*

(1) the equation is state-independent, i.e., $\lambda(t,\psi,\xi)$ *is independent of* ψ,

or

(2) *in the state-dependent case*

(a) $f(0,x,y,d)$ *is independent of* d *for* $x \in \Omega_1$, $y \in \Omega_2$, *and*

(b) $\varphi \in C^1$ *and* $\dot{\varphi}(0-) = f(0,\varphi(0),\Lambda(0,\varphi),d)$.

Then

(i) *the solution,* $x(t;d)$*, of IVP (4.114)-(4.115) is differentiable wrt* d *for all* $t \in [0,\alpha]$ *and* $d \in \Omega_5$,

(ii) $\dfrac{x(t;d+h)-x(t;d)}{|h|_{\mathbb{R}^m}}$ *converges uniformly to* $\frac{\partial x}{\partial d}(t;d)$ *on* $t \in [0,\alpha]$,

(iii) *the derivative is* $\frac{\partial x}{\partial d}(t;d)h = z(t;h)$*, where* $z(t;h)$ *is the solution of the linear IVP (4.119)-(4.120).*

Corollary 4.46 *Assuming the conditions of Theorem 4.45, the function*

$$\Big(\Omega_5 \subset \mathbb{R}^m\Big) \to C, \quad d \mapsto x(\cdot,d)_t$$

is differentiable for all $t \in [0,\alpha]$.

Theorem 4.47 *Assume that the conditions of Theorem 4.45 are satisfied. Then the function* $\Big(\Omega_5 \subset \mathbb{R}^m\Big) \to W^{1,\infty}$, $d \mapsto x(\cdot;d)_t$ *is differentiable for all* $t \in [0,\alpha]$.

4.3.2 General case, differentiability in $W^{1,p}$

In this subsection we show that assumptions (A1)–(A7') and (A8b) imply that the solution is differentiable wrt d in the state-space $W^{1,p}$. As in Sections 4.1.3 and 4.2.2, we transform (4.114)-(4.115) into

$$y(t) = \begin{cases} 0, & t \in [-r,0] \\ \displaystyle\int_0^t f\Big(u,\ y(u)+\tilde{\varphi}(u),\ \Lambda(u, y_u+\tilde{\varphi}_u), d\Big)\, du, & t \in [0,T], \end{cases} \tag{4.121}$$

and introduce the operator

$$S(y,d)(t) = \begin{cases} 0, & t \in [-r,0] \\ \displaystyle\int_0^t f\Big(u,\ y(u)+\tilde{\varphi}(u),\ \Lambda(u, y_u+\tilde{\varphi}_u), d\Big)\, du, & t \in [0,T], \end{cases} \tag{4.122}$$

where we consider S as

$$S \ : \ \overline{\mathcal{G}}_{\mathbb{Y}^p_{\bar{\alpha}}}(\bar{\beta}) \times \overline{\mathcal{G}}_{\mathbb{R}^m}(\bar{d};\, \delta) \to \mathbb{Y}^p_{\bar{\alpha}}$$

where $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\delta > 0$ is specified by the next lemma.

Lemma 4.48 *Let $1 \le p < \infty$, $\bar d \in \Omega_5$ and $R > 0$. Then there exist $\delta > 0$, $\bar\alpha > 0$ and $\bar\beta > 0$ such that $\mathcal{G}_{\mathbb{R}^m}(\bar d;\, \delta) \subset \Omega_5$, and the operator S defined by (4.122) satisfies*

(i) $S \;:\; \overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta) \times \mathcal{G}_{\mathbb{R}^m}(\bar d;\, \delta) \to \overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta)$,

(ii) S is a uniform contraction on $\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta) \cap \overline{\mathcal{G}}_{W^{1,\infty}_{\bar\alpha}}(R)$ both in $|\cdot|_{\Upsilon^p_{\bar\alpha}}$ and $|\cdot|_{\Upsilon^\infty_{\bar\alpha}}$ norms, i.e., there exists $0 \le \theta < 1$ such that for all $d \in \mathcal{G}_{\mathbb{R}^m}(\bar d;\, \delta)$, $y, \bar y \in \overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta) \cap \overline{\mathcal{G}}_{W^{1,\infty}_{\bar\alpha}}(R)$

$$|S(y,d) - S(\bar y, d)|_{\Upsilon^\infty_{\bar\alpha}} \le \theta |y - \bar y|_{\Upsilon^\infty_{\bar\alpha}},$$

and

$$|S(y,d) - S(\bar y, d)|_{\Upsilon^p_{\bar\alpha}} \le \theta |y - \bar y|_{\Upsilon^p_{\bar\alpha}}.$$

The proof is an obvious modification that of Lemma 4.19.

The next lemma guarantees continuous differentiability of $S(y,d)$ wrt y and d.

Lemma 4.49 *Let $\bar d \in \Omega_5$, $1 \le p < \infty$ be fixed, and $R > 0$ given, and assume (A1'), (A2), (A3), (A4'), (A5), (A6), (A7') and (A8b). Let δ, $\bar\alpha$, $\bar\beta$ be the constants from Lemma 4.48, i.e., such that the operator S defined by (4.122) satisfies*

$$S \;:\; \overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta) \times \mathcal{G}_{\mathbb{R}^m}(\bar d;\, \delta) \to \overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta),$$

and it is a uniform contraction on $\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta) \cap \overline{\mathcal{G}}_{\mathbb{R}^m}(R)$. Assume that there exists $\mathcal{W} \subset \Upsilon^p_{\bar\alpha}$ such that

(i) $\mathcal{W} \subset (\overline{\mathcal{G}}_{\Upsilon^p_{\bar\alpha}}(\bar\beta) \cap \overline{\mathcal{G}}_{\mathbb{R}^m}(R))$,

(ii) for $y \in \mathcal{W}$ it follows that $y + \tilde\varphi \in \mathcal{K}$.

Then the operator

$$S(y,d) \;:\; \Big(\mathcal{W} \times \mathcal{G}_{\mathbb{R}^m}(\bar d;\, \delta) \subset (\mathcal{W} \cap \Upsilon^p_{\bar\alpha}) \times \mathbb{R}^m\Big) \to \Upsilon^p_{\bar\alpha}$$

has continuous partial derivatives wrt y and d on its domain, and for $y \in \mathcal{W}$, $d \in \mathcal{G}_{\mathbb{R}^m}(\bar d;\, \delta)$, $h \in \Upsilon^p_{\bar\alpha}$ we have that

$$\Big(\frac{\partial S}{\partial y}(y,d)h\Big)(t) = \begin{cases} 0, & t \in [-r,0], \\ \displaystyle\int_0^t \frac{\partial f}{\partial x}\Big(u, y(u) + \tilde\varphi(u), \Lambda(u, y_u + \tilde\varphi_u), d\Big)h(u) & \\ \quad + \displaystyle\frac{\partial f}{\partial y}\Big(u, y(u) + \tilde\varphi(u), \Lambda(u, y_u + \tilde\varphi_u), d\Big)\Big(\frac{\partial B_\Lambda}{\partial x}(y + \tilde\varphi)h\Big)(u)\,du, & t \in [0,\bar\alpha], \end{cases} \tag{4.123}$$

and for $y \in \mathcal{W}$, $d \in \mathcal{G}_{\mathbb{R}^m}(\bar d;\, \delta)$, $h \in \mathbb{R}^m$ it follows that

$$\Big(\frac{\partial S}{\partial d}(y,d)h\Big)(t) = \begin{cases} 0, & t \in [-r,0], \\ \displaystyle\int_0^t \frac{\partial f}{\partial d}\Big(u, y(u) + \tilde\varphi(u), \Lambda(u, y_u + \tilde\varphi_u), d\Big)h\,du. & t \in [0,\bar\alpha]. \end{cases} \tag{4.124}$$

Proof Relation (4.123) follows from (4.50) by applying Lemma 4.22 for f^d, only the continuity of $\frac{\partial S}{\partial d}(y,d)$ wrt y and d requires a proof, which is omitted, since it follows like that in Lemma 4.22.

Let $h \in \mathbb{R}^m$, $y \in \mathcal{W}$, $d \in \Omega_5$. Assumption (A7') (ii), and (4.117) yield that

$$\left|\frac{\partial S}{\partial d}(y,d)h\right|_{\mathbb{Y}^p_{\bar\alpha}} \le N\bar\alpha^{1/p}|h|_{\mathbb{R}^m},$$

which proves the boundedness of $\frac{\partial S}{\partial d}(y,d)$. Consider

$$\begin{aligned}
&\frac{1}{|h|_{\mathbb{R}^m}}\left|S(y,d+h)-S(y,d)-\frac{\partial S}{\partial d}(y,d)h\right|_{\mathbb{Y}^p_{\bar\alpha}}\\
&= \frac{1}{|h|_{\mathbb{R}^m}}\bigg(\int_0^{\bar\alpha}\Big|f(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u),d+h)\\
&\qquad - f(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u),d)\\
&\qquad - \frac{\partial f}{\partial d}\Big(u,y(u)+\tilde\varphi(u),\Lambda(u,y_u+\tilde\varphi_u),d\Big)h\Big|^p\,du\bigg)^{1/p},
\end{aligned}$$

from which (4.124) follows, using differentiability of f wrt d, and the Lebesgue Dominant Convergence Theorem. It is easy to see the continuity of $\frac{\partial S}{\partial d}(y,d)$. □

The following theorem is based on Lemmas 4.48 and 4.49 and Theorem 4.14. The proof is omitted, since it is the same as that of Theorem 4.23.

Theorem 4.50 *Assume that φ, μ, f and $\bar c$ satisfy (A1'), (A2), (A3), (A4'), (A5), (A6), (A7') and (A8b). Then there exist $\alpha > 0$, $\delta > 0$ such that IVP (4.114)-(4.115) has unique solution, $x(t;d)$, on $[0,\alpha]$ corresponding to any parameter $d \in \mathcal{G}_{\mathbb{R}^m}(\bar d;\,\delta)$. Assume that $x(\cdot;\bar d) \in \mathcal{K}$, then $x(t;d)$ is continuously differentiable wrt d, as a function*

$$\mathcal{G}_{\mathbb{R}^m}(\bar d;\,\delta) \to \mathbb{X}^p_\alpha, \qquad d \mapsto x(\cdot;d).$$

Chapter 5
STABILITY BY LINEARIZATION

Stability properties of solutions of a modeling differential equation are of great importance in applications. For linear delay equations the stability of the trivial ($x(t) = 0$) solution is characterized by the location of the zeros of its characteristic equation. Necessary and sufficient conditions for stability in terms of the parameters (coefficients, delays) of the equation are known only for the simplest equations, even in the case of linear constant delay equations. There are numerous sufficient conditions for guaranteeing stability for special equations (see e.g. [31]). One possible approach to find sufficient stability conditions is, analogously to the ODEs case, by Liapunov's method. But, unfortunately, there is no general strategy to construct a Liapunov functional for a given equation, and if the equation is complicated (nonlinear, with several time- or state-dependent delays), obtaining a Liapunov functional can be very difficult if not impossible.

For nonlinear autonomous ODEs the linearization method is a very useful one, since we can deduce stability properties of the solution of the nonlinear equation from that of the corresponding linear equation, which is significantly easier to check. Recently, Cooke and Huang ([13]) introduced this method for nonlinear delay equations with state-dependent delays of the form (3.64). Since this technique is a very powerful tool to discuss local stability properties of a nonlinear delay equation, in this chapter we shall obtain a similar linearization test for the autonomous version of our equation, (3.1). Note, that despite the significant technical differences between our presentation and that of [13] due to the different form of the two equations, the main ideas are of course the same, since both follow the steps of the proof of the ODEs case (see e.g. [39]), and the two results are equivalent in the sence that they both provide the same linear equation for nonlinear equations which can be rewritten in both forms. Example 5.8 will show an equation, which is not included in (3.64), but is covered by (3.1), and of course, examples can be constructed for the opposite direction as well.

We note, that the main difficulty to obtain linearization results for state-dependent delay equations is that it is difficult to differentiate the delayed term in the presence of state-dependent delays (see in Chapter 4). We shall define a bounded linear operator, $\mathcal{F} : C \to \mathbb{R}^n$ (see (5.6) below), as a candidate for the linearized equation about the trivial solution. This is not the "true" linearization at zero, since the delayed term is not necessarily differentiable at zero (in the space C), but using assumption (A5), we can get an estimate on the error replacing the right hand side of the equation by $\mathcal{F}x_t$ (see Lemma 5.2 below), which turns out to be sufficient to prove that the asymptotic stability of the corresponding linearized equation, (5.9), implies that of the nonlinear equation, (5.1).

Section 5.1 contains the main results, and in Section 5.2 we illustrate the method on several examples with constant, time- and state-dependent delays.

5.1 Main results

Consider the autonomous version of (3.1)

$$\dot{x}(t) = f\Big(x(t), \Lambda(x_t)\Big), \qquad t \geq 0, \tag{5.1}$$

with corresponding initial condition

$$x(t) = \varphi(t), \qquad -r \leq t \leq 0. \tag{5.2}$$

In this section we use the notations

$$\lambda(\psi, \xi) \equiv \int_{-r}^{0} d_s\mu(s, \psi)\xi(s), \qquad \psi \in \Omega_3,\ \xi \in C,$$

and

$$\Lambda(\psi) \equiv \lambda(\psi, \psi), \qquad \psi \in \Omega_3,$$

i.e., we use the notations of the previous chapters but omitting t from the arguments. We assume hypotheses (A1)–(A7) (with the understanding that t is missing from the arguments of f and μ), and we also assume that

(H) $0 \in \Omega_1 \cap \Omega_2$, and $f(0, 0) = 0$,

i.e., $x^* = 0$ is an equilibrium point of equation (5.1). Note, that by Theorems 3.8 and 3.14, IVP (5.1)–(5.2) has a unique solution on $[-r, \alpha]$ for some $\alpha > 0$.

First we introduce constants which we shall use throughout this section.

It follows from the assumption that Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$ and $0 \in \Omega_1 \cap \Omega_2$ that there exists a constant $\delta_1 > 0$ such that $\overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1) \subset \Omega_1 \cap \Omega_2$. Assumption (A4) (or (A7)) implies that there exists a constant $L_1 = L_1(\delta_1)$ such that

$$|f(x, y) - f(\bar{x}, \bar{y})| \leq L_1(|x - \bar{x}| + |y - \bar{y}|), \qquad \text{for } x, \bar{x}, y, \bar{y} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1). \tag{5.3}$$

Inequality (2.5) and $|x(t)| \leq |x_t|_C$ yield that

$$x(t) \in \overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1) \quad \text{and} \quad \Lambda(x_t) \in \overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1) \qquad \text{for } x_t \in \overline{\mathcal{G}}_C(\delta_2), \tag{5.4}$$

where $\delta_2 \equiv \delta_1 \min\{1, 1/\|\mu\|\}$.

We shall need the following estimate.

Lemma 5.1 *Assume (A1)–(A7) and (H). Let x be the solution of (5.1)-(5.2) corresponding to initial function φ satisfying $|\varphi|_C \leq \delta_2$. Assume that $\alpha > 0$ is such that $|x_t| \leq \delta_2$ for $0 \leq t \leq \alpha$. Then the solution x satisfies the inequality*

$$|x_t| \leq |\varphi|_C \exp\Big(L_1(1 + \|\mu\|)t\Big), \qquad t \in [0, \alpha].$$

Proof Let $\alpha > 0$ satisfy the condition of the lemma, and let $t \in [0, \alpha]$. The integrated form of (5.1), and relations (5.3), (5.4) and (H) yield the following estimates.

$$
\begin{aligned}
|x(t)| &\leq |\varphi(0)| + \int_0^t |f(x(u), \Lambda(x_u))|\, du \\
&\leq |\varphi|_C + L_1 \int_0^t |x(u)| + |\Lambda(x_u)|\, du \\
&\leq |\varphi|_C + L_1 \int_0^t |x(u)| + \|\mu\| |x_u|_C\, du.
\end{aligned}
\tag{5.5}
$$

Lemma 2.14, the assumption $|\varphi|_C \leq \delta_2$ and (5.5) imply that

$$
\max_{-r \leq v \leq t} |x(v)| \leq |\varphi|_C + L_1(1 + \|\mu\|) \int_0^t \max_{-r \leq v \leq u} |x(v)|\, du, \qquad t \in [0, \alpha],
$$

which, using Gronwall-Bellman inequality, yields the statement of the lemma. □

Define the linear operator

$$
\mathcal{F} \,:\, C \to \mathbb{R}^n, \qquad \mathcal{F}\psi \equiv \frac{\partial f}{\partial x}(0,0)\psi(0) + \frac{\partial f}{\partial y}(0,0)\lambda(0, \psi) \tag{5.6}
$$

and the function

$$
G \,:\, C \to \mathbb{R}^n, \qquad G(\psi) \equiv f(\psi(0), \Lambda(\psi)) - \mathcal{F}\psi. \tag{5.7}
$$

Note, that $\mathcal{F}$ is a bounded operator, since by (2.5) it follows that

$$
|\mathcal{F}\psi| \leq \left(\left\| \frac{\partial f}{\partial x}(0,0) \right\| + \left\| \frac{\partial f}{\partial y}(0,0) \right\| \|\mu\| \right) |\psi|_C.
$$

By this notation we can rewrite (5.1) as

$$
\dot{x}(t) = \mathcal{F}x_t + G(x_t), \qquad t \geq 0, \tag{5.8}
$$

and therefore we can consider it as a perturbation of the constant delay equation

$$
\dot{x}(t) = \mathcal{F}x_t, \qquad t \geq 0 \tag{5.9}
$$

by the function G.

We shall need the following estimate of G.

Lemma 5.2 *Assume (A1)–(A7) and (H). There exists a constant $N > 0$ such that for every $\eta > 0$ there exists a constant $\theta = \theta(\eta) > 0$ such that*

$$
|G(\psi)| \leq N\Big(\eta + |\psi|_{W^{1,\infty}}\Big)|\psi|_C \tag{5.10}
$$

for all $\psi \in W^{1,\infty}$ such that $|\psi|_C \leq \theta$.

Proof The definition of $\mathcal{F}$, (A7), (H), Lemmas 2.16 and 2.17 imply

$$\begin{aligned}
|G(\psi)| &\leq \left| f\Big(\psi(0), \Lambda(\psi)\Big) - \frac{\partial f}{\partial x}(0,0)\psi(0) - \frac{\partial f}{\partial y}(0,0)\lambda(0,\psi)\right| \\
&= \left| f\Big(\psi(0), \Lambda(\psi)\Big) - f(0,0) - \frac{\partial f}{\partial x}(0,0)\psi(0) - \frac{\partial f}{\partial y}(0,0)\lambda(0,\psi)\right| \\
&\leq \sup_{0\leq\nu\leq 1} \left\| \frac{\partial f}{\partial x}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial x}(0,0)\right\| |\psi(0)| + \left\|\frac{\partial f}{\partial y}(0,0)\right\| |\Lambda(\psi) - \lambda(0,\psi)| \\
&\quad + \sup_{0\leq\nu\leq 1} \left\| \frac{\partial f}{\partial y}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial x}(0,0)\right\| |\Lambda(\psi)|. \qquad (5.11)
\end{aligned}$$

By the continuous differentiability of f guaranteed by (A7), for every $\eta > 0$ there exists $0 < \theta_1(\eta) \leq \delta_1$ such that if $|x|, |y| < \theta_1(\eta)$ then

$$\left\|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right\| < \eta \quad \text{and} \quad \left\|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(0,0)\right\| < \eta$$

It follows from (2.5), $\theta_1(\eta) \leq \delta_1$ and the definition of δ_2 that the constant $\theta = \theta(\eta) \equiv \theta_1(\eta)\min\{1, 1/\|\mu\|\}$ satisfies $\theta \leq \delta_2$, and if $\psi \in \overline{\mathcal{G}}_C(\theta)$ then

$$\left\|\frac{\partial f}{\partial x}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial x}(0,0)\right\| < \eta \quad \text{and} \quad \left\|\frac{\partial f}{\partial y}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial y}(0,0)\right\| < \eta \qquad (5.12)$$

for all $0 \leq \nu \leq 1$. It follows from assumption (A5) with $L_2 = L_2(\delta_1)$, $\theta \leq \delta_2$ and (5.4), that for $\psi \in \overline{\mathcal{G}}_C(\theta) \cap W^{1,\infty}$

$$\begin{aligned}
|\Lambda(\psi) - \lambda(0,\psi)| &= |\lambda(\psi,\psi) - \lambda(0,\psi)| \\
&\leq L_2(\delta_1)|\psi|_{W^{1,\infty}}|\psi|_C. \qquad (5.13)
\end{aligned}$$

By combining (5.11), (5.12) and (5.13) we get for $\psi \in \overline{\mathcal{G}}_C(\theta) \cap W^{1,\infty}$ that

$$\begin{aligned}
|G(\psi)| &\leq \eta|\psi|_C + \eta\|\mu\||\psi|_C + \left\|\frac{\partial f}{\partial y}(0,0)\right\| L_2(\delta_1)|\psi|_{W^{1,\infty}}|\psi|_C \\
&\leq N(\eta + |\psi|_{W^{1,\infty}})|\varphi|_C,
\end{aligned}$$

where $N \equiv \max\Big\{1 + \|\mu\|, \left\|\frac{\partial f}{\partial y}(0,0)\right\| L_2(\delta_1)\Big\}$. □

Let $S(t)$ be the semigroup generated by the linear constant-delay equation (5.9), and ω_0 be the supremum of the real part of the characteristic roots of equation (5.9). (See Section 2.3 for the definition of $S(t)$ and ω_0.) We show that the stability properties of the trivial solution of the nonlinear state-dependent autonomous equation (5.1) can be obtained by that of the linear constant-delay equation (5.9).

Theorem 5.3 *Assume (A1)–(A7) and (H), and that the semigroup $S(t)$ is asymptotically stable, i.e., $\omega_0 < 0$. Then for every $\omega > \omega_0$ there exist $K = K(\omega) > 0$ and $\delta = \delta(\omega) > 0$ such that for all $\varphi \in G_C(\delta)$ the corresponding solution, $x(t)$, of IVP (5.1)-(5.2) is defined for $t \in [0, \infty)$, and satisfies*

$$|x(t)| \leq Ke^{\omega t}|\varphi|_C, \qquad t \geq 0.$$

Proof Fix an arbitrary $\omega_0 < \omega < 0$ and fix ω^* such that $\omega_0 < \omega^* < \omega$. Then by Lemma 2.18, there exists a constant $M = M(\omega^*) \geq 1$ such that

$$|S(t)\varphi|_C \leq M e^{\omega^* t} |\varphi|_C, \qquad t \geq 0, \quad \varphi \in C. \tag{5.14}$$

Let $x(t)$ be the solution of (5.8) (or equivalently (5.1)) corresponding to an initial function $\varphi \in C$. By Lemma 2.20 we get that

$$x_t = S(t-r)x_r + \int_0^{t-r} S(t-r-s)X_0 G(x_{s+r})\, ds, \qquad t \geq r, \tag{5.15}$$

where X_0 is defined by (2.14).

Let $N > 0$ be the constant given by Lemma 5.2, define

$$\eta \equiv \frac{\omega - \omega^*}{4MN},$$

and let $\theta(\eta)$ be the constant corresponding to this η from Lemma 5.2. Finally, define two more constants

$$\delta_3 \equiv \min\left\{\delta_2, \frac{\omega - \omega^*}{4MN}, \frac{\omega - \omega^*}{4MNL_1(1 + \|\mu\|)\delta_2}, \theta(\eta)\right\},$$

and

$$\delta \equiv \delta_3 \exp\Big(-L_1(1 + \|\mu\|)r\Big)\frac{1}{M}e^{\omega^* r}.$$

We comment, that $\frac{1}{M}e^{\omega^* r} \leq 1$ since $M \geq 1$ and $\omega^* < 0$, and hence $\delta \leq \delta_3 \leq \delta_2$.

Let $|\varphi|_C < \delta$. Then by (5.4) and $\delta \leq \delta_2$ it follows that $\varphi(0) \in \Omega_1$ and $\Lambda(\varphi) \in \Omega_2$, and therefore Theorem 3.8 implies that there exists a solution if IVP (5.1)-(5.2) $x(t)$ corresponding to φ on an interval $[0, \alpha]$. Since, by (5.4) and Theorem 3.8, the solution is continuable till $x_t \in \mathcal{G}_C(\delta_2)$, and since Lemma 5.1 and the definition of δ imply the relation $|x_r|_C < \delta_3 \leq \delta_2$, it follows that there exists $r < t_1 \leq \alpha$ such that $|x_t|_C < \delta_3$ on $t \in [0, t_1)$. Suppose that there exists t_2 such that $r < t_2 \leq \alpha$ and the solution satisfies

$$|x_t|_C < \delta_3 \qquad \text{for } t \in [0, t_2), \quad \text{and} \quad |x_{t_2}|_C = \delta_3. \tag{5.16}$$

For $t \in [r, t_2)$ and $|\varphi|_C \leq \delta$, estimate (5.3), (2.5), (5.16), $\delta_3 \leq \delta_2$ and the definition of δ_3 imply that

$$\begin{aligned} |\dot{x}(t)| &= |f(x(t), \Lambda(x_t))| \\ &\leq L_1(|x(t)| + |\Lambda(x_t)|) \\ &\leq L_1(1 + \|\mu\|)|x_t| \\ &\leq L_1(1 + \|\mu\|)\delta_3 \\ &\leq \frac{\omega - \omega^*}{4MN}. \end{aligned} \tag{5.17}$$

Then (5.17) yields that

$$\sup_{t-r \leq s \leq t} |\dot{x}(s)| \leq \frac{\omega - \omega^*}{4MN},$$

and hence, by using (5.16), we also have

$$|x_t|_{W^{1,\infty}} \leq \frac{\omega - \omega^*}{4MN}, \qquad \text{for } t \in [r, t_2), \quad |\varphi|_C \leq \delta. \tag{5.18}$$

Since for $t \in [r, t_2)$, $|\varphi|_C < \delta_3$ and $0 \le s \le t$ relation (5.16) yields that $|x_{s+r}|_C \le \delta_3 \le \theta(\eta)$, then Lemma 5.2, (5.14), (5.15), (5.18) and the relation $|X_0 z|_C = |z|$ (for $z \in \mathbb{R}^n$) imply that

$$\begin{aligned} |x_t|_C &\le \|S(t-r)\||x_r|_C + \int_0^{t-r} \|S(t-r-s)\||G(x_{s+r})|\,ds \\ &\le Me^{\omega^*(t-r)}|x_r|_C + \int_0^{t-r} MNe^{\omega^*(t-r-s)}\Big(\eta + |x_{s+r}|_{W^{1,\infty}}\Big)|x_{s+r}|_C\,ds \\ &\le Me^{\omega^*(t-r)}|x_r|_C + \int_r^t MNe^{\omega^*(t-s)}\Big(\eta + \frac{\omega-\omega^*}{4MN}\Big)|x_s|_C\,ds. \end{aligned}$$

By multiplying both sides by $e^{-\omega^* t}$ and changing variable in the integral we get

$$|x_t|_C e^{-\omega^* t} \le Me^{-\omega^* r}|x_r|_C + \int_r^t MNe^{-\omega^* s}\Big(\eta + \frac{\omega-\omega^*}{4MN}\Big)|x_s|_C\,ds.$$

By applying Gronwall-Bellman inequality for the function $|x_t|_C e^{-\omega^* t}$ we get

$$|x_t|_C e^{-\omega^* t} \le Me^{-\omega^* r}|x_r|_C \exp\left(MN\left(\eta + \frac{\omega-\omega^*}{4MN}\right)t\right), \qquad r \le t \le t_2,$$

or equivalently, for $r \le t \le t_2$

$$|x_t|_C \le Me^{-\omega^* r}|x_r|_C \exp\left(\left(MN\left(\eta + \frac{\omega-\omega^*}{4MN}\right) + \omega^*\right)t\right).$$

From the definition of η it follows that

$$\begin{aligned} |x_t|_C &\le Me^{-\omega^* r}|x_r|_C \exp\left(\left(\frac{\omega-\omega^*}{2} + \omega^*\right)t\right) \\ &< Me^{-\omega^* r}|x_r|_C e^{\omega t}, \qquad r \le t \le t_2. \end{aligned} \tag{5.19}$$

Then this estimate, Lemma 5.1 and the definition of δ imply for $|\varphi|_C < \delta$ that

$$\begin{aligned} |x_t|_C &< Me^{-\omega^* r}|\varphi|_C e^{L_1(1+\|\mu\|)r} e^{\omega t} \\ &< \delta_3, \qquad r \le t \le t_2, \end{aligned}$$

which contradicts to the definition of t_2. Therefore $|x_t| < \delta_3$ for $r \le t \le \alpha$, but this implies that $\alpha = \infty$, and (5.19) holds for all $t \ge r$, therefore, by (5.16) and (5.19), the statement of the theorem is proved with $K \equiv Me^{\omega^* r}\delta_3$. □

Remark 5.4 *We note, that if $\omega_0 > 0$, i.e., the trivial solution of the linear equation is unstable, then so is the trivial solution of the nonlinear equation. Since unstability results are of less interest in applications, and the detailed proof is rather lengthy, technical, and also similar to the state-independent case, we omit it. (See Section 10.1 in [31] for the state-independent case.)*

5.2 Applications

In this section we show examples, when by the linearization technique of the previous section, we can find conditions implying asymptotic stability of a nonlinear delay equation. The applicability of this linearization method depends on whether we are able to check the asymptotic stability of the linearized equation, which is a difficult problem in general, but in the examples we present in this section we can refer to existing conditions from the literature.

Example 5.5 Consider the scalar constant delay equation

$$\dot{x}(t) = -ax(t-1)(1+x(t)), \qquad t \geq 0, \quad (a > 0). \tag{5.20}$$

This equation arises as we transform the delayed logistic equation

$$\dot{x}(t) = \tau x(t)(1 - x(t-\tau)/K)$$

by the new variable $y(t) = -1 + x(t)/K$, and change the time scale. (See e.g. [38].) It is known (e.g. [38]), that the trivial solution of (5.20) is asymptotically stable for $a < \pi/2$, and unstable for $a > \pi/2$. We can obtain this result by using Theorem 5.3. Equation (5.20) has the form (5.1) with $r = 1$, $f(x,y) = -ay(1+x)$ and $\lambda(\psi,\xi) = \xi(-1)$. Since $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = -a$, the linearized equation (5.9) for this equation is

$$\dot{x}(t) = -ax(t-1), \qquad t \geq 0. \tag{5.21}$$

Since the trivial solution of (5.21) is asymptotically stable for $a < \pi/2$, and unstable for $a > \pi/2$ (see e.g. [31]), the same result holds for the trivial solution of (5.20) by Theorem 5.3 and Remark 5.4.

Example 5.6 Consider the scalar delay equation

$$\dot{x}(t) = x(t)\Big(a + bx(t-\tau) - cx^2(t-\tau)\Big), \qquad t \geq 0,$$

where $a > 0$ and $c > 0$. This is a delayed Lotka-Volterra type population model introduced by Gopalsamy and Ladas (see e.g. in [38]). The equation has a unique positive equilibrium point, $\bar{x} = (b + \sqrt{b^2 + 4ac})/(2c)$. By the new variable $y(t) = x(t) - \bar{x}$ we can transform the equilibrium point to zero, and get the equation

$$\dot{y}(t) = -(y(t) + \bar{x})\Big((2c\bar{x} - b)y(t-\tau) + cy^2(t-\tau)\Big), \qquad t \geq 0. \tag{5.22}$$

We can rewrite (5.22) in the form (5.1) with $f(u,v) = -(u+\bar{x})\Big((2c\bar{x}-b)v + cv^2\Big)$ and $\lambda(\psi,\xi) = \xi(-\tau)$. Since $\frac{\partial f}{\partial u}(0,0) = 0$ and $\frac{\partial f}{\partial v}(0,0) = -\bar{x}(2c\bar{x} - b)$, the linearized form of (5.22) is

$$\dot{x}(t) = -\bar{x}(2c\bar{x} - b)x(t-\tau), \qquad t \geq 0,$$

which is asymptotically stable if $0 < \bar{x}(2c\bar{x} - b)\tau < \pi/2$, or equivalently,

$$\frac{b\sqrt{b^2 + 4ac} + b^2 + 4ac}{2c}\tau < \frac{\pi}{2},$$

and therefore under this assumption the trivial solution of (5.22) is asymptotically stable as well.

Example 5.7 Consider the scalar delay equation with state-dependent delay

$$\dot{x}(t) = x(t)\Big(a - bx(t) - \sum_{i=1}^{m} b_i x(t-\tau_i) - cx(t-\tau(x_t))\Big), \qquad t \geq 0,$$

where

$$a > 0, \quad \text{and} \quad b > \sum_{i=1}^{m} |b_i| + |c|. \tag{5.23}$$

This population model with state-dependent delay term was studied in [12], where it was shown that (5.23) yields that the unique positive equilibrium, $\bar{x} = a/(b + \sum_{i=1}^{m} b_i + c)$, of the equation is globally asymptotically stable (for initial functions $\varphi(s) > M$ with some $M > 0$). We can show this result (for local asymptotic stability) by using linearization technique. By the new variable $y(t) = x(t) - \bar{x}$ we transform the equilibrium point to the origin, and the corresponding equation is

$$\dot{y}(t) = -(y(t) + \bar{x})\Big(by(t) + \sum_{i=1}^{m} b_i y(t-\tau_i) + cy(t - \tau(y_t + \bar{x}))\Big), \tag{5.24}$$

which has the form (5.1) with $f(u,v) = -(u+\bar{x})(bu+v)$, $\lambda(\psi,\xi) = \sum_{i=1}^{m} b_i\xi(-\tau_i) + c\xi(-\tau(\psi + \bar{x}))$. (Here and later, $\bar{x}$ in the argument of τ denotes a constant function with value equal to $\bar{x}$.) We have that $\frac{\partial f}{\partial u}(0,0) = -b\bar{x}$, $\frac{\partial f}{\partial v}(0,0) = -\bar{x}$, and $\lambda(0,\xi) = \sum_{i=1}^{m} b_i\xi(-\tau_i) + c\xi(-\tau(\bar{x}))$. Therefore the linearized equation of (5.24) is

$$\dot{x}(t) = -b\bar{x}x(t) - \bar{x}\left(\sum_{i=1}^{m} b_i x(t-\tau_i) + cx(t-\tau(\bar{x}))\right). \tag{5.25}$$

By a result from [31] (page 154) it follows that (5.23) yields the asymptotic stability of the trivial solution of (5.25), for arbitrary delay function $\tau(\cdot)$, which, by Theorem 5.3, implies that the trivial solution of (5.24) is asymptotically stable as well.

Example 5.8 Consider the scalar constant delay equation

$$\dot{x}(t) = \gamma x(t)\left(1 - \sum_{i=1}^{m} \frac{a_i x(t-\tau_i)}{1 + c_i x(t-\tau_i)}\right). \tag{5.26}$$

This is the so-called Michaelis-Menton single species growth equation (see e.g. in [38]). We assume that

$$\gamma > 0,\ a_i > 0,\ c_i > 0,\ \tau_i > 0, \quad \text{and} \quad \sum_{i=1}^{m} \frac{a_i}{1+c_i} = 1.$$

The last assumption yields that $\bar{x} = 1$ is a positive equilibrium point of (5.26). It was shown in [38] that $\gamma r \leq 1$ implies the global asymptotic stability of $\bar{x}$, where $r = \max_{i=1,\dots,m} \tau_i$.

By letting $y(t) = x(t) - 1$, we get

$$\dot{y}(t) = -\gamma(y(t)+1)\sum_{i=1}^{m} \frac{a_i y(t-\tau_i)}{(1+c_i)(1+c_i+c_i y(t-\tau_i))}. \tag{5.27}$$

We can rewrite (5.27) in the form of (5.1), by selecting $f(u,v) = -\gamma(u+1)v$, and

$$\lambda(\psi,\xi) = \sum_{i=1}^{m} \frac{a_i}{(1+c_i)(1+c_i+c_i\psi(-\tau_i))}\xi(-\tau_i).$$

Note, that this delayed term is not covered by Example 1.4, but it is clear, that we can replace $A_k(t)$ of Example 1.4 by functions $A_k(t,\psi)$, and assuming that each $A_k(t,\psi)$ is continuous on $[0,T]\times\Omega_3$, and locally Lipschitz-continuous in ψ, we can extend the example for this case as well, i.e., we can rewrite the corresponding delay term in the form (1.2), and the corresponding λ satisfies (A2) and (A5). It is clear that the function λ defined above has this properties, therefore (5.27) has the form (5.1) with this f and λ. We have that $\frac{\partial f}{\partial u}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = -\gamma$, therefore the corresponding linearized equation is

$$\dot{x}(t) = -\gamma \sum_{i=1}^{m} \frac{a_i}{(1+c_i)^2} x(t-\tau_i). \tag{5.28}$$

By a condition from e.g. [28] or [37], it follows that the trivial solution of (5.27) is asymptotically stable if

$$\gamma \sum_{i=1}^{m} \frac{a_i}{(1+c_i)^2}\tau_i < 1.$$

It follows from the assumptions $\sum_{i=1}^{m} \frac{a_i}{1+c_i} = 1$, $c_i > 0$ and $r = \max_{i=1,\dots,m}\tau_i$ that

$$\gamma \sum_{i=1}^{m} \frac{a_i}{(1+c_i)^2}\tau_i < \gamma r \sum_{i=1}^{m} \frac{a_i}{1+c_i} = \gamma r,$$

therefore the condition $\gamma r \le 1$ implies that trivial solution of (5.28), and hence that of (5.27) is asymptotically stable.

Note, that the delayed term of (5.27) can not be written in the form given by the Stieltjes-integral in (3.64), and hence this equation is not included in (3.64) (without multiple delay terms).

Example 5.9 In [38] the scalar equation

$$\dot{x}(t) = f\left(\int_{-r}^{-\sigma} x(t+s)\,d\mu(s)\right) - g(x(t))$$

has been studied, where $r > \sigma > 0$, and

(i) $\mu(s)$ is nondecreasing and $\mu(-\sigma) - \mu(-r) = 1$,

(ii) $f(x)$ is strictly decreasing, $f(0) > 0$, $\lim_{x\to\infty} f(x) = 0$,

(iii) $g(x)$ is strictly increasing, $g(0) = 0$, $\lim_{x\to\infty} g(x) = \infty$,

and a condition was derived for the global asymptotic stability of the unique positive equilibrium.

We study the local asymptotic stability of the state-dependent version of this equation, i.e., consider

$$\dot{x}(t) = f\left(\int_{-r}^{-\sigma} x(t+s)\,d\mu(s,x_t)\right) - g(x(t)), \tag{5.29}$$

where we assume $r > \sigma > 0$, (ii), (iii) above and modify (i) as (i') for all $\psi \in C$, the function $\mu(\cdot, \psi)$ is nondecreasing and $\mu(-\sigma, \psi) - \mu(-r, \psi) = 1$.

Under this assumptions, (5.29) has a unique positive equilibrium point, $\bar{x}$, since the function

$$\begin{aligned} f\left(\int_{-r}^{-\sigma} \bar{x}\, d\mu(s, \bar{x})\right) - g(\bar{x}) &= f\Big(\bar{x}(\mu(-\sigma, \bar{x}) - \mu(-r, \bar{x}))\Big) - g(\bar{x}) \\ &= f(\bar{x}) - g(\bar{x}) \end{aligned}$$

has a unique positive zero. (Here and later, $\bar{x}$ in the second argument of μ denotes a constant function with value $\bar{x}$.) Using $y(t) = x(t) - \bar{x}$ and an argument similar to the one above, we get

$$\dot{y}(t) = f\left(\int_{-r}^{-\sigma} y(t+s)\, d\mu(s, x_t) + \bar{x}\right) - g\Big(y(t) + \bar{x}\Big). \tag{5.30}$$

We can rewrite (5.30) in the form (5.1) with $F(u, v) = f(v + \bar{x}) - g(u + \bar{x})$, and $\lambda(\psi, \xi) = \int_{-r}^{-\sigma} \xi(s)\, d\mu(s, \psi)$. We have that $\frac{\partial F}{\partial u}(0,0) = -g'(\bar{x})$ and $\frac{\partial F}{\partial v}(0,0) = f'(\bar{x})$. Therefore the linearized version of (5.30) is

$$\dot{x}(t) = -g'(\bar{x})x(t) + f'(\bar{x}) \int_{-r}^{-\sigma} x(t+s)\, d\mu(s, \bar{x}). \tag{5.31}$$

Note that $g'(\bar{x}) > 0$ and $f'(\bar{x}) < 0$ by the assumptions. Theorem 1.1 of [37] yields that the trivial solution of (5.31) is asymptotically stable if

$$-f'(\bar{x}) \int_{-r}^{-\sigma} s\, d\mu(s, \bar{x}) < \frac{3}{2},$$

and therefore by our theorem, if this condition is satisfied, then the trivial solution of (5.30) is asymptotically stable as well.

Chapter 6

APPROXIMATION OF SOLUTIONS IN C

Numerical methods for solving delay equations have been investigated by many authors (without completeness, we refer to [3], [15], [22], [34], [42], [46]).

In this chapter we define a sequence of delay equations with piecewise constant arguments which approximate equation (3.1), and obtain a discrete recurrence relation for the approximate solution, which can be evaluated numerically easily. We comment that this method is identical to Euler's method, and hence it is a one step method guaranteeing only first order convergence. It was introduced for point state-dependent equations in [21] (see also [46]), but with the aid of the approximating equations we can obtain a nice new proof (different from those in [21] or [46]) for the convergence of the approximate solutions.

The usage of equations with piecewise constant arguments for approximating delay equations was originally introduced in [26] for linear delay equations with constant delays, and it has been generalized for nonlinear delay equations with point state-dependent delay terms in [27]. Note, that the methods of [27] and that of this chapter are not exactly the same for the point delay case, but we can use the same technique to prove the convergence results.

We present the convergence results in Section 6.1, and show numerical experiments in Section 6.2. In Chapter 7 we shall apply the method defined in this chapter for parameter identification.

By using these approximating piecewise constant equations and following the steps of the Cauchy-Peano existence theorem for ODEs (see e.g. in [11]), it is possible to obtain an alternative proof for the existence and uniqueness of solutions of (3.1) (see this method for the point state-dependent case in [27]).

6.1 Theoretical convergence

Throughout this section we shall use the notation $[t]_h \equiv [t/h]h$, where $[\cdot]$ is the greatest integer function. For later reference we mention some elementary properties of this function:

$$t - h < [t]_h \leq t, \tag{6.1}$$

$$|[t]_h - t| \leq h, \tag{6.2}$$

$$\lim_{h\to 0}[t]_h = t. \tag{6.3}$$

Let h be a positive number. We associate the following FDE with piecewise constant right-hand side to (3.1).

$$\dot{y}_h(t) = f\left([t]_h, y_h([t]_h), \int_{-r}^{0} d_s\mu(s,[t]_h,(y_h)_{[t]_h})\, y_h([t]_h+s)\right), \qquad t \in [0,T]. \tag{6.4}$$

The subscript h of $y_h(t)$ emphasizes that $y_h(t)$ is the solution of (6.4) corresponding to the discretization parameter h. The notation $(y_h)_{[t]_h}$ denotes the solution segment of the function $y_h(\cdot)$ at time $[t]_h$, i.e., $(y_h)_{[t]_h} : [-r, 0] \to \mathbb{R}^n$, $(y_h)_{[t]_h}(s) \equiv y_h([t]_h + s)$. The associated initial condition to (6.4) is

$$y_h(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{6.5}$$

By a solution of the initial value problem (6.4)-(6.5) we mean a function $y_h : [-r, T] \to \mathbb{R}^n$, which is defined on $[-r, 0]$ by (6.5) and satisfies the following properties on $[0, T]$:

(i) the function y_h is continuous on $[0, T]$,

(ii) the derivative $\dot{y}_h(t)$ exists at each point $t \in [0, \infty)$ with the possible exception of the points ih $(i = 0, 1, 2, \ldots)$ where finite one-sided derivatives exist,

(iii) the function y_h satisfies (6.4) on each interval $[ih, (i + 1)h) \cap [0, T]$ for $i = 0, 1, 2, \ldots$

Note, that by using the notation $\Lambda(t, \psi)$, we can rewrite (6.4) as

$$\dot{y}_h(t) = f\Big([t]_h, y_h([t]_h), \Lambda([t]_h, (y_h)_{[t]_h})\Big), \qquad t \in [0, T].$$

Lemma 6.1 *Let $\gamma \in \Pi(T, \Omega_1, \Omega_2, \Omega_3)$.*

(i) *Let $y_h(t)$ be a continuous function on $[0, T]$ such that $(y_h)_{[t]_h} \in \Omega_3$ for $t \in [0, \alpha]$ $(\alpha \le T)$. Then the function $t \mapsto \Lambda\Big([t]_h, (y_h)_{[t]_h}\Big)$ is defined and piecewise-constant on $[0, \alpha]$.*

(ii) *(6.4) is equivalent to the integral equation*

$$y_h(t) = \begin{cases} \varphi(t), & t \in [-r, 0], \\ \varphi(0) + \int_0^t f\Big([u]_h, y_h([u]_h), \Lambda([u]_h, (y_h)_{[u]_h})\Big)du, & t \in [0, T]. \end{cases} \tag{6.6}$$

(iii) *For an arbitrary fixed $h > 0$ there exists a constant $0 < \alpha \le T$ such that IVP (6.4)-(6.5) has unique solution on $[-r, \alpha]$, which is piecewise linear on $[0, \alpha]$.*

Proof Part (i) is obvious, part (ii) follows from (i). Using part (i) and the method of steps on intervals $[ih, (i + 1)h)$ we get existence and uniqueness of solution of IVP (6.4)-(6.5) for fixed $h > 0$ while $y_h([u]_h) \in \Omega_1$, the third argument of f in (6.4) remains in Ω_2, and $(y_h)_{[u]_h} \in \Omega_3$. □

The following lemma shows that the solutions corresponding to $h > 0$ on a compact time interval form a uniformly bounded, equicontinuous family of functions.

Lemma 6.2 *Assume (A1)-(A6) and let $\gamma \in \Pi(T, \Omega_1, \Omega_2, \Omega_3)$.*

(i) *For an arbitrary finite $\alpha \le T$ there exists a constant $K_1 = K_1(\alpha, \gamma) > 0$ such that for every $h > 0$*

$$|y_h(t)| \le K_1, \qquad t \in [-r, \alpha]. \tag{6.7}$$

(ii) *For an arbitrary finite $\alpha \le T$ there exists a constant $K_2 = K_2(\alpha, \gamma) > 0$ such that for every $h > 0$*

$$|y_h(t) - y_h(\bar{t})| \le K_2|t - \bar{t}|, \qquad t, \bar{t} \in [-r, \alpha], \tag{6.8}$$

and

$$|(y_h)_t - (y_h)_{\bar{t}}|_C \le K_2|t - \bar{t}|, \qquad t, \bar{t} \in [0, \alpha]. \tag{6.9}$$

(iii) *There exists $\alpha \leq T$ such that IVP (6.4)-(6.5) has a unique solution on $[0,\alpha]$ for an arbitrary $h > 0$.*

Proof It is easy to see that $K_1 \equiv |\varphi|_C + \|f\|\alpha$ satisfies (6.7).

To find K_2, let $t, \bar{t} \in [-r, 0]$. Then by (A6) and Lemma 2.3 we have that

$$|y_h(t) - y_h(\bar{t})| = |\varphi(t) - \varphi(\bar{t})| \leq |\varphi|_{W^{1,\infty}}|t - \bar{t}|.$$

For $t, \bar{t} \geq 0$ it follows from (6.4) that

$$|y_h(t) - y_h(\bar{t})| \leq \|f\||t - \bar{t}|.$$

For $-r \leq t \leq 0 \leq \bar{t} \leq \alpha$, using the previous two estimates, we get

$$\begin{aligned} |y_h(t) - y_h(\bar{t})| &\leq |y_h(t) - y_h(0)| + |y_h(0) - y_h(\bar{t})| \\ &\leq |\varphi|_{W^{1,\infty}}|t| + \|f\|\bar{t} \\ &\leq \max\{|\varphi|_{W^{1,\infty}}, \|f\|\}|t - \bar{t}|. \end{aligned}$$

Therefore $K_2 \equiv \max\{|\varphi|_{W^{1,\infty}}, \|f\|\}$ satisfies (6.8), and thus (6.9) as well.

Inequality (6.8) yields for arbitrary $h > 0$ that $|y_h(t) - \varphi(0)| \leq K_2 t$, which, using that $\varphi(0) \in \Omega_1$ and Ω_1 is open, implies that there exists $\alpha_1 \leq T$ such that $y_h(t) \in \Omega_1$ for $t \in [0, \alpha_1]$ and for all $h > 0$.

Similarly to that in the proof of Lemma 3.13, we can show that for all $u \in [0, \alpha_1]$ it follows that $(y_h)_u \in W^{1,\infty}$, and $|(y_h)_u|_{1,\infty} \leq M_1 \equiv \max\{\|f\|, |\varphi|_{W^{1,\infty}}, K_1\}$. By (6.2), (6.7), (6.8), (6.9) with $K_2 = K_2(\alpha_1, \gamma)$, and Lemma 3.12 with the constant $L_2 = L_2(\alpha, M_1)$ we have that

$$\begin{aligned} &|\Lambda([u]_h, (y_h)_{[u]_h}) - \Lambda(0, (y_h)_0)| \\ &\quad\leq |\Lambda([u]_h, (y_h)_{[u]_h}) - \Lambda([u]_h, (y_h)_0)| + |\Lambda([u]_h, (y_h)_0) - \Lambda(0, (y_h)_0)| \\ &\quad\leq (\|\mu\| + L_2(\alpha_1, M_1)|\varphi|_{W^{1,\infty}})|(y_h)_{[u]_h} - (y_h)_0|_C + |\Lambda([u]_h, \varphi) - \Lambda(0, \varphi)| \\ &\quad\leq (\|\mu\| + L_2(\alpha_1, M_1)|\varphi|_{W^{1,\infty}})K_2|[u]_h| + |\Lambda([u]_h, \varphi) - \Lambda(0, \varphi)| \\ &\quad\leq (\|\mu\| + L_2(\alpha_1, M_1)|\varphi|_{W^{1,\infty}})K_2 u + |\Lambda([u]_h, \varphi) - \Lambda(0, \varphi)|, \end{aligned}$$

therefore, using (6.1), Lemma 2.8 and that Ω_2 is open, there exists $\alpha_2 \leq \alpha_1$ such that the third argument of f in (6.4) remains in Ω_2 for $t \in [0, \alpha_2]$ and for all $h > 0$. Finally, it follows from (6.9) that

$$|(y_h)_t - \varphi|_C \leq K_2 t, \qquad t \in [0, \alpha_2],$$

therefore there exists $\alpha \leq \alpha_2$ such that $(y_h)_t \in \Omega_3$ for $t \in [0, \alpha]$ and for all $h > 0$. With this α by repeating the proof of Lemma 6.1 part (iii) we can finish the proof of this Lemma. □

The next theorem shows that the solutions of IVP (6.4)-(6.5) uniformly approximate the solution of IVP (3.1)-(3.2) on compact time intervals as $h \to 0^+$.

Theorem 6.3 *Assume (A1)-(A6) and let $\gamma \in \Pi(T, \Omega_1, \Omega_2, \Omega_3)$. Then the solutions of IVP (6.4)-(6.5) uniformly approximate the solution of IVP (3.1)-(3.2) on compact time intervals as $h \to 0^+$, i.e.,*

$$\lim_{h \to 0^+} \max_{0 \leq t \leq \alpha} |x(t) - y_h(t)| = 0, \tag{6.10}$$

where $\alpha \leq T$ is a finite positive number satisfying Lemma 6.2 (iii). Moreover, assume that *(i) f is locally Lipschitz-continuous in all of its arguments, i.e., for every $\alpha > 0$, $M > 0$ there exists a constant $\tilde{L}_1 = \tilde{L}_1(\alpha, M)$ such that for all $t, \bar{t} \in [0, \alpha]$, $x, \bar{x} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_1$, $y, \bar{y} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_2$ it follows that*

$$|f(t,x,y) - f(\bar{t},\bar{x},\bar{y})| \le \tilde{L}_1\Big(|t-\bar{t}| + |x-\bar{x}| + |y-\bar{y}|\Big),$$

(ii) for every $\alpha > 0$, $M > 0$ there exists a constant $\tilde{L}_2 = \tilde{L}_2(\alpha, M)$ such that for all $\xi \in W^{1,\infty} \cap \Omega_3$, $t, \bar{t} \in [0, \alpha]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M_2) \cap \Omega_3$ it follows that

$$|\lambda(t,\psi,\xi) - \lambda(\bar{t},\bar{\psi},\xi)| \le \tilde{L}_2 |\xi|_{W^{1,\infty}}\Big(|t-\bar{t}| + |\psi-\bar{\psi}|_C\Big),$$

then the convergence is linear in h, i.e., there exists a constant $M_3(\alpha,\gamma) > 0$ such that

$$|x(t) - y_h(t)| \le M_3 h, \qquad t \in [0,\alpha], \quad h > 0. \tag{6.11}$$

Proof Let $\alpha > 0$ be a finite constant satisfying Lemma (6.2) part (iii), and define

$$M \equiv \max\{\|\mu\|, 1\} \cdot \max\Big\{|x|_{W^{1,\infty}_\alpha}, K_1(\alpha,\gamma)\Big\},$$

where $K_1(\alpha,\gamma)$ is the constant from Lemma 6.2 (i). Then the definition of M, inequalities (6.7) and (2.5) imply that x_t, $(y_h)_t$ and $\Lambda(t, x_t)$, $\Lambda(t,(y_h)_t)$ remain in $\bar{G}_C(M)$ for $t \in [0,\alpha]$. Let $L_1 = L_1(\alpha, M)$ be the constant given by (A4). Then equation (6.6), assumption (A4) and standard estimates yield the following inequalities

$$\begin{aligned}
|x(t) - y_h(t)| &\le \int_0^t \Big| f\Big(u, x(u), \Lambda(u,x_u)\Big) - f\Big([u]_h, x(u), \Lambda(u,x_u)\Big)\Big| \, du \\
&\quad + \int_0^t \Big| f\Big([u]_h, x(u), \Lambda(u,x_u)\Big) - f\Big([u]_h, y_h([u]_h), \Lambda([u]_h, (y_h)_{[u]_h})\Big)\Big| \, du \\
&\le \int_0^t \Big| f\Big(u, x(u), \Lambda(u,x_u)\Big) - f\Big([u]_h, x(u), \Lambda(u,x_u)\Big)\Big| \, du \\
&\quad + \int_0^t L_1 \Big(|x(u) - y_h([u]_h)| + \Big|\Lambda(u,x_u) - \Lambda([u]_h, (y_h)_{[u]_h})\Big|\Big) \, du.
\end{aligned} \tag{6.12}$$

Inequalities (6.8) with $K_2 = K_2(\alpha,\gamma)$ and (6.2) imply

$$\begin{aligned}
|x(u) - y_h([u]_h)| &\le |x(u) - y_h(u)| + |y_h(u) - y_h([u]_h)| \\
&\le |x(u) - y_h(u)| + K_2|u - [u]_h| \\
&\le |x(u) - y_h(u)| + K_2 h,
\end{aligned} \tag{6.13}$$

and therefore

$$|x_u - (y_h)_{[u]_h}|_C \le |x_u - (y_h)_u|_C + K_2 h. \tag{6.14}$$

Using (6.13), (6.14), Lemma 3.12 with $L_2 = L_2(\alpha, M)$, we can estimate the last term in the right hand side of (6.12) as follows:

$$\begin{aligned}
&\Big|\Lambda(u,x_u) - \Lambda([u]_h, (y_h)_{[u]_h})\Big| \\
&\le |\Lambda(u,x_u) - \Lambda([u]_h, x_u)| + \Big|\Lambda([u]_h, x_u) - \Lambda([u]_h, (y_h)_{[u]_h})\Big| \\
&\le |\Lambda(u,x_u) - \Lambda([u]_h, x_u)| + (\|\mu\| + L_2|x_u|_{W^{1,\infty}})|x_u - (y_h)_{[u]_h}|_C \\
&\le |\Lambda(u,x_u) - \Lambda([u]_h, x_u)| + (\|\mu\| + L_2 M)|x_u - (y_h)_u|_C + (\|\mu\| + L_2 M)K_2 h.
\end{aligned} \tag{6.15}$$

By combining (6.12), (6.13) and (6.15) we get

$$|x(t)-y_h(t)| \le g_h(t) + \int_0^t L_1(1+L_2M+\|\mu\|) \max_{0\le s\le u} |x(s)-y_h(s)|\,du, \tag{6.16}$$

where

$$\begin{aligned} g_h(t) \;&\equiv\; \int_0^t \Big| f\Big(u, x(u), \Lambda(u,x_u)\Big) - f\Big([u]_h, x(u), \Lambda(u,x_u)\Big)\Big| du \\ &\quad + \int_0^t |\Lambda(u,x_u) - \Lambda([u]_h, x_u)|\,du + L_1(1+L_2M+\|\mu\|)K_2ht. \end{aligned}$$

The function $g_h(t)$ is monotone increasing in t, therefore Lemma 2.14 and (6.16) imply for $t\in[0,\alpha]$ that

$$\max_{0\le s\le t} |x(s)-y_h(s)| \le g_h(\alpha) + \int_0^t L_1(1+L_2M+\|\mu\|) \max_{0\le s\le u} |x(s)-y_h(s)|\,du, \tag{6.17}$$

which, by Gronwall-Bellman inequality, implies that

$$\max_{0\le s\le t} |x(s)-y_h(s)| \le g_h(\alpha)\exp\Big(L_1(1+L_2M+\|\mu\|)t\Big), \qquad t\in[0,\alpha]. \tag{6.18}$$

To finish the proof of (6.10), it is enough to show that $g_h(\alpha)\to 0$ as $h\to 0^+$. Relation (6.3) and the continuity of f yield that for all $u\ge 0$

$$f\Big([u]_h, x(u), \Lambda(u,x_u)\Big) \to f\Big(u, x(u), \Lambda(u,x_u)\Big), \qquad \text{as } h\to 0^+,$$

and by Lemma 2.8 and (6.3) we have for all $u\ge 0$ that $\Lambda([u]_h, x_u)\to\Lambda(u,x_u)$, as $h\to 0^+$, therefore by the Lebesgue Dominated Convergence Theorem we get

$$\int_0^\alpha \Big| f\Big(u, x(u), \Lambda(u,x_u)\Big) - f\Big([u]_h, x(u), \Lambda(u,x_u)\Big)\Big| du \to 0, \qquad \text{as } h\to 0^+.$$

Similarly, $\int_0^\alpha \Big|\Lambda(u,x_u)-\Lambda([u]_h,x_u)\Big|\,du\to 0$ as $h\to 0^+$, hence $g_h(\alpha)\to 0$ as $h\to 0^+$.

By assumption (i), (ii) and (6.2) we have

$$\begin{aligned} |g_h(\alpha)| \;&\le\; \int_0^\alpha (\tilde L_1+\tilde L_2M)|u-[u]_h|\,du + L_1(1+L_2M+\|\mu\|)K_2ht \\ &\le\; (\tilde L_1+\tilde L_2M + L_1(1+L_2M+\|\mu\|)K_2)h\alpha. \end{aligned} \tag{6.19}$$

Relations (6.18) and (6.19) yield that

$$M_3 = (\tilde L_1+\tilde L_2M+L_1(1+L_2M+\|\mu\|)K_2)\alpha\exp\Big(L_1(1+L_2M+\|\mu\|)\alpha\Big)$$

satisfies (6.11). □

Next we address the issue of computing the solutions of (6.4). Fix $N\in\mathbb{N}$, and let $h=r/N$. By integrating (6.4) from kh to $(k+1)h$ for some $k\in\mathbb{N}$ and using that the right hand side of (6.4) is constant on $[kh,(k+1)h)$, we get

$$y((k+1)h) = y(kh) + hf(kh, (y_h)_{kh}, \Lambda(kh, (y_h)_{kh})), \tag{6.20}$$

By introducing the notation $a(k) \equiv y_h(hk)$ we can reformulate (6.20) as

$$a(k+1) = a(k) + hf(kh, (y_h)_{kh}, \Lambda(kh, (y_h)_{kh})), \qquad k = 0, 1, 2, \ldots, \tag{6.21}$$

which is a recursive formula for $a(k)$, i.e., for $y_h(kh)$, $k \in \mathbb{N}$. The question is reduced to computing $\Lambda(kh, (y_h)_{kh})$ in (6.21). In the case when the delayed term contains only point delays, i.e., $\Lambda(t, \psi) = \sum_{i=1}^{m} A_i(t)\psi(-\tau_i(t, \psi(0)))$, we get

$$\Lambda(kh, (y_h)_{kh}) = \sum_{i=1}^{m} A_i(kh) y_h(kh - \tau_i(kh, a(k))),$$

which is easy to evaluate, using the linearity of $y_h(t)$ on the intervals $[jh, (j+1)h]$. The general case is similar, we get

$$\begin{aligned}
\Lambda(kh, (y_h)_{kh}) &= \int_{-r}^{0} d_s\mu(s, kh, (y_h)_{kh}) y_h(kh+s) \\
&= \sum_{j=0}^{N-1} \int_{(j-N)h}^{(j+1-N)h} d_s\mu(s, kh, (y_h)_{kh}) y_h(kh+s) \\
&= \sum_{j=0}^{N-1} \int_{(j-N)h}^{(j+1-N)h} d_s\mu(s, kh, (y_h)_{kh}) \Big(a(k+j-N) \\
&\qquad + \frac{a(k+j+1-N) - a(k+j-N)}{h} \big(s - (j-N)h \big) \Big),
\end{aligned}$$

which shows how to compute $\Lambda(kh, (y_h)_{kh})$ for a given μ, assuming that it is easy to compute the integral of a constant and the function s with respect to $\mu(\cdot, kh, (y_h)_{kh})$.

6.2 Numerical examples

In this section we present numerical examples for the approximating scheme described in Section 6.1.

Example 6.4 Consider the nonlinear scalar initial value problem

$$\dot{x}(t) = -x^2\Big(t - \tau(x(t))\Big) + \sin 2t + \sin^4(t - \sin^2 t), \qquad t \ge 0, \tag{6.22}$$

$$x(t) = \sin^2 t, \qquad t \in [-2, 0], \tag{6.23}$$

where $\tau(x) \equiv \min\{|x|, 2\}$. Clearly, equation (6.22) can be written in the form (3.1) by choosing $r = 2$, $f(t, x, y) \equiv -y^2 + \sin 2t + \sin^4(t - \sin^2 t)$, and $\mu(s, t, \psi) \equiv \chi_{[-\tau(\psi(0)),0]}(s)$. It is easy to verify that assumptions (A1)–(A6) are satisfied, and $x(t) = \sin^2 t$ is the unique solution of (6.22)-(6.23). The approximating initial value problem is

$$\begin{aligned}
\dot{y}_h(t) &= -y_h^2\Big([t]_h - \tau(y_h([t]_h))\Big) + \sin(2[t]_h) + \sin^4([t]_h - \sin^2([t]_h)), \quad t \ge 0, \\
y_h(t) &= \sin^2(t), \qquad t \in [-2, 0].
\end{aligned}$$

Table 6.1

h	t	$x(t)$	$y_h(t)$	$\|x(t) - y_h(t)\|$
10^{-2}	20.0	0.833469	0.828241	5.228e-03
	40.0	0.555194	0.565731	1.054e-02
	60.0	0.092910	0.088392	4.518e-03
	80.0	0.987815	0.985422	2.393e-03
10^{-3}	20.0	0.833469	0.832948	5.212e-04
	40.0	0.555194	0.556241	1.047e-03
	60.0	0.092910	0.092457	4.526e-04
	80.0	0.987815	0.987578	2.362e-04
10^{-3}	20.0	0.833469	0.833417	5.210e-05
	40.0	0.555194	0.555298	1.046e-04
	60.0	0.092910	0.092864	4.526e-05
	80.0	0.987815	0.987791	2.359e-05

The corresponding numerical runnings are printed out in Table 6.1. This experiment shows that, in agreement with the theoretical expectations, the approximating sequence converges linearly to the true solution of the initial value problem.

Example 6.5 Consider the scalar distributed delay equation with constant delays

$$\dot{x}(t) = 4\pi \int_{-1}^{0} \sin(2\pi s) x(t+s)\, ds, \qquad t \geq 0, \tag{6.24}$$

$$x(t) = \cos(2\pi t), \qquad -1 \leq t \leq 0. \tag{6.25}$$

Note that (6.24) has the form (3.1) with $f(t,x,y) = 4\pi y$ and $d_s\mu(s,t,\psi) = \sin(2\pi s)\, ds$. It is easy to check that $x(t) = \cos(2\pi t)$ the analytical solution of IVP (6.24)-(6.25). Table 6.2 contains the corresponding approximate solutions and the error of the approximation.

Example 6.6 Our next example is a scalar equation

$$\dot{x}(t) = -\int_{-1}^{0} (t+s+2) x(t+s)\, ds + 1 - \frac{1}{(t+2)^2}, \qquad t \geq 0, \tag{6.26}$$

$$x(t) = \frac{1}{(t+2)}, \qquad -1 \leq t \leq 0. \tag{6.27}$$

With $f(t,x,y) = 1 - 1/(t+2)^2 - y$ and $d_s\mu(s,t,\psi) = (t+s+2)\, ds$, (6.26) has the form (3.1), and clearly, the conditions (A1)–(A6) are satisfied. The solution of IVP (6.26)-(6.27) is $x(t) = 1/(t+2)$, and the numerical results are presented in Table 6.3. The numerical approximation exhibits a first order convergence.

Example 6.7 Finally, consider

$$\dot{x}(t) = \int_{-1}^{0} x^2(t+s)\, ds - \frac{1}{2} + \pi\cos(\pi t), \qquad t \geq 0, \tag{6.28}$$

$$x(t) = \sin(\pi t), \qquad -1 \leq t \leq 0. \tag{6.29}$$

We can rewrite (6.28) in the form of (3.1), by choosing $f(t,x,y) = -1/2 + \pi\cos(\pi t) + y$, and $d_s\mu(s,t,\psi) = \psi(s)\, ds$. The solution of IVP (6.28)-(6.29) is $x(t) = \sin(\pi t)$. The approximate

Table 6.2

h	t	$x(t)$	$y_h(t)$	$\lvert x(t) - y_h(t)\rvert$
10^{-2}	0.50	-1.000000	-1.017396	1.740e-02
	1.00	1.000000	1.028012	2.801e-02
	1.50	-1.000000	-1.037858	3.786e-02
	2.00	1.000000	1.047843	4.784e-02
	2.50	-1.000000	-1.056361	5.636e-02
	3.00	1.000000	1.065118	6.512e-02
10^{-3}	0.50	-1.000000	-1.001863	1.863e-03
	1.00	1.000000	1.003022	3.022e-03
	1.50	-1.000000	-1.004231	4.231e-03
	2.00	1.000000	1.005431	5.431e-03
	2.50	-1.000000	-1.006619	6.619e-03
	3.00	1.000000	1.007810	7.810e-03
10^{-4}	0.50	-1.000000	-1.000188	1.876e-04
	1.00	1.000000	1.000304	3.043e-04
	1.50	-1.000000	-1.000427	4.274e-04
	2.00	1.000000	1.000549	5.492e-04
	2.50	-1.000000	-1.000671	6.710e-04
	3.00	1.000000	1.000793	7.929e-04

solutions of this IVP for different values of h are presented in Table 6.4, and they show linear convergence to the true solution.

Table 6.4

h	t	$x(t)$	$y_h(t)$	$\lvert x(t) - y_h(t)\rvert$
10^{-2}	1.5	-1.000000	-0.970243	2.976e-02
	3.0	0.000000	0.020626	2.063e-02
	4.5	1.000000	1.000778	7.780e-04
	6.0	0.000000	-0.015040	1.504e-02
	7.5	-1.000000	-0.999821	1.791e-04
10^{-3}	1.5	-1.000000	-0.997056	2.944e-03
	3.0	0.000000	0.001982	1.982e-03
	4.5	1.000000	1.000070	6.982e-05
	6.0	0.000000	-0.001513	1.513e-03
	7.5	-1.000000	-0.999982	1.779e-05
10^{-4}	1.5	-1.000000	-0.999706	2.941e-04
	3.0	0.000000	0.000197	1.974e-04
	4.5	1.000000	1.000007	6.920e-06
	6.0	0.000000	-0.000151	1.514e-04
	7.5	-1.000000	-0.999998	1.784e-06

Table 6.3

h	t	$x(t)$	$y_h(t)$	$\lvert x(t) - y_h(t) \rvert$
10^{-2}	2.0	0.250000	0.250163	1.625e-04
	4.0	0.166667	0.166823	1.563e-04
	6.0	0.125000	0.125195	1.954e-04
	8.0	0.100000	0.099868	1.317e-04
	10.0	0.083333	0.081792	1.542e-03
10^{-3}	2.0	0.250000	0.250016	1.564e-05
	4.0	0.166667	0.166681	1.467e-05
	6.0	0.125000	0.125017	1.703e-05
	8.0	0.100000	0.099984	1.552e-05
	10.0	0.083333	0.083207	1.263e-04
10^{-4}	2.0	0.250000	0.250002	1.558e-06
	4.0	0.166667	0.166668	1.458e-06
	6.0	0.125000	0.125002	1.680e-06
	8.0	0.100000	0.099998	1.570e-06
	10.0	0.083333	0.083321	1.237e-05

Chapter 7
IDENTIFICATION OF PARAMETERS

In this chapter we study the parameter identification (estimation) problem for IVP (3.1)-(3.2). We assume that some parameters (γ) of the equation are unknown, but we have measurements $(X_0, X_1, \ldots, X_l)$ at discrete time values $(t_0, t_1, \ldots, t_l)$ for the solution of the IVP. The goal is to find the parameter value, which minimizes the least squares fit-to-data criterion

$$J(\gamma) = \sum_{i=0}^{l} |x(t_i;\gamma) - X_i|^2, \qquad \gamma \in \Gamma,$$

i.e., which is the best-fit parameter for the measurements. (Denote this problem by $\mathcal{P}$). Problem $\mathcal{P}$ has been studied by many authors, for different classes of differential equations (see e.g. [4] and the references therein), including delay equations as well ([5], [41]).

All the above cited papers use the same idea to find the solution of the optimization problem $\mathcal{P}$:

1) First take finite dimensional approximations of the parameters, γ^N, (i.e., $\gamma^N \in \Gamma^N \subset \Gamma$, $\dim \Gamma^N < \infty$, $\gamma^N \to \gamma$ as $N \to \infty$).

2) Take approximate initial value problems (for $M = 1, 2, \ldots,$) corresponding to parameters from Γ^N, ($N = 1, 2, \ldots$), with solutions $y^M(\cdot;\gamma^N)$, such that $y^M(t,\gamma^N) \to x(t,\gamma)$ as $N, M \to \infty$, uniformly on compact time intervals.

3) Define the least square minimization problems ($\mathcal{P}^{N,M}$) for each $N, M = 1, 2, \ldots$, i.e., find $\gamma^{N,M} \in \Gamma^N$, which minimizes the least squares fit-to-data criterion

$$J^{N,M}(\gamma^N) = \sum_{i=0}^{l} |y^M(t_i;\gamma^N) - X_i|^2, \qquad \gamma^N \in \Gamma^N.$$

4) Assuming that the actual parameters belong to a compact subset of Γ, argue, that the sequence of solutions, $\gamma^{N,M}$ ($N, M = 1, 2, \ldots$), of the finite dimensional minimization problems $\mathcal{P}^{N,M}$ has a convergent subsequence with limit $\bar{\gamma} \in \Gamma$.

5) Show that $\bar{\gamma}$ is the solution of the minimization problem $\mathcal{P}$.

Note, that step 4) and 5) can be argued without using the particular approximation method of the initial value problem, using only compactness arguments and step 2) (see e.g. in [41]).

In Section 7.1 we show that the approximation scheme defined in Chapter 6 has the property required in step 2), and in Section 7.2 we present numerical examples for estimating parameters of IVP (3.1)-(3.2) by applying our approximation scheme and the method described above. We note that the proof of step 2), by using the approximating technique of Chapter 6, is elementary, and it is an easy modification of the proof of Theorem 6.3. On the other hand, in [41], the same proof, using first order spline scheme, requires long and technical argument, especially for the point state-dependent case.

We assume throughout this chapter, that only a part of the delay function Λ and the function f, represented by vector parameters c and d, respectively, and the initial function are unknown in the equation.

7.1 Main results

Consider the delay equation

$$\dot{x}(t) = f\Big(t, x(t), \Lambda(t, x_t, c), d\Big), \qquad t \in [0, T], \tag{7.1}$$

where $c \in \Omega_4$, $d \in \Omega_5$, Ω_4 and Ω_5 are open subsets of $\mathbb{R}^m$, and the corresponding initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{7.2}$$

In this section we use the notations of Sections 4.2 and 4.3, i.e., Λ, λ, $\|\mu\|$ and $\|f\|$ are defined by (4.94), (4.95), (4.97) and (4.116), respectively. We assume that f and μ are given in the equation, but the parts of f and Λ represented by d and c, and the initial function are unknown, i.e., considered as parameters.

Define the parameter space in this section by

$$\Gamma_2 \equiv W^{1,\infty} \times \mathbb{R}^m \times \mathbb{R}^m,$$

and the set of feasible parameters by

$$\Pi_2 \equiv \Big\{(\varphi, c, d) \in W^{1,\infty} \times \Omega_4 \times \Omega_5 \;:\; \varphi(0) \in \Omega_1,\; \varphi \in \Omega_3,\; \int_{-r}^{0} d_s\mu(s, 0, \varphi, c)\varphi(s) \in \Omega_2\Big\}.$$

(See also (3.46).)

We assume that f, φ and μ satisfy (A1'), (A2'), (A3), (A4'), (A5') and (A6). These conditions imply by Theorems 3.8, 4.32 and 4.44 that IVP (7.1)-(7.2) has unique solution on an interval $[0, \alpha]$ for parameters from a neighborhood of (φ, c, d).

Theorem 7.1 *Assume that f, μ and $(\bar{\varphi}, \bar{c}, \bar{d}) \in \Pi_2$ satisfy (A1'), (A2'), (A3), (A4'), (A5') and (A6). Then there exist constants $\alpha > 0$, $\delta > 0$ and $L_3 = L_3(\alpha, \bar{\varphi}, \bar{c}, \bar{d}, \delta)$, such that IVP (7.1)-(7.2) has unique solution on $[0, \alpha]$ for all $\varphi \in W^{1,\infty}$, $c \in \Omega_4$ and $d \in \Omega_5$ with $|\varphi - \bar{\varphi}|_{W^{1,\infty}} + |c - \bar{c}|_{\mathbb{R}^m} + |d - \bar{d}|_{\mathbb{R}^m} < \delta$, and*

$$|x(\cdot; \varphi, c, d)_t - x(\cdot; \bar{\varphi}, \bar{c}, \bar{d})_t|_{W^{1,\infty}} \le L_3\Big(|\varphi - \bar{\varphi}|_{W^{1,\infty}} + |c - \bar{c}|_{\mathbb{R}^m} + |d - \bar{d}|_{\mathbb{R}^m}\Big), \quad t \in [0, \alpha].$$

Let h be a positive constant, and assume that for each $k \in \mathbb{N}$ given a finite dimensional subspace Φ^k of $W^{1,\infty}$, such that for each $\varphi \in W^{1,\infty}$, the projection of φ onto Φ^k, denoted by φ^k, satisfies that $|\varphi^k - \varphi|_{W^{1,\infty}} \to 0$, as $k \to \infty$. Let $c^k, d^k \in \mathbb{R}^m$. Then define the following delay equation with piecewise constant arguments

$$\dot{y}_{h,k}(t) = f\Big([t]_h, y_{h,k}([t]_h), \Lambda([t]_h, (y_{h,k})_{[t]_h}, c^k), d^k\Big), \qquad t \in [0, T], \tag{7.3}$$

with initial condition

$$y_{h,k}(t) = \varphi^k(t), \qquad t \in [-r, 0]. \tag{7.4}$$

Here, to emphasize that the solution corresponds to a given $h > 0$ and (φ^k, c^k, d^k), we denote the solution and the solution segment function of IVP (7.3)-(7.4) by $y_{h,k}(t)$ and $(y_{h,k})_t$, respectively. Lemma 6.1 implies, that for each fixed $h > 0$ and $(\varphi^k, c^k, d^k) \in \Pi_2$, IVP (7.1)-(7.2) has unique solution on some interval $[0, \alpha]$.

We shall need the following lemma.

Lemma 7.2 *Assume that f and μ satisfy (A1'), (A2'), (A4') and (A5'), and $(\varphi, c, d) \in \Pi_2$. Fix sequences $\varphi^k \in \Phi^k$, $c^k \in \Omega_4$ and $d^k \in \Omega_5$ such that $|\varphi^k - \varphi|_{W^{1,\infty}} + |c^k - c|_{\mathbb{R}^m} + |d^k - d|_{\mathbb{R}^m} < \delta$, where $\delta > 0$ is such that $\overline{\mathcal{G}}_{\Gamma_2}((\varphi, c, d); \delta) \subset \Pi_2$. Then we have that:*

(i) *For an arbitrary finite $\alpha \le T$ there exists a constant $K_1 = K_1(\alpha, \delta, \varphi, c, d) > 0$ such that for every $h > 0$ and $k \in \mathbb{N}$ it follows that*

$$|y_{h,k}(t)| \le K_1, \qquad t \in [-r, \alpha]. \tag{7.5}$$

(ii) *For an arbitrary finite $\alpha \le T$ there exists a constant $K_2 = K_2(\alpha, \delta, \varphi, c, d) > 0$ such that for every $h > 0$ and $k \in \mathbb{N}$ it follows that*

$$|y_{h,k}(t) - y_{h,k}(\bar{t})| \le K_2|t - \bar{t}|, \qquad t, \bar{t} \in [-r, \alpha], \tag{7.6}$$

and

$$|(y_{h,k})_t - (y_{h,k})_{\bar{t}}|_C \le K_2|t - \bar{t}|, \qquad t, \bar{t} \in [0, \alpha]. \tag{7.7}$$

(iii) *There exists $\alpha \le T$ and $\bar{\delta} > 0$ such that IVP (7.3)-(7.4) has a unique solution on $[0, \alpha]$ for every $h > 0$ and k such that $|\varphi^k - \varphi|_{W^{1,\infty}} + |c^k - c|_{\mathbb{R}^m} + |d^k - d|_{\mathbb{R}^m} < \bar{\delta}$.*

Proof The proof follows the steps of that of Lemma 6.2. It is easy to see that $K_1 \equiv |\varphi|_C + \delta + \|f\|\alpha$ satisfies (7.5).

To find K_2, let $t, \bar{t} \in [-r, 0]$. Then by (A6) and Lemma 2.3 we have that

$$\begin{aligned} |y_{h,k}(t) - y_{h,k}(\bar{t})| &= |\varphi^k(t) - \varphi^k(\bar{t})| \\ &\le |\varphi^k|_{W^{1,\infty}}|t - \bar{t}| \\ &\le (|\varphi|_{W^{1,\infty}} + \delta)|t - \bar{t}|. \end{aligned}$$

For $t, \bar{t} \ge 0$ it follows from (7.3) that $|y_{h,k}(t) - y_{h,k}(\bar{t})| \le \|f\||t - \bar{t}|$. Then, clearly, $K_2 \equiv \max\{|\varphi|_{W^{1,\infty}} + \delta, \|f\|\}$ satisfies (7.6), and thus (7.7) as well.

Inequality (7.6) yields for arbitrary $h > 0$ that $|y_{h,k}(t) - \varphi(0)| \le K_2 t$, which, by using that $\varphi(0) \in \Omega_1$ and Ω_1 is open, implies that there exists $\alpha_1 \le T$ such that $y_{h,k}(t) \in \Omega_1$ for $t \in [0, \alpha_1]$ and for all $h > 0$ and $k \in \mathbb{N}$.

Since $y_{h,k}$ is a piecewise linear function, it follows that $(y_{h,k})_u \in W^{1,\infty}$ for all $u \in [0, \alpha_1]$, and it is easy to see that $|(y_{h,k})_u|_{1,\infty} \le M_1 \equiv \max\{\|f\|, |\varphi|_{W^{1,\infty}} + \delta, K_1\}$. By (6.2), (7.5), (7.6), (7.7), and Lemma 4.31 with the constant $L_2 = L_2(\alpha, M_1, |c|_{\mathbb{R}^m} + \delta)$ we have that

$$\begin{aligned} &|\Lambda([u]_h, (y_{h,k})_{[u]_h}, c^k) - \Lambda(0, (y_{h,k})_0, c^k)| \\ &\le |\Lambda([u]_h, (y_{h,k})_{[u]_h}, c^k) - \Lambda([u]_h, \varphi, c)| + |\Lambda([u]_h, \varphi, c) - \Lambda(0, \varphi, c)| \\ &\quad + |\Lambda(0, \varphi, c) - \Lambda(0, (y_{h,k})_0, c^k)| \\ &\le (\|\mu\| + L_2|\varphi|_{W^{1,\infty}})\Big(|(y_{h,k})_{[u]_h} - \varphi|_C + |c - c^k|_{\mathbb{R}^m}\Big) \\ &\quad + |\Lambda([u]_h, \varphi, c) - \Lambda(0, \varphi, c)| + (\|\mu\| + L_2|\varphi|_{W^{1,\infty}})\Big(|\varphi - \varphi^k|_C + |c - c^k|_{\mathbb{R}^m}\Big) \\ &\le (\|\mu\| + L_2 M_1)\Big(|(y_{h,k})_{[u]_h} - (y_{h,k})_0|_C + |\varphi^k - \varphi|_C + |c - c^k|_{\mathbb{R}^m}\Big) \\ &\quad + |\Lambda([u]_h, \varphi, c) - \Lambda(0, \varphi, c)| + (\|\mu\| + L_2 M_1)\Big(|\varphi - \varphi^k|_C + |c - c^k|_{\mathbb{R}^m}\Big) \\ &\le (\|\mu\| + L_2 M_1)\Big(K_2[u]_h + 2|c - c^k|_{\mathbb{R}^m} + 2|\varphi - \varphi^k|_{W^{1,\infty}}\Big) + |\Lambda([u]_h, \varphi, c) - \Lambda(0, \varphi, c)|, \end{aligned}$$

therefore, using (6.1), Lemma 2.8 and the facts that Ω_2 and Ω_5 are open, there exist $0 < \alpha_2 \leq \alpha_1$ and $\bar{\delta} > 0$ such that the third and fourth arguments of f in (7.3) remain in Ω_2 and Ω_5, respectively, for $t \in [0, \alpha_2]$, $h > 0$ and for k such that $|\varphi^k - \varphi|_{W^{1,\infty}} + |c^k - c|_{\mathbb{R}^m} + |d^k - d|_{\mathbb{R}^m} < \bar{\delta}$. It follows from (7.7) that

$$|(y_{h,k})_t - \varphi|_C \leq K_2 t, \qquad t \in [0, \alpha_2],$$

therefore there exist $\alpha \leq \alpha_2$ such that $(y_{h,k})_t \in \Omega_3$ for $t \in [0, \alpha]$ and for all $h > 0$. Finally, it is easy to show, by using the method of steps, that for k such that $|\varphi^k - \varphi|_{W^{1,\infty}} + |c^k - c|_{\mathbb{R}^m} + |d^k - d|_{\mathbb{R}^m} < \bar{\delta}$, and for all $h > 0$, IVP (7.3)-(7.4) has unique solution on $[0, \alpha]$. □

The following theorem guarantees step 2) of the identification method described in the introduction of this chapter, using the approximation method of Chapter 6.

Theorem 7.3 *Assume that f, μ and φ satisfy (A1'), (A2'), (A3), (A4'), (A5') and (A6). Let $(\varphi, c, d) \in \Pi_2$, and fix sequences $\varphi^k \in \Phi^k$, $c^k \in \Omega_4$, and $d^k \in \Omega_5$ such that $|\varphi^k - \varphi|_{W^{1,\infty}} \to 0$, $|c^k - c|_{\mathbb{R}^m} \to 0$, and $|d^k - d|_{\mathbb{R}^m} \to 0$ as $k \to \infty$, and let $\alpha > 0$ be the constant from Lemma 7.2 (iii). Then the solution, $y_{h,k}$, of IVP (7.3)-(7.4) converges uniformly on $[0, \alpha]$ to the solution, x, of IVP (7.1)-(7.2) as $h \to 0^+$ and $k \to \infty$, i.e.,*

$$\lim_{\substack{h\to 0^+ \\ k\to\infty}} \max_{0\leq t\leq\alpha} |x(t) - y_{h,k}(t)| = 0.$$

Proof We follow the steps of the proof of Theorem 6.3.

Let $\bar{\delta}$ be the constant from Lemma 7.2 (iii), and we assume throughout the proof that k is large enough that $|\varphi^k - \varphi|_{W^{1,\infty}} + |c^k - c|_{\mathbb{R}^m} + |d^k - d|_{\mathbb{R}^m} < \bar{\delta}$. Let $K_1 = K_1(\alpha, \bar{\delta}, \varphi, c, d)$ and $K_2 = K_2(\alpha, \bar{\delta}, \varphi, c, d)$ be the constants from Lemma 7.2 (i) and (ii), respectively. Define

$$M \equiv \max\{\|\mu\|, 1\} \cdot \max\Big\{|x|_{W^{1,\infty}_\alpha}, K_1\Big\},$$

Then the definition of M, inequalities (6.7) and (2.5) imply that x_t, $(y_{h,k})_t$ and $\Lambda(t, x_t)$, $\Lambda(t, (y_{h,k})_t)$ remain in $\bar{G}_C(M)$ for $t \in [0, \alpha]$. Let $L_1 = L_1(\alpha, M)$ be the constant given by (A4'). Then equation (6.6), assumption (A4') and standard estimates yield the following inequalities

$$\begin{aligned}
&|x(t) - y_{h,k}(t)| \\
&\quad\leq |\varphi(0) - \varphi^k(0)| + \int_0^t \Big| f\Big(u, x(u), \Lambda(u, x_u, c), d\Big) - f\Big([u]_h, x(u), \Lambda(u, x_u, c), d\Big) \Big| \, du \\
&\qquad + \int_0^t \Big| f\Big([u]_h, x(u), \Lambda(u, x_u, c), d\Big) - f\Big([u]_h, y_{h,k}([u]_h), \Lambda([u]_h, (y_{h,k})_{[u]_h}, c^k), d^k\Big) \Big| \, du \\
&\quad\leq |\varphi - \varphi^k|_C + \int_0^t \Big| f\Big(u, x(u), \Lambda(u, x_u, c), d\Big) - f\Big([u]_h, x(u), \Lambda(u, x_u, c), d\Big) \Big| \, du \qquad (7.8) \\
&\qquad + \int_0^t L_1 \Big(|x(u) - y_{h,k}([u]_h)| + \Big|\Lambda(u, x_u, c) - \Lambda([u]_h, (y_{h,k})_{[u]_h}, c^k)\Big| + |d - d^k|_{\mathbb{R}^m} \Big) \, du.
\end{aligned}$$

Similarly to (6.13) and (6.14), inequalities (7.6) and (6.2) imply that

$$|x(u) - y_{h,k}([u]_h)| \leq |x(u) - y_{h,k}(u)| + K_2 h, \qquad u \in [-r, \alpha], \tag{7.9}$$

and

$$|x_u - (y_{h,k})_{[u]_h}|_C \le |x_u - (y_{h,k})_u|_C + K_2 h, \qquad u \in [0,\alpha]. \tag{7.10}$$

Using (7.9), (7.10), Lemma 4.31 with $L_2 = L_2(\alpha, M, |c|_{\mathbb{R}^m} + \bar{\delta})$, we can estimate the last term in the right hand side of (7.8) as follows.

$$\begin{aligned}
&\Big|\Lambda(u, x_u, c) - \Lambda([u]_h, (y_{h,k})_{[u]_h}, c^k)\Big| \\
&\le |\Lambda(u, x_u, c) - \Lambda([u]_h, x_u, c)| + \Big|\Lambda([u]_h, x_u, c) - \Lambda([u]_h, (y_{h,k})_{[u]_h}, c^k)\Big| \\
&\le |\Lambda(u, x_u, c) - \Lambda([u]_h, x_u, c)| + (\|\mu\| + L_2|x_u|_{W^{1,\infty}})\Big(|x_u - (y_{h,k})_{[u]_h}|_C + |c - c^k|_{\mathbb{R}^m}\Big) \\
&\le |\Lambda(u, x_u, c) - \Lambda([u]_h, x_u, c)| + (\|\mu\| + L_2 M)|x_u - (y_{h,k})_u|_C \\
&\quad + (\|\mu\| + L_2 M)\Big(K_2 h + |c - c^k|_{\mathbb{R}^m}\Big).
\end{aligned} \tag{7.11}$$

By combining (7.8), (7.9) and (7.11) we get

$$|x(t) - y_{h,k}(t)| \le g_{h,k}(t) + \int_0^t L_1(1 + L_2 M + \|\mu\|) \max_{0\le s\le u} |x(s) - y_{h,k}(s)|\, du, \tag{7.12}$$

where

$$\begin{aligned}
g_{h,k}(t) \;&\equiv\; \int_0^t \Big| f\Big(u, x(u), \Lambda(u, x_u, c), d\Big) - f\Big([u]_h, x(u), \Lambda(u, x_u, c), d\Big)\Big| du \\
&\quad + \int_0^t |\Lambda(u, x_u, c) - \Lambda([u]_h, x_u, c)|\, du + L_1(1 + L_2 M + \|\mu\|) K_2 h t \\
&\quad + L_1\Big((L_2 M + \|\mu\|)|c - c^k|_{\mathbb{R}^m} + |d - d^k|_{\mathbb{R}^m}\Big) t + |\varphi - \varphi^k|_C.
\end{aligned} \tag{7.13}$$

Then (7.12), Lemma 2.14 and the Gronwall-Bellman inequality imply that

$$\max_{0\le s\le t} |x(s) - y_{h,k}(s)| \le g_{h,k}(\alpha) \exp\Big(L_1(1 + L_2 M + \|\mu\|)t\Big), \qquad t \in [0,\alpha]. \tag{7.14}$$

As in the proof of Theorem 6.3, by using the Lebesgue Dominated Convergence Theorem for the first two integrals in (7.13), and the assumptions that $\psi^k \to \psi$, $c^k \to c$, and $d^k \to d$ as $k \to \infty$, we get that $g_{h,k}(\alpha) \to 0$ as $h \to 0^+$ and $k \to \infty$, which finishes the proof of the theorem. □

7.2 Numerical examples

In this section we present applications of the identification method described in the introduction and in Section 7.1. Consider an identification problem corresponding to IVP (7.1)-(7.2), then we define the approximating IVPs by (7.3)-(7.4). Define the corresponding finite dimensional minimization problems, and find the solutions of them. Choose small enough h and large enough k, and use the solution of the minimization problem corresponding to this h and k as an approximation of the solution of the original identification problem.

We note, that in each example we used the built in numerical minimization routine of Mathematica (which does not require the knowledge of the derivative of the minimizing function) for solving the finite dimensional minimization problems, i.e., for computing the minimum of

Table 7.1

t_i	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
X_i	1.1300	1.5003	1.9921	2.6451	3.5121	4.6632	6.1917	8.2212	10.915

Table 7.2

h	$\bar{\tau}$	$J_h(\bar{\tau})$	steps
0.050	0.968794	0.0174081	21
0.010	0.988730	0.0174045	21
0.005	0.991233	0.0174043	21
0.001	0.993233	0.0174042	21

the least square cost functions, and we also used Mathematica for evaluating the cost function for each required value of the parameter, i.e., for computing the solution of IVP (7.3)-(7.4).

Example 7.4 Consider the scalar delay equation

$$\dot{x}(t) = x(t-\tau), \qquad t \in [0,4], \tag{7.15}$$

where we assume that $\tau \in [0.2, 3.0]$, with initial condition

$$x(t) = 1, \qquad t \in [-3,0]. \tag{7.16}$$

The solution of this IVP corresponding to $\tau = 1.0$ is

$$x(t;1) = \sum_{i=0}^{[t]} \frac{(t-i)^i}{i!}.$$

We used this formula to generate the "measured data" corresponding to the following time values presented in Table 7.1.

Since the parameter is one dimensional, there is no need for discretizing the parameter space. Let $h > 0$ and define the approximating IVP

$$\dot{y}_h(t) = y_h([t]_h - \tau), \qquad t \in [0,4], \tag{7.17}$$

where we assume that $\tau \in [0.2, 3.0]$, with initial condition

$$x(t) = 1, \qquad t \in [-3,0]. \tag{7.18}$$

Consider the minimization problem: minimize

$$J_h(\tau) = \sum_{i=1}^{9} (y_h(t_i;\tau) - X_i)^2, \qquad \tau \in [0.2, 3.0],$$

where $y_h(t;\tau)$ is the solution of (7.17)-(7.18). We present the numerical solution of these minimization problems in Table 7.2 for different h values. We print out the computed $\bar{\tau}$, which minimizes the cost function $J_h(\tau)$, the value of the cost function at $\bar{\tau}$, and the number of steps

Table 7.3

t_i	0.0	0.5	1.0	1.5	2.0	2.5	3.0
X_i	0.000000	0.229849	0.708073	0.994996	0.826822	0.358169	0.0199149

Table 7.4

h	$\bar{a}$	$\bar{b}$	$J_h(\bar{a}, \bar{b})$	steps
0.050	-1.02202	2.02076	0.00093899	92
0.010	-1.00427	2.00415	0.00003634	95
0.005	-1.00213	2.00208	0.00000905	93
0.001	-1.00042	2.00042	0.00000036	94

done by the numerical minimization routine to reach the minimum value (in each case the starting two value (required by the routine) for τ are 2.5 and 1.5).

Example 7.5 Consider the scalar delay equation with state-dependent delay

$$\dot{x}(t) = ax^2(t - |x(t)|) + \sin(bt) + \sin^4(t - \sin^2(t)), \qquad t \in [0, 3], \tag{7.19}$$

with initial condition

$$x(t) = \sin^2(t), \qquad t \le 0, \tag{7.20}$$

where a and b are unknown parameters, but we assume that $a, b \in [-5, 5]$. It is easy to see that the solution of IVP (7.19)-(7.20) corresponding to parameter values $a = -1.0$ and $b = 2.0$ is $x(t; -1, 2) = \sin^2(t)$. We used this function to generate data shown in Table 7.3.

The approximating equation corresponding to (7.19) is

$$\dot{y}_h(t) = ay_h^2([t]_h - |y_h([t]_h)|) + \sin(b[t]_h) + \sin^4\Big([t]_h - \sin^2([t]_h)\Big), \qquad t \in [0, 3].$$

The minimizing function is

$$J_h(a, b) = \sum_{i=1}^{7} (y_h(t_i; a, b) - X_i)^2, \qquad a, b, \in [-5, 5].$$

Table 7.4 contains the numerical runnings corresponding to this equation. We used the starting values 2.5 and 1.5 for both a and b in the numerical optimization routine of Mathematica in each cases.

Example 7.6 Consider the scalar equation

$$\dot{x}(t) = x\left(t - 1 - \frac{1}{t+1}\right), \qquad t \in [0, 2], \tag{7.21}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-2, 0]. \tag{7.22}$$

It is easy to check that the solution of IVP (7.21)-(7.22) with initial function

$$\varphi(t) = \begin{cases} \frac{2}{3}(t+2), & -2 \le t \le -0.5, \\ 1, & -0.5 \le t \le 0 \end{cases} \tag{7.23}$$

Table 7.5

t_i	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75
X_i	1.00000	1.02311	1.10469	1.26755	1.5379	1.7879	2.0379	2.2879

Table 7.6

h	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$J_h(\bar{a}_1, \bar{a}_2, \bar{a}_3)$	steps
0.050	-0.296021	0.880291	1.01774	0.002681	145
0.010	-0.350648	0.863814	1.01823	0.002874	150
0.005	-0.357622	0.861696	1.01828	0.002897	148
0.001	-0.363224	0.859983	1.01832	0.002915	140

is

$$x(t) = \begin{cases} 1 + \frac{2}{3}t + \frac{t^3}{3} - \frac{2}{3}\log(t+1), & t \in [0,1], \\ 1 - \frac{2}{3}\log 2 + t, & t \in [1,2]. \end{cases}$$

We generate measurements by using this function (see Table 7.5).

Consider the corresponding approximate equation

$$\dot{y}_h(t) = y_h\left([t]_h - 1 - \frac{1}{[t]_h + 1}\right), \qquad t \in [0,2].$$

First we approximate the unknown initial function on $[-2, 0]$ by linear spline functions with three node points at -2, -1 and 0, with corresponding values a_1, a_2 and a_3 at the node points. (It is known that sufficiently smooth functions can be approximated by linear spline functions in the $W^{1,\infty}_\alpha$ norm, see e.g. [44].) We assume that the parameter values satisfy $a_i \in [-4, 4]$, $i = 1, 2, 3$. Then the parameter space is three dimensional. The corresponding minimizing function is of three variables:

$$J_h(a_1, a_2, a_3) = \sum_{i=1}^{8}(y_h(t_i; a_1, a_2, a_3) - X_i)^2, \qquad a_1, a_2, a_3, \in [-4, 4].$$

We present the numerical solution of this problem in Table 7.6 for several h values.

Next we consider linear spline approximation of the initial function with node points at -2, -1.5, -1, -0.5 and at 0, and with the corresponding values a_i ($i = 1, 2, \ldots, 5$) at these points. Then the parameter space is five dimensional, and the minimizing function is

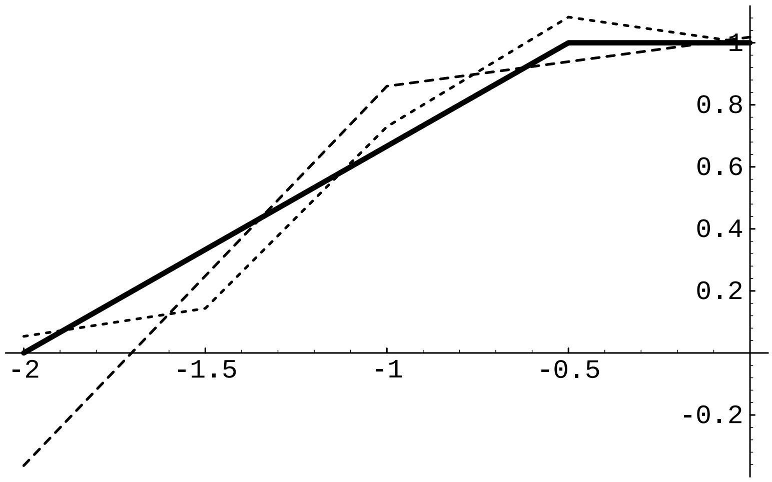
$$J_h(a_1, \ldots, a_5) = \sum_{i=1}^{8}(y_h(t_i; a_1, \ldots, a_5) - X_i)^2, \qquad a_i \in [-4, 4], \quad i = 1, 2, \ldots, 5.$$

The following numerical results are shown in Table 7.7. In Figure 7.1 we plotted the true initial function, defined by (7.23) (solid line), and the computed approximate initial functions with three and five node points (dotted linear splines) corresponding to $h = 0.001$.

Table 7.7

h	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$J_h(\bar{a}_1, \ldots, \bar{a}_5)$	steps
0.050	0.0565933	0.165824	0.782140	1.07451	0.997993	0.00051494	317
0.010	0.0536655	0.147732	0.739048	1.08165	0.997858	0.00047607	337
0.005	0.0540436	0.145239	0.733611	1.08239	0.997840	0.00047021	322
0.001	0.0534458	0.143636	0.729410	1.08284	0.997831	0.00046543	314

Figure 7.1



Chapter 8

WELL-POSEDNESS IN L^p

In many applications (e.g. in control theory) we can not assume the continuity of the function f in equation (3.1), therefore it is important to extend well-posedness results for the case when the functions $f(\cdot, x, y)$ and $\tau(\cdot, \psi)$ are L^p functions only. In this case it turns out, that the natural state-space for solutions is a product space of the form $\mathbb{R}^n \times L^p$ (see e.g. [8], [9] or [35] for L^p theory of delay equations).

In [34] Ito and Kappel studied the well-posedness and approximation of semilinear Cauchy problems, in particular, the delay system

$$\dot{x}(t) = f\Big(t,\ x(t),\ x(t - \tau(t, x_t))\Big) \tag{8.1}$$

in the state-space $\mathbb{R}^n \times L^p$. They proved an abstract well-posedness result, and used it to prove well-posedness of (8.1). We state their results in Section 8.1, and in Section 8.2 we show how it can be applied to our problem, to the state-dependent delay system

$$\dot{x}(t) = f\left(t, x(t), \int_{-r}^{0} d_s\mu(s, t, x_t)\, x(t + s)\right), \qquad t \in [0, T], \tag{8.2}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{8.3}$$

8.1 An abstract well-posedness result of Ito and Kappel

In this section we state the abstract well-posedness result of [34].

Fix $T > 0$ and $1 \le p < \infty$. Let $W \subset V \subset H$ and U be Banach spaces such that the embedding $V \subset H$ is dense and continuous, the embedding $W \subset V$ is just continuous. Consider the equation in the space V:

$$x(t) = S(t)\varphi + \int_0^t S(t - s)BF(s, x(s))\, ds, \qquad 0 \le t \le T, \quad \varphi \in W. \tag{8.4}$$

A function $x : [0, T] \to V$ is called a solution of (8.4) if x is continuous and satisfies (8.4) on $[0, T]$.

We have the following assumptions:

(B1) $\{S(t) : t \ge 0\}$ is a C_0-semigroup on H, which leaves the spaces V and W invariant. Moreover, $S(t)|_V$, $t \ge 0$ is a C_0-semigroup on V and we define $M_0 \equiv \max_{0 \le t \le T} \|S(t)\|_{\mathcal{L}(V)}$. We also assume that there exists a constant $M_1 > 0$ such that $\|S(t)\|_{\mathcal{L}(W)} \le M_1$, $0 \le t \le T$, (B2) $B \in \mathcal{L}(U, H)$ and there exist nonnegative constants M_2 and M_3 such that

(i) $\left| \int_0^t S(t-s)Bf(s)\,ds \right|_V \le M_2 |f|_{L^p([0,T],U)}, \quad 0 \le t \le T, \quad f \in L^p([0,T],U),$

and

(ii) $\left| \int_0^t S(t-s)Bf(s)\,ds \right|_W \le M_3 |f|_{L^\infty([0,T],U)}, \quad 0 \le t \le T, \quad f \in L^\infty([0,T],U),$

(B3) F is a mapping $[0,T] \times V \to U$. For any $M > 0$ there exists a constant $K = K(M) > 0$ such that

(i) $|F(t,\psi) - F(t,\bar{\psi})|_U \le K(1 + |\bar{\psi}|_W)|\psi - \bar{\psi}|_V$ a.e. on $0 \le t \le T$ for all $\psi, \bar{\psi} \in V$ with $\bar{\psi} \in W$ and $\psi, \bar{\psi} \in \overline{\mathcal{G}}_V(M)$,

and

(ii) $|F(t,\psi)|_U \le K(1 + |\psi|_V)$ a.e. on $0 \le t \le T$ for all $\psi \in \overline{\mathcal{G}}_V(M)$. Moreover, for any $\psi \in V$ the mapping $t \to F(t,\psi)$ is strongly measurable on $[0,T]$,

(B4) For any $M > 0$ there exists a constant $K = K(M) > 0$ such that

$$|F(t,\psi) - F(\bar{t},\psi)|_U \le K(1 + |\psi|_W)|t - \bar{t}| \quad \text{for a.e. } t, \bar{t} \in [0,T], \quad \text{all } \psi \in \overline{\mathcal{G}}_V(M) \cap W.$$

We note, that the inequality

$$|F(t,\psi) - F(\bar{t},\bar{\psi})|_U \le K(1 + |\bar{\psi}|_W)\left(|t - \bar{t}| + |\psi - \bar{\psi}|_V\right), \tag{8.5}$$

for a.e. $0 \le t, \bar{t} \le T, \quad \bar{\psi} \in W, \psi \in V, \quad \psi, \bar{\psi} \in \overline{\mathcal{G}}_V(M)$ implies (B3) (i) and (B4).

The infinitesimal generator of $S(\cdot)$ (considered as a C_0-semigroup over H) and its domain are denoted by A_H and $\mathrm{dom} A_H$, respectively.

Under the above assumptions the following theorems hold:

Theorem 8.1 (see Theorem 2.2 in [34]) *Assume that (B1)–(B3) are satisfied and let $R > 0$ be given. Then there exists $\alpha = \alpha(R) > 0$ such that equation (8.4) for any $\varphi \in W$ satisfying $|\varphi|_W \le R$ has a unique solution $x(\cdot;\varphi) = \mathcal{S}\varphi \in C([0,\alpha],V) \cap L^\infty([0,\alpha],W)$. Moreover, $\mathcal{S}$ is a Lipschitzean mapping on $\{\varphi \in W : |\varphi|_W \le R\}$ into $C([0,\alpha],V)$ and also into $L^\infty([0,\alpha],W)$.*

Theorem 8.2 (see Theorem 2.4 in [34]) *Assume that (B1)–(B4) are satisfied and the space U is reflexive. Furthermore let $\varphi \in W \cap \mathrm{dom} A_H$ such that $A_H\varphi + BF(0,\varphi) \in V$. Then the unique solution of (8.4) is in $C^1([0,\alpha],V) \cap C([0,\alpha],\mathrm{dom} A_H)$ for any closed subinterval $[0,\alpha]$ of the maximal interval of existence for $x(\cdot)$, and*

$$\dot{x}(t) = A_H x(t) + BF(t, x(t)), \qquad 0 \le t \le \alpha$$

in H.

8.2 Well-posedness

Next we list the assumptions on the parameters of IVP (8.2)-(8.3) which guarantee the well-posedness of IVP (8.2)-(8.3).

(C1) The function $f : [0,T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ satisfies:

(i) For any $M > 0$ there exists a constant $L_1 = L_1(M) > 0$ such that

$$|f(t,x,y) - f(\bar{t},\bar{x},\bar{y})| \le L_1\Big(|t-\bar{t}| + |x-\bar{x}| + |y-\bar{y}|\Big),$$

for a.e. $0 \le t, \bar{t} \le T$, and all $x, \bar{x} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_1$ and $y, \bar{y} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_2$,

(ii) there exists a constant $N \ge 0$ such that

$$|f(t,0,0)| \le N, \qquad \text{a.e. } 0 \le t \le T,$$

and

(iii) the function $t \mapsto f(t,x,y)$ is measurable on $[0,T]$ for any $x \in \Omega_1$, $y \in \Omega_2$.

(C2) The function $\mu(\cdot,t,\psi) : [-r,0] \to \mathbb{R}^{n\times n}$ is defined, and is of bounded-variation for all $t \in [0,T]$ and $\psi \in \Omega_3$ and it satisfies:

(i) $\|\mu\| \equiv \operatorname{ess\,sup}\{|\lambda(t,\psi,\xi)| \;:\; \text{a.e. } t \in [0,T], \text{ all } \psi \in \Omega_3,\ \xi \in \overline{\mathcal{G}}_C(1)\} < \infty$,

(ii) for every finite $\alpha > 0$ with $\alpha \le T$, and $M > 0$ there exists a constant $L_2 = L_2(\alpha, M) > 0$ such that for a.e. $t, \bar{t} \in [0,\alpha]$, and for all $\xi \in W^{1,\infty} \cap \Omega_3$, and $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M) \cap \Omega_3$ it follows that

$$|\lambda(t,\psi,\xi) - \lambda(\bar{t},\bar{\psi},\xi)| \le L_2|\xi|_{W^{1,\infty}}\Big(|t-\bar{t}| + |\psi-\bar{\psi}|_C\Big),$$

and

(iii) for all $\psi, \xi \in \Omega_3$ the function $t \mapsto \lambda(t,\psi,\xi)$ is measurable on $[0,T]$.

(C3) The initial function $\varphi \in W^{1,\infty} \cap C$.

Next we show that under natural conditions the functions f and μ defined in Examples 1.1–1.4 satisfy assumptions (C1) and (C2), respectively.

Example 8.3 Consider Example 1.1, where

$$\mu(s,t,\psi) = \sum_{k=1}^{m} A_k \chi_{[-\tau_k,0]}(s)$$

and

$$f(t,x,y) = A_0 x + y.$$

Then, clearly, f satisfies (C1), and μ is of bounded variation, and satisfies (C2) (i). We have that $\lambda(t,\psi,\xi) = \sum_{k=1}^{m} A_k \xi(-\tau_k)$ is independent of t and ψ, therefore (C2) (ii) and (iii) are satisfied as well.

Example 8.4 Consider the functions

$$\mu(s,t,\psi) = \sum_{k=1}^{m} A_k(t)\chi_{[-\tau_k(t),0]}(s) + \tilde{\mu}(s,t),$$

where

$$\tilde{\mu}(s,t) \equiv \begin{cases} 0, & s \in [-r,-\tau_0], \\ \int_{-\tau_0}^{s} G(u,t)\,du, & s \in (-\tau_0,0], \end{cases}$$

and

$$f(t,x,y) = A_0(t)x + y,$$

as defined in Example 1.2. The assumptions

(i) $A_k(\cdot) \in W^{1,\infty}([0,T],\mathbb{R}^{n\times n}), \qquad k = 0,1,\ldots,m,$

(ii) $\tau_k(\cdot) \in W^{1,\infty}([0,T],\mathbb{R}), \qquad k = 1,2,\ldots,m,$

(iii) the function $[-\tau_0,0]\times[0,T] \to \mathbb{R}^{n\times n} \;:\; (s,t) \mapsto G(s,t)$ is measurable,

(iv) $\|G(s,t) - G(s,\bar{t})\| \le g(s)|t-\bar{t}|$, for $s \in [-\tau_0,0]$, $t,\bar{t} \in [0,T]$, where $g \in L^1([-\tau_0,0];\mathbb{R})$,

(v) $\|G(s,t)\| \le g_0(s)$, for $s \in [-\tau_0,0]$, $t \in [0,T]$, where $g_0 \in L^1([-\tau_0,0];\mathbb{R})$

imply conditions (C1) and (C2). This example is included in Example 8.6, therefore the proof is omitted here.

Example 8.5 Consider

$$\mu(s,t,\psi) = \chi_{[-\tau(t,\psi),0]}(s)I$$

as defined in Example 1.3 with the corresponding function

$$\lambda(t,\psi,\xi) = \xi(-\tau(t,\psi)).$$

Then it is easy to see that (C2) (i) is satisfied. Assume that

(i) the function $\tau(\cdot,\psi)$ is measurable for all $\psi \in \Omega_3$,

(ii) τ is locally Lipschitz-continuous in ψ, i.e., for all $M > 0$ there exists a constant $L_\tau = L_\tau(M)$ such that $|\tau(t,\psi) - \tau(\bar{t},\bar{\psi})| \le L_\tau(|t-\bar{t}| + |\psi-\bar{\psi}|_C)$, for a.e. $0 \le t,\bar{t} \le T$ and all $\psi,\bar{\psi} \in \overline{\mathcal{G}}_C(M)\cap\Omega_3$.

Then it follows from (ii) and Lemma 2.3 for $\xi \in W^{1,\infty}$ and $\psi,\bar{\psi} \in \overline{\mathcal{G}}_C(M)\cap\Omega_3$ that

$$\begin{aligned} |\lambda(t,\psi,\xi) - \lambda(\bar{t},\bar{\psi},\xi)| &= |\xi(-\tau(t,\psi)) - \xi(-\tau(\bar{t},\bar{\psi}))| \\ &\le |\xi|_{W^{1,\infty}}|\tau(t,\psi) - \tau(\bar{t},\bar{\psi})| \\ &\le |\xi|_{W^{1,\infty}} L_\tau\Big(|t-\bar{t}| + |\psi-\bar{\psi}|_C\Big), \qquad \text{for a.e. } t,\bar{t} \in [0,T], \end{aligned}$$

so condition (C2) (ii) holds. Assumption (i) implies condition (C2) (iii).

Example 8.6 Let

$$\mu(s,t,\psi) = \sum_{k=1}^{m} A_k(t)\chi_{[-\tau_k(t,\psi),0]}(s) + \tilde{\mu}(s,t,\psi),$$

with

$$\tilde{\mu}(s,t,\psi) \equiv \begin{cases} 0, & s \in [-r, -\tau_0], \\ \int_{-\tau_0}^{s} G(u,t,\psi)\,du, & s \in (-\tau_0, 0], \end{cases}$$

and $f(t,x,y) = A_0(t)x + y$, as in Example 1.4.

We shall show that the assumptions

(i) $A_k(\cdot) \in W^{1,\infty}([0,T], \mathbb{R}^{n\times n}), \qquad k = 0,1,\ldots,m,$

(ii) the functions $\tau_k(\cdot,\psi) : [0,T] \to \mathbb{R}$ are measurable for all $\psi \in \Omega_3, \quad k = 1,2,\ldots,m,$

(iii) τ_k is locally Lipschitz-continuous in ψ, for all $k = 1,2,\ldots$, i.e., for all $M > 0$ there exists a constant $L_\tau = L_\tau(M)$ such that $|\tau_k(t,\psi) - \tau_k(\bar{t},\bar{\psi})| \le L_\tau(|t-\bar{t}| + |\psi - \bar{\psi}|_C)$, for a.e. $0 \le t, \bar{t} \le T$ and all $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M) \cap \Omega_3, \quad k = 1,2,\ldots,m,$

(iv) the function $[-\tau_0, 0] \times [0,T] \to \mathbb{R}^{n\times n} : (s,t) \mapsto G(s,t,\psi)$ is measurable for all $\psi \in \Omega_3$,

(v) $\|G(s,t,\psi) - G(s,\bar{t},\bar{\psi})\| \le g(s)\Big(|t-\bar{t}| + |\psi - \bar{\psi}|_C\Big)$, for $s \in [-\tau_0, 0]$, $t, \bar{t} \in [0,T]$, and $\psi, \bar{\psi} \in \Omega_3$, where $g \in L^1([-\tau_0,0];\mathbb{R})$,

(vi) $\|G(s,t,\psi)\| \le g_0(s)$, for all $t \in [0,T]$, $\psi \in \Omega_3$, where $g_0 \in L^1([-\tau_0,0];\mathbb{R})$

imply conditions (C1) and (C2).

Let $x, \bar{x} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_1$ and $y, \bar{y} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(M) \cap \Omega_2$. Then the triangle inequality and the definition of the norm $\|\cdot\|_{W^{1,\infty}([0,T],\mathbb{R}^{n\times n})}$ yield the following inequalities

$$\begin{aligned} &|f(t,x,y) - f(\bar{t},\bar{x},\bar{y})| \\ &\quad\le \|A_0(t) - A_0(\bar{t})\||x| + \|A_0(\bar{t})\||x-\bar{x}| + |y-\bar{y}| \\ &\quad\le \|A_0(t) - A_0(\bar{t})\|M + \|A_0(\bar{t})\||x-\bar{x}| + |y-\bar{y}| \\ &\quad\le \|A_0\|_{W^{1,\infty}([0,T],\mathbb{R}^{n\times n})} M|t-\bar{t}| + \|A_0\|_{W^{1,\infty}([0,T],\mathbb{R}^{n\times n})}|x-\bar{x}| + |y-\bar{y}|. \end{aligned}$$

Therefore condition (C1) (i) holds. Condition (C1) (ii) is satisfied with $N = 0$. The assumed measurability of A_0 implies (C1) (iii).

We have seen in Example 1.2 that if $A_k(\cdot)$ are bounded functions on $[0,T]$ and $\|G(s,t,\psi)\| \le g_0(s)$ for $s \in [-\tau_0, 0]$, $t \in [0,T]$, $\psi \in \Omega_3$, where $g_0(s)$ is integrable on $[-\tau_0, 0]$, then μ satisfies (C2) (i). The corresponding λ is

$$\lambda(t,\psi,\xi) = \sum_{k=1}^{m} A_k(t)\xi(-\tau_k(t,\psi)) + \int_{-\tau_0}^{0} G(s,t,\psi)\xi(s)\,ds,$$

therefore for $\xi \in W^{1,\infty}$ simple estimates yield

$$\begin{aligned} &|\lambda(t,\psi,\xi) - \lambda(\bar{t},\bar{\psi},\xi)| \\ &\quad\le \sum_{k=1}^{m} \|A_k(t)\||\xi(-\tau_k(t,\psi)) - \xi(-\tau_k(\bar{t},\bar{\psi}))| + \sum_{k=1}^{m} \|A_k(t) - A_k(\bar{t})\||\xi(-\tau_k(\bar{t},\bar{\psi}))| \\ &\qquad + \int_{-\tau_0}^{0} \|G(s,t,\psi) - G(s,\bar{t},\bar{\psi})\||\xi(s)|\,ds \end{aligned}$$

$$
\begin{aligned}
\leq\ & \sum_{k=1}^{m} \|A_k(t)\| |\xi|_{W^{1,\infty}} |\tau_k(t,\psi) - \tau_k(\bar{t},\bar{\psi})| + \sum_{k=1}^{m} \|A_k(t) - A_k(\bar{t})\| |\xi|_C \\
& + \int_{-\tau_0}^{0} \|G(s,t,\psi) - G(s,\bar{t},\bar{\psi})\| ds |\xi|_C \\
\leq\ & \Big(\sum_{k=1}^{m} \|A_k\|_{W^{1,\infty}([0,T],\mathbb{R}^{n\times n})} L_\tau \Big(|t-\bar{t}| + |\psi - \bar{\psi}|_C \Big) + \sum_{k=1}^{m} \|A_k\|_{W^{1,\infty}([0,T],\mathbb{R}^{n\times n})} |t-\bar{t}| \\
& + \int_{-\tau_0}^{0} |g(s)| ds \Big(|t-\bar{t}| + |\psi - \bar{\psi}|_C \Big)\Big) |\xi|_{W^{1,\infty}}, \qquad \text{for a.e. } t,\bar{t} \in [0,T].
\end{aligned}
$$

This inequality shows that (C2) (ii) is satisfied. By elementary properties of measurable functions and by Tonelli's theorem we get that assumptions (i), (ii) and (iv) yield condition (C2) (iii). This completes the discussion of Example 8.6.

Next we define the spaces H, V, W and U used in the abstract formulation in Section 8.1. As in [34], let

$$
\begin{aligned}
H &= \mathbb{R}^n \times L^p, \\
V &= \{(\varphi(0),\psi) : \psi \in C\}, \\
W &= \{(\psi(0),\psi) : \psi \in W^{1,\infty} \cap C\}
\end{aligned}
$$

and $U = \mathbb{R}^n$, with the norms

$$
\begin{aligned}
|(\eta,\psi)|_H^p &\equiv |\eta|^p + |\psi|_{L^p}^p, \\
|(\psi(0),\psi)|_V &\equiv |\psi|_C, \\
|(\psi(0),\psi)|_W &\equiv |\psi|_{W^{1,\infty}}.
\end{aligned}
$$

Then, clearly, $W \subset V \subset H$ is satisfied with dense embeddings. Simple estimates show for $(\psi(0),\psi) \in V$ that

$$
\begin{aligned}
|(\psi(0),\psi)|_H &= (|\psi(0)|^p + |\psi|_{L^p}^p)^{1/p} \\
&\leq (|\psi|_C^p + r|\psi|_C^p)^{1/p} \\
&= (1+r)^{1/p} |\psi|_C \\
&= (1+r)^{1/p} |(\psi(0),\psi)|_V,
\end{aligned}
$$

and for $(\psi(0),\psi) \in W$ we have that

$$
\begin{aligned}
|(\psi(0),\psi)|_V &= |\psi|_C \\
&\leq \sup_{-r\leq t\leq 0} |\psi(t) - \psi(0)| + |\psi(0)| \\
&\leq r \cdot \operatorname*{ess\,sup}_{-r\leq t\leq 0} |\dot{\psi}(t)| + |\psi(0)| \\
&\leq (r+1)|\psi|_{W^{1,\infty}} \\
&= (r+1)|(\psi(0),\psi)|_W,
\end{aligned}
$$

therefore both embeddings in $W \subset V \subset H$ are continuous as well.

Define $B : U \to H$ by

$$Bu = (u, 0), \tag{8.6}$$

and $F : [0,T] \times V \to U$ by

$$F(t,\psi) = f\Big(t, \psi(0), \Lambda(t,\psi)\Big). \tag{8.7}$$

Lemma 8.7 *Assume (C1) and (C2). Then the function F defined by (8.7) satisfies assumptions (B3) and (B4).*

Proof Let $M > 0$ and $(\psi(0),\psi), (\bar\psi(0),\bar\psi) \in \overline{\mathcal{G}}_V(M)$. The definition of $\|\mu\|$ implies that

$$|\Lambda(t,\psi)| \le \|\mu\| |\psi|_C, \qquad \text{for a.e. } t \in [0,T].$$

Define $M_1 \equiv M \max\{1, \|\mu\|\}$. Then the second and third argument of f in (8.7) remains in $\overline{\mathcal{G}}_{\mathbb{R}^n}(M_1) \cap \Omega_1$ and in $\overline{\mathcal{G}}_{\mathbb{R}^n}(M_1) \cap \Omega_2$ for a.e. $t \in [0,T]$, respectively, therefore by the Lipschitz-continuity of f with Lipschitz-constant $L_1 = L_1(M_1)$ we get for a.e. $t, \bar t \in [0,T]$ that

$$\begin{aligned}
&|F(t,\psi) - F(\bar t,\bar\psi)|_U \\
&\quad = \Big| f\Big(t,\psi(0),\Lambda(t,\psi)\Big) - f\Big(\bar t, \bar\psi(0), \Lambda(\bar t,\bar\psi)\Big)\Big| \\
&\quad \le L_1\Big(|t-\bar t| + |\psi(0) - \bar\psi(0)| + |\Lambda(t,\psi) - \Lambda(\bar t,\bar\psi)|\Big) \\
&\quad \le L_1\Big(|t-\bar t| + |\psi(0) - \bar\psi(0)| + |\lambda(t,\psi,\psi) - \lambda(\bar t,\bar\psi,\psi)| + |\lambda(\bar t,\bar\psi,\psi) - \lambda(\bar t,\bar\psi,\bar\psi)|\Big) \\
&\quad \le L_1\Big(|t-\bar t| + |\psi - \bar\psi|_C + L_2(M)|\psi|_{W^{1,\infty}}\Big(|t-\bar t| + |\psi-\bar\psi|_C\Big) + \|\mu\||\psi-\bar\psi|_C\Big) \\
&\quad \le L_1 \max\{1+\|\mu\|, L_2(M)\}(1 + |\psi|_{W^{1,\infty}})\Big(|t-\bar t| + |\psi-\bar\psi|_C\Big),
\end{aligned}$$

therefore $K_1 \equiv L_1 \max\{1+\|\mu\|, L_2(M)\}$ satisfies the constant K in (8.5), and consequently we have proved (B3) (i) and (B4). To show (B3) (ii), consider

$$\begin{aligned}
|F(t,\psi)|_U &\le |F(t,\psi) - F(t,0)|_U + |F(t,0)|_U \\
&\le K_1|\psi|_C + |f(t,0,0)| \\
&= K_1|\psi|_C + N \\
&\le \max\{K_1, N\}(1 + |\psi|_C),
\end{aligned}$$

hence $K \equiv \max\{K_1, N\}$ is good in both part of (B3) and in (B4).

Finally, we have to show that the function

$$t \mapsto F(t,\psi) = f\Big(t,\psi(0),\Lambda(t,\psi)\Big)$$

is measurable for all fixed $\psi \in \Omega_3$. First we show that assumption (C2) (iii) implies that the function $t \mapsto f(t,x,g(t))$ is measurable for all simple function $g : [0,T] \to \Omega_2$, $x \in \Omega_1$. Let B_i $(i = 1,2,\ldots,k)$ be disjoint measurable subsets of $[0,T]$ such that $\bigcup_{i=1}^k B_i = [0,T]$, and $g(t) = \sum_{i=1}^k y_i \chi_{B_i}(t)$ where $y_i \in \Omega_2$. It is easy to see that

$$f(t,x,g(t)) = \sum_{i=1}^k \chi(t) f(t,x,y_i),$$

therefore assumption (C1) (iii) yields that the function $t \mapsto f(t,x,g(t))$ is measurable. By (C2) (iii) the function $t \mapsto \Lambda(t,\psi)$ is measurable for all $\psi \in \Omega_3$. Approximate it by simple functions $\Lambda_i(t,\psi)$, i.e., $\lim_{i\to\infty} \Lambda_i(t,\psi) = \Lambda(t,\psi)$ for all $t \in [0,T]$, $\psi \in \Omega_3$. We have shown that the functions $t \mapsto f\Big(t,\psi(0),\Lambda_i(t,\psi)\Big)$ are measurable functions, and using assumption (C1) (i) we can see that

$$\Big|f\Big(t,\psi(0),\Lambda_i(t,\psi)\Big) - f\Big(t,\psi(0),\Lambda(t,\psi)\Big)\Big| \le |\Lambda_i(t,\psi) - \Lambda(t,\psi)|, \qquad \text{for a.e. } t \in [0,T],$$

hence

$$\lim_{i\to\infty} f\Big(t,\psi(0),\Lambda_i(t,\psi)\Big) = f\Big(t,\psi(0),\Lambda(t,\psi)\Big),$$

for a. e. $t \in [0,T]$, $\psi \in \Omega_3$, and therefore we get that $F(\cdot,\psi)$ is measurable. □

Define the C_0-semigroup on H by

$$S(t)(\eta,\psi) \equiv (\eta, g_t), \tag{8.8}$$

where $g : [-r,\infty) \to \mathbb{R}^n$ is defined by

$$g(s) \equiv \begin{cases} \varphi(s), & -r \le s < 0, \\ \eta, & 0 \le s. \end{cases}$$

(I.e., $S(\cdot)$ is the solution semigroup of the Cauchy-problem $\dot{x}(t) = 0$, $x(0) = \eta$, $x(s) = \varphi(s)$, $-r \le s < 0$.) Lemma 3.2 in [34] yields that $S(\cdot)$ is a C_0-semigroup defined on H, and assumptions (B1) and (B2) are satisfied. Let A be the infinitesimal generator of $S(\cdot)$. It is known (see e.g. [9]) that

$$\begin{aligned} \mathrm{dom} A_H &= \{(\psi(0),\psi) \ : \ \psi \in W^{1,p}\}, \\ A_H(\psi(0),\psi) &= (0,\dot{\psi}). \end{aligned}$$

We conclude, that with this particular choice of the spaces U, H, V and W, the semigroup $S(t)$, the function F, and the assumed conditions (C1)–(C3), IVP (8.2)-(8.3) can be written in abstract form as (8.4), and Theorems 8.1 and 8.2 give the following local existence, uniqueness and continuous dependence on initial data result for IVP (8.2)-(8.3).

Theorem 8.8 *Assume that conditions (C1)–(C3) hold. Then for an arbitrary $R > 0$ there exists $\alpha = \alpha(R) > 0$ such that for all $\varphi \in W^{1,\infty} \cap C$ with $|\varphi|_{W^{1,\infty}} \le R$ it follows that IVP (8.2)-(8.3) has a unique solution $x(\cdot;\varphi)$ on the interval $[0,\alpha]$. Moreover, there exists $L = L(R) > 0$ such that*

$$\max\Big\{\sup_{0\le t\le\alpha} |x(t;\varphi) - x(t;\bar{\varphi})|, \sup_{0\le t\le\alpha} |\dot{x}(t;\varphi) - \dot{x}(t;\bar{\varphi})|\Big\} \le L|\varphi - \bar{\varphi}|_{W^{1,\infty}}.$$

for $\varphi, \bar{\varphi} \in W^{1,\infty} \cap C$, $\quad |\varphi|_{W^{1,\infty}}, |\bar{\varphi}|_{W^{1,\infty}} \le R$.

We close this chapter by noting that Ito and Kappel presented an abstract approximation framework for the integral equation (8.4), and constructed a particular approximation scheme using first order spline functions, and showed that the scheme satisfies the requirements of the abstract framework, and therefore it provides an approximation method for equation (8.1). Since (8.2) can be written in abstract form as (8.4), the spline scheme defined in [34] can be applied for (8.2) as well.

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