

On numerical solutions for a class of nonlinear delay equations with time- and state-dependent delays

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Abstract. In this paper, based on a general approximation framework developed by the authors for nonlinear functional differential equations with state- and time-dependent delays, we present numerical experiments for the above class of equations using approximating equations with piecewise constant arguments (EPCA).

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1. Introduction

In this paper we present numerical experiments for a class of functional differential equations (FDEs) with time- and state-dependent delays. The computational scheme used here is based on approximation of functional differential equations by equations with piecewise constant arguments (EPCA). Related (theoretical) convergence and rate of convergence results of this method for FDEs with time- and state-dependent delays are discussed by the authors in [7]. (See also Theorem 2.1. below.)

Numerical findings summarized in Section 3 show good agreement with the theoretical predictions concerning the convergence properties of our method (Example 3.1. – 3.4.) and furthermore they reveal interesting dynamics of the particular equations considered (see e.g., Example 3.8.). For the sake of comparison we constructed and implemented a second order scheme, but as Example 3.3. shows the accuracy of higher order schemes can break down if certain “jump discontinuities” are not properly tracked (see [9] for details) by the numerical method. In Examples 3.5. – 3.7. we study initial value problems where the Lipschitz-continuity of the initial function is violated (and thus uniqueness of the solution is not guaranteed by Theorem 2.1.) and in Example 3.7 we see how numerical solutions approach asymptotically the stable branch of the solution. We observed the most interesting dynamic behavior in Example 3.8, where the delay equation has a unique unstable periodic solution which our method approximates on finite intervals (the length of which depends on the discretization parameter), but afterward the numerical solutions approach $\pm\infty$ in a very “regular” fashion. In fact, in [8] we establish results on asymptotic behavior of the solutions, corresponding to small initial data, of the delay equation studied in Example 3.8.

(see also Theorem 3.9. below) which shows that the numerical solution produced by our method in that case approximate the stable unbounded solutions of the delay equation.

2. Approximation schemes for FDEs with time- and state-dependent delays

In this section we consider the delay differential equation

$$\dot{x}(t) = f\left(t, x(t), x(t - \tau(t, x(t)))\right), \quad t \geq 0 \quad (2.1)$$

with initial condition

$$x(t) = \Phi(t), \quad t \in [-\lambda, 0], \quad (2.2)$$

where $\lambda \equiv -\inf\{t - \tau(t, u) : t \geq 0, u \in \mathbf{R}\}$. In the case when λ is not finite, $[-\lambda, 0]$ denotes the interval $(-\infty, 0]$.

Furthermore, we assume, that

$$\tau(t, u) \geq 0, \quad \text{for all } t \geq 0, \quad u \in \mathbf{R}, \quad (2.3)$$

i.e., equation (2.1) is a delay differential equation.

Fix a positive number h . Replacing t by $[t/h]h$ in the right hand side of (2.1) ($[\cdot]$ is the greatest integer function) and discretizing the delay function as well we get the following equation with piecewise constant argument.

$$\dot{y}_h(t) = f\left([t/h]h, y_h([t/h]h), y_h\left(\left[\frac{t}{h}\right]h - \left[\frac{\tau([t/h]h, y_h([t/h]h))}{h}\right]h\right)\right), \quad t \geq 0, \quad (2.4)$$

with initial condition corresponding to (2.2)

$$y_h(-kh) = \Phi(-kh), \quad k = 0, 1, 2, \dots, \quad -\lambda \leq -kh \leq 0. \quad (2.5)$$

It is easy to check that for all $t \geq 0$ we have

$$-\lambda \leq \left[\frac{t}{h}\right]h - \left[\frac{\tau([t/h]h, y_h([t/h]h))}{h}\right]h \leq 0,$$

i.e., (2.4) is also a delay equation and it has the same initial interval as equation (2.1).

By a solution of the initial value problem (2.1)-(2.2) we mean a function y_h defined on $\{-kh : k = 0, 1, \dots, -\lambda \leq -kh \leq 0\}$ by (2.5), which satisfies the following properties on $[0, \infty)$:

- (i) the function y_h is continuous on $[0, \infty)$,
- (ii) the derivative $\dot{y}_h(t)$ exists at each point $t \in [0, \infty)$ with the possible exception of the points kh ($k = 0, 1, \dots$) where finite one-sided derivatives exist
- (iii) the function y_h satisfies (2.4) on each interval $[kh, (k+1)h)$ for $k = 0, 1, \dots$

In [7] we have shown that the solutions of this equations approximate uniformly the solution of (2.1) on finite time intervals as $h \rightarrow 0$. We have shown also existence and uniqueness of the solution of initial value problem (2.1)-(2.2) under some smoothness conditions. In particular, we have the following theorem:

Theorem 2.1. (see also [7]): Fix $T > 0$. If

- (i) $f \in C([0, \infty) \times \mathbf{R}^2, \mathbf{R})$, $\Phi \in C([-\lambda, 0], \mathbf{R})$, Φ is bounded on $[-\lambda, 0]$,
 $\tau \in C([0, \infty) \times \mathbf{R}, [0, \infty))$

then the IVP (2.1)-(2.2) has a solution.

Moreover, if in addition

- (ii) $f(t, u, v)$ is Lipschitz-continuous on $[0, T] \times \mathbf{R}^2$, i.e., there exists constant $L_1 > 0$ such that for all $t_1, t_2 \in [0, T]$, $u_1, u_2, v_1, v_2 \in \mathbf{R}$

$$|f(t_1, u_1, v_1) - f(t_2, u_2, v_2)| \leq L_1(|t_1 - t_2| + |u_1 - u_2| + |v_1 - v_2|)$$

- (iii) The initial function $\Phi(t)$ is Lipschitz-continuous on $[-\lambda, 0]$,

- (iv) The delay function is Lipschitz-continuous on $[0, T] \times \mathbf{R}^2$, i.e., there exists constant $L_2 > 0$ such that for all $t_1, t_2 \in [0, T]$, $u_1, u_2 \in \mathbf{R}$

$$|\tau(t_1, u_1) - \tau(t_2, u_2)| \leq L_2(|t_1 - t_2| + |u_1 - u_2|)$$

then the solution of IVP (2.1)-(2.2) is unique and there exists a constant $M(T, \Phi) > 0$ such that

$$|x(t) - y_h(t)| \leq Mh, \quad t \in [0, T], \quad h > 0, \quad (2.6)$$

where $x(\cdot)$ is the solution of IVP (2.1)-(2.2) and $y_h(\cdot)$ is the solution of IVP (2.4)-(2.5).

Using the method of steps on the intervals $[kh, (k+1)h]$ one can easily see that IVP (2.4)-(2.5) has a unique solution on $[0, \infty)$. Let $t \in [kh, (k+1)h)$. Integrating (2.4) we get

$$\begin{aligned} & y_h(t) \\ &= y_h(kh) + \int_{kh}^t f \left([s/h]h, y_h([s/h]h), y_h \left(\left[\frac{s}{h} \right] h - \left[\frac{\tau([s/h]h, y_h([s/h]h))}{h} \right] h \right) \right) ds \\ &= y_h(kh) + f \left(kh, y_h(kh), y_h \left(\left(k - \left[\frac{\tau(kh, y_h(kh))}{h} \right] \right) h \right) \right) \cdot (t - nk). \end{aligned} \quad (2.7)$$

Introducing the notation $a(k) \equiv y_h(kh)$ from (2.7) we obtain the following difference equation for the sequence $a(k)$:

$$\begin{aligned} a(k+1) &= a(k) + f \left(kh, a(k), a(k-d_k) \right) \cdot h, \quad k = 0, 1, 2, \dots, \\ a(-k) &= \Phi(-kh), \quad k = 0, 1, 2, \dots, \quad -\lambda \leq -kh \leq 0, \end{aligned} \quad (2.8)$$

where we have used the notation

$$d_k \equiv \left[\frac{\tau(kh, a(k))}{h} \right]. \quad (2.9)$$

From computational point of view method (2.8)-(2.9) is very simple, because it requires only function evaluations at mesh points, and we have an explicit difference equation for the approximate values at the mesh points. From (2.6) we see that (2.8)-(2.9) defines a first order scheme for the approximation solutions of (2.1)-(2.2). Similar method works for the several delay case (see [7]).

Remark 2.2. Using a modification of Heun method for the class of IVPs described by (2.1)-(2.2) and a piecewise linear approximation between mesh points we obtain the following numerical scheme

$$x_{n+1} = x_n + \left(f(t_n, x_n, y_n) + f(t_{n+1}, \tilde{x}_{n+1}, \tilde{y}_{n+1}) \right) \frac{h}{2}, \quad n = 0, 1, 2, \dots \quad (2.10)$$

where for $n = 0, 1, 2, \dots$

$$\begin{aligned} t_n &\equiv nh \\ x_0 &\equiv \Phi(0) \\ \tilde{x}_{n+1} &\equiv x_n + f(t_n, x_n, y_n)h \\ l_n &\equiv t_n - \tau(t_n, x_n) \\ k_n &\equiv [l_n/h] \\ y_n &\equiv \begin{cases} \Phi(l_n), & l_n \leq 0 \\ \frac{x_{k_{n+1}} - x_{k_n}}{h} (l_n - k_n h) + x_{k_n}, & l_n > 0 \end{cases} \\ \tilde{l}_{n+1} &\equiv t_{n+1} - \tau(t_{n+1}, \tilde{x}_{n+1}) \\ \tilde{k}_{n+1} &\equiv [\tilde{l}_{n+1}/h] \\ \tilde{y}_{n+1} &\equiv \begin{cases} \Phi(\tilde{l}_{n+1}), & \tilde{l}_{n+1} \leq 0 \\ \frac{x_{\tilde{k}_{n+1}+1} - x_{\tilde{k}_{n+1}}}{h} (\tilde{l}_{n+1} - \tilde{k}_{n+1}h) + x_{\tilde{k}_{n+1}}, & 0 < \tilde{l}_{n+1} \leq t_n \\ \frac{\tilde{x}_{n+1} - x_n}{h} (\tilde{l}_{n+1} - t_n h) + x_n, & t_n < \tilde{l}_{n+1} < t_{n+1} \\ \tilde{x}_{n+1}, & \tilde{l}_{n+1} = t_{n+1} \end{cases} \end{aligned} \quad (2.11)$$

Our numerical studies indicate that the method has second order convergence. Higher order methods could be constructed similarly. We refer the interested reader to [10] and the references therein for an exhaustive survey on that topic.

3. Numerical Examples

In this section, as indicated in the introduction, we present case studies to test the performance of our numerical scheme. Note that all runs were performed on SUN workstations at the University of Texas at Dallas. (Program listings are available upon request.)

Example 3.1.

$$\begin{aligned}\dot{x}(t) &= \frac{8}{t+1} \cdot x\left(t - \left(\frac{t}{2} + \frac{1}{2}\right)\right), & t \geq 0 \\ x(t) &= (t+1)^2, & t \in \left[-\frac{1}{2}, 0\right]\end{aligned}$$

Analytic solution of the IVP is $x(t) = (t+1)^2$. Applying our method we obtain the following approximation equation with piecewise constant arguments:

$$\begin{aligned}\dot{y}(t) &= \frac{8}{[t/h]h+1} \cdot y\left(\left[\frac{t}{h} - \left[\frac{[t/h]h+1}{2h}\right]\right]h\right), & t \geq 0 \\ y(-ih) &= (-ih+1)^2, & -\frac{1}{2} \leq -ih \leq 0, \quad i \in \mathbf{N}.\end{aligned}$$

The resulting difference equation is

$$\begin{aligned}a(k+1) &= a(k) + \frac{8}{kh+1} \cdot a(k-d_k)h \\ a(-k) &= (-kh+1)^2 \\ d_k &\equiv \left[\frac{k}{2} + \frac{1}{2h}\right].\end{aligned}$$

We list numerical results for the first (see above) and second order methods (see Remark 2.2.) in Table 1.

Table 1. Example 3.1.

h	t	1st order method		2nd order method	
		error	rel. error	error	rel. error
10^{-2}	100	6.10411e+01	5.98383e-03	5.34280e-02	5.23753e-06
	150	1.36729e+02	5.99664e-03	1.19423e-01	5.23761e-06
	200	2.42558e+02	6.00377e-03	2.11600e-01	5.23749e-06
10^{-3}	100	5.82027e+00	5.70559e-04	5.34287e-04	5.23759e-08
	150	1.30371e+01	5.71778e-04	1.19424e-03	5.23768e-08
	200	2.31211e+01	5.72290e-04	2.11603e-03	5.23756e-08
10^{-4}	100	7.64026e-01	7.48972e-05	5.34288e-06	5.23760e-10
	150	1.71034e+00	7.50115e-05	1.19425e-05	5.23770e-10
	200	3.03396e+00	7.50961e-05	2.11603e-05	5.23756e-10

Example 3.2. (see also [9]):

$$\begin{aligned}\dot{x}(t) &= \cos t \cdot x(x(t) - 2), & t \in [0, T] \\ x(t) &= 1, & t \in [-2, 0]\end{aligned}$$

Note that in general it is not possible to determine in advance if this equation is a delay equation because of the state-dependent delay (i.e., $x(t) - 2$ could be greater than t). To prevent advanced arguments, our program checks at every step if the approximate delay is nonnegative and stops with error message if it is negative. In this case the exact solution

is $x(t) = \sin t + 1$, i.e., we have a delay equation, which is preserved by the approximation scheme. For numerical results see Table 2.

Table 2. Example 3.2.

h	t	$x(t)$	$y_h(t)$	error
10^{-2}	10.0	4.55979e-01	4.65168e-01	9.18946e-03
	20.0	1.91295e+00	1.91587e+00	2.92592e-03
	30.0	1.19684e-02	1.61671e-02	4.19871e-03
	40.0	1.74511e+00	1.75338e+00	8.26352e-03
10^{-3}	10.0	4.55979e-01	4.56898e-01	9.19535e-04
	20.0	1.91295e+00	1.91324e+00	2.95784e-04
	30.0	1.19684e-02	1.23912e-02	4.22788e-04
	40.0	1.74511e+00	1.74595e+00	8.33183e-04
10^{-4}	10.0	4.55979e-01	4.56071e-01	9.19538e-05
	20.0	1.91295e+00	1.91297e+00	2.95947e-05
	30.0	1.19684e-02	1.20107e-02	4.22876e-05
	40.0	1.74511e+00	1.74520e+00	8.33455e-05
10^{-5}	10.0	4.55987e-01	4.55996e-01	9.19539e-06
	20.0	1.91294e+00	1.91294e+00	2.95949e-06
	30.0	1.19637e-02	1.19680e-02	4.22890e-06
	40.0	1.74514e+00	1.74515e+00	8.33453e-06

Example 3.3.

$$\dot{x}(t) = x \left(t - 1 - \frac{1}{t+1} \right), \quad t \geq 0$$

$$x(t) = \begin{cases} \frac{2}{3}(t+2), & -2 \leq t \leq -0.5 \\ 1, & -0.5 \leq t \leq 0 \end{cases}$$

The exact solution is $x(t) = 1 + \frac{2}{3}t + \frac{t^3}{3} - \frac{2}{3} \log(t+1)$ on $[0, 1]$ and $x(t) = 1 - \frac{2}{3} \log 2 + t$ on $[1, 2]$. The first derivative of the solution is not continuous at $t = -0.5$ and at $t = 0$, therefore the second derivative also has a jump at the points where the time lag is equal to -0.5 and 0 , i.e. when $t - 1 - \frac{1}{t+1} = -0.5$ and when $t - 1 - \frac{1}{t+1} = 0$, or equivalently, at $t = 1$ and at $t = \sqrt{2}$. It is known ([9]) that discontinuities of the second derivative may lead to the loss of the second order convergence if discontinuities occur at points which are not mesh points. In our example $t = 1$ is a mesh point, and the second order method keeps the second order convergence at $t = 1$, on the other hand $t = \sqrt{2}$ is not a mesh point, and a breakdown in the order of convergence occurs at $t = \sqrt{2}$. (See [4], [9] and [10] for a detailed discussion about propagation of the jump discontinuities and correction techniques for the preservation of higher order convergence.)

Table 3. Example 3.3.

h	t	$x(t)$	1st order method		2nd order method	
			$y_{1,h}(t)$	error	$y_{2,h}(t)$	error
10^{-2}	0.5	1.1463e+00	1.1451e+00	1.2232e-03	1.1463e+00	3.0863e-06
	1.0	1.5379e+00	1.5361e+00	1.7685e-03	1.5379e+00	4.1666e-06
	1.4	1.9379e+00	1.9361e+00	1.7685e-03	1.9379e+00	4.1666e-06
	1.5	2.0379e+00	2.0362e+00	1.6125e-03	2.0380e+00	1.8447e-04
	2.0	2.5379e+00	2.5870e+00	4.9096e-02	2.5894e+00	5.1553e-02
10^{-3}	0.5	1.1463e+00	1.1462e+00	1.1859e-04	1.1463e+00	3.0864e-08
	1.0	1.5379e+00	1.5377e+00	1.7921e-04	1.5379e+00	4.1666e-08
	1.4	1.9379e+00	1.9377e+00	1.7921e-04	1.9379e+00	4.1666e-08
	1.5	2.0379e+00	2.0379e+00	5.1257e-06	2.0380e+00	1.8657e-04
	2.0	2.5379e+00	2.5892e+00	5.1304e-02	2.5894e+00	5.1547e-02
10^{-4}	0.5	1.1463e+00	1.1463e+00	1.1367e-05	1.1463e+00	3.0864e-10
	1.0	1.5379e+00	1.5378e+00	1.7113e-05	1.5379e+00	4.1667e-10
	1.4	1.9379e+00	1.9378e+00	1.7113e-05	1.9379e+00	4.1671e-10
	1.5	2.0379e+00	2.0380e+00	1.6929e-04	2.0380e+00	1.8659e-04
	2.0	2.5379e+00	2.5894e+00	5.1524e-02	2.5894e+00	5.1547e-02

Example 3.4. (Dynamics of a one block compartmental model with pipe [6].)

$$\dot{x}(t) = -(a_0 + a(t))x(t) + (1 - \dot{\tau}(t))a(t - \tau(t))x(t - \tau(t))$$

$$x(t) = 1, \quad t \in [-1, 0]$$

$$a(t) \equiv 1 + \frac{1}{2} \sin(t)$$

$$a_0 = 0.1$$

We examined the dynamics of this equation with constant, $\tau(t) = 1$, and time-dependent, $\tau(t) = 1 + \frac{1}{1+t}$, delays (see Figure 1). The solutions are in good agreement with physical expectations.

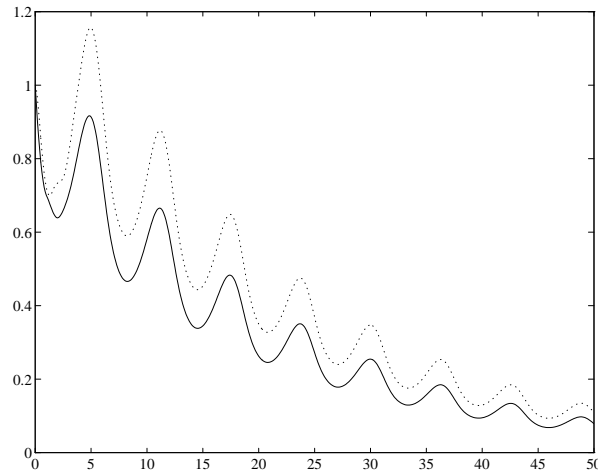


Figure 1. Example 3.4., solid line: $\tau(t) = 1$, dotted line: $\tau(t) = 1 + \frac{1}{t+1}$

Example 3.5. (see also [1], originally introduced by J. A. Yorke):

$$\begin{aligned} \dot{x}(t) &= -x(t - \tau(t)), & t \in [0, T] \\ x(0) &= 1 \\ \tau(t) &\equiv \begin{cases} t - 2 + \sqrt{4 - 2t}, & 0 \leq t \leq 2 \\ 0, & t > 2. \end{cases} \end{aligned}$$

The solution of this problem is

$$x(t) = \begin{cases} \frac{(t-2)^2}{4}, & 0 \leq t \leq 2 \\ 0, & t > 2. \end{cases}$$

In this example the delay function is not Lipschitz-continuous, but the approximating method converges to the unique solution, see Table 4.

Table 4. Example 3.5.

h	t	$x(t)$	$y_h(t)$	error
10^{-2}	1.0	2.50000e-01	2.51437e-01	1.43735e-03
	2.0	0.00000e+00	2.68572e-04	2.68572e-04
	3.0	0.00000e+00	9.83061e-05	9.83061e-05
10^{-3}	1.0	2.50000e-01	2.50134e-01	1.34471e-04
	2.0	0.00000e+00	1.38094e-05	1.38094e-05
	3.0	0.00000e+00	5.07766e-06	5.07766e-06
10^{-4}	1.0	2.50000e-01	2.50013e-01	1.28346e-05
	2.0	0.00000e+00	5.26486e-07	5.26486e-07
	3.0	0.00000e+00	1.93674e-07	1.93674e-07

Example 3.6. Our method may work if the initial function is not continuous. Consider e.g.,

$$\begin{aligned} \dot{x}(t) &= x(t-1), & t \geq 0 \\ x(t) &= \begin{cases} 1, & t = 0 \\ 0, & t < 0. \end{cases} \end{aligned}$$

The analytic solution is

$$x(t) = \sum_{i=0}^{[t]} \frac{(t-i)^i}{i!}.$$

Numerical results, indicating convergence to the true solution, are listed in Table 5.

Table 5. Example 3.6.

h	t	$x(t)$	$y_h(t)$	error	rel. error
10^{-2}	5.00	1.08750e+01	1.08376e+01	3.73878e-02	3.43795e-03
	10.00	1.85338e+02	1.83756e+02	1.58103e+00	8.53055e-03
	15.00	3.15863e+03	3.11569e+03	4.29424e+01	1.35952e-02
	20.00	5.38314e+04	5.28283e+04	1.00310e+03	1.86341e-02
10^{-3}	5.00	1.08750e+01	1.08713e+01	3.74888e-03	3.44724e-04
	10.00	1.85338e+02	1.85179e+02	1.58949e-01	8.57618e-04
	15.00	3.15863e+03	3.15431e+03	4.32754e+00	1.37007e-03
	20.00	5.38314e+04	5.37300e+04	1.01324e+02	1.88225e-03
10^{-4}	5.00	1.08750e+01	1.08746e+01	3.74989e-04	3.44817e-05
	10.00	1.85338e+02	1.85322e+02	1.59034e-02	8.58076e-05
	15.00	3.15863e+03	3.15820e+03	4.33089e-01	1.37113e-04
	20.00	5.38314e+04	5.38212e+04	1.01426e+01	1.88415e-04

Example 3.7.

$$\dot{x}(t) = -x(t - |x(t)|), \quad t \geq 0$$

$$x(t) = \begin{cases} -1, & t \leq -1 \\ 1.5(t+1)^{1/3} - 1, & -1 \leq t \leq -\frac{7}{8} \\ \frac{10}{7}t + 1, & -\frac{7}{8} \leq t \leq 0 \end{cases}$$

The initial function is not Lipschitz-continuous, therefore the uniqueness is not guaranteed by Theorem 2.1. In fact, the IVP has two solutions: $t+1$ is solution for all $t \geq 0$ and the analytic expression on $[0, 0.25]$ for the other solution (which eventually goes to zero [2]) is $t+1-t^{1.5}$.

Assuming exact function evaluations, our method should follow the solution $x(t) = t+1$, because $a(0) = 1$ and $a(k) = kh+1$ imply that

$$\begin{aligned} a(k+1) &= a(k) - a\left(k - \left\lceil \frac{a(k)}{h} \right\rceil\right) h = a(k) - a\left(k - \left\lceil \frac{kh+1}{h} \right\rceil\right) h \\ &= a(k) - a\left(-\left\lceil \frac{1}{h} \right\rceil\right) h = a(k) + h = (k+1)h + 1 \end{aligned}$$

where we have used the fact that $\lceil 1/h \rceil \geq 1$ for $h < 1$. Moreover, if $h = 1/N$, where N is a fixed positive integer, then $\lceil 1/h \rceil = N$, and $a(k - \lceil (kh+1)/h \rceil) = a(-N) = \Phi(-Nh) = \Phi(-1)$. In our runs, due to a computer round-off error in the evaluation of the $\lceil \cdot \rceil$ function, the method actually picks up one value of the initial function on the interval $(-1, -7/8)$ and then it follows the solution which goes to 0. For $h = 0.01$ the round-off error occurs later, so the approximate solution initially follows $x(t) = t+1$ (see Figure 2.)

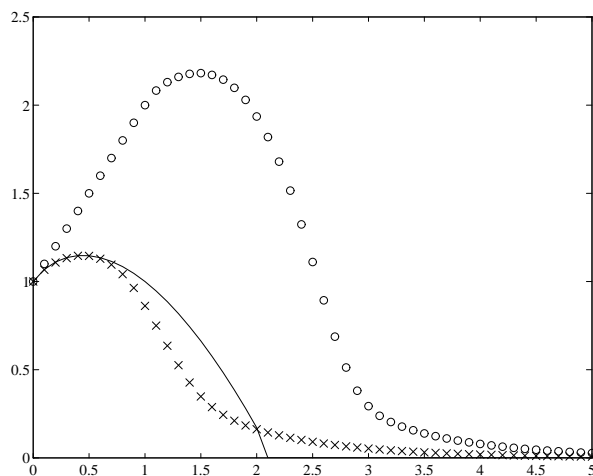


Figure 2. Example 3.7., solid line: $t + 1 - t^{1.5}$, o: $h = 0.01$, x: $h = 0.001$

Example 3.8.

$$\begin{aligned} \dot{x}(t) &= x(t - |x(t)|) + \sin 2t - \sin^2(t - \sin^2 t), & t \geq 0 \\ x(t) &= \sin^2 t, & t \leq 0. \end{aligned}$$

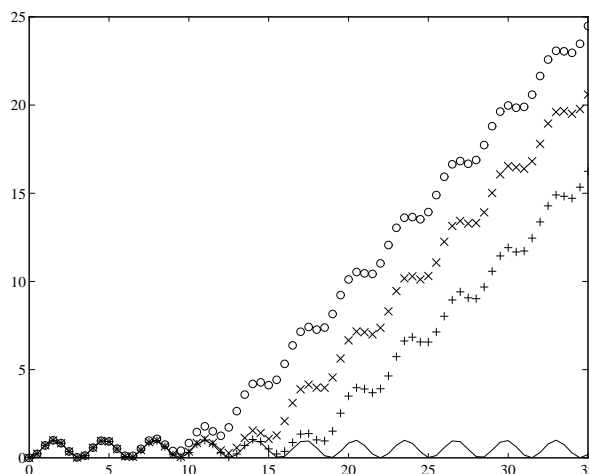


Figure 3. Example 3.8., solid line: $\sin^2 t$, o: $h = 0.01$, x: $h = 0.001$, +: $h = 0.0001$

The analytic solution is $\sin^2 t$. If we look at the graph of the approximating solutions, we can see an interesting behavior (see Figure 3). The approximate solutions diverge from the analytic solution “very regularly”. Next we consider the homogeneous version of the previous equation, i.e.,

$$\begin{aligned} \dot{z}(t) &= z(t - |z(t)|), & t \geq 0 \\ z(t) &= \Phi(t), & t \leq 0 \end{aligned} \tag{3.1}$$

For zero initial function the unique solution of this problem is $z(t) = 0$. If the initial function is not zero, even if it is very small in the sup-norm, the corresponding solution is unbounded, moreover, the same type of asymptotic behavior can be observed for various initial functions. In our experimentations we used the following initial functions:

$$\Phi_1(t) = t + 0.2$$

$$\Phi_2(t) = 0.2 \sin 5t + .01$$

$$\Phi_3(t) = 0.4 \cos 2t$$

Corresponding solutions are shown on Figure 4. Note that for $\Phi(0) < 0$, similar behavior can be observed with the exception that the solutions are decreasing in that case. As a matter of fact, our numerical findings suggested the following conjecture (for details and proofs see [8]):

Theorem 3.9. *Let $z_0 \equiv \Phi(0) > 0$, where $\Phi(\cdot)$ is the initial function in IVP (3.1). Moreover, we assume that either $\Phi(-z_0) \leq 1$ or $\Phi(-z_0) > 1$ and there exists $L > z_0$ such that $\Phi(-L) = 1$. Then there exist a constant α and a function $\beta(t)$ such that the solution of IVP (3.1) has the form $z(t) = t + \alpha + \beta(t)$, where $\lim_{t \rightarrow \infty} \beta(t) = 0$ and $\lim_{t \rightarrow \infty} \dot{\beta}(t) = 0$.*

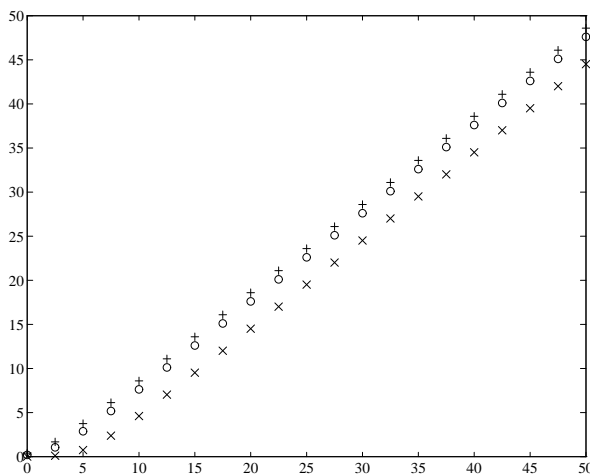


Figure 4. IVP (3.1) with initial function o: $\Phi_1(t)$, x: $\Phi_2(t)$, +: $\Phi_3(t)$

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