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ON THE EFFECTS OF DELAY PERTURBATIONS ON THE STABILITY OF DELAY DIFFERENCE EQUATIONS

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Abstract We consider a class of linear delay difference equations with perturbed time lags and present conditions which guarantee that the asymptotic stability of the trivial solution of the equation at hand is preserved under these perturbations. As an application of this perturbation result, we give sufficient conditions for asymptotic stability of scalar linear delay difference equations.

1. Introduction

In this paper we study the effects of perturbations of time delays on the stability of a class of linear delay difference systems. Our goal is to obtain a "practical" condition, i.e., a norm bound on the perturbations corresponding to the particular system under consideration, which guarantees the preservation of asymptotic stability under perturbations. It turns out that such condition can be formulated using the infinite sum of the fundamental solution of the unperturbed system (see Theorem 2.3 below). Since asymptotic stability of the unperturbed system implies that the components of its fundamental solution go to zero exponentially at infinity, it is possible to get "good" numerical estimates of the infinite sum, and consequently obtain norm bounds on the allowable perturbations.

We present our main results in Section 2, and in Section 3 we consider numerical examples. In Section 4, as an application of our perturbation result, we obtain sufficient conditions for asymptotic stability of scalar linear delay difference equations.

To conclude this section we note, that perturbation related issues for delay differential equations, and in particular, delay perturbations, have been studied by many authors. We refer the interested reader to [3], [4], [9], [10], [15] and the references therein for related articles, and also for [5], which contains the continuous counterpart of the results of this paper.

2. Main Results

First we introduce some notations used throughout this paper. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denotes the set of nonnegative integers, integers, and real numbers, respectively. For a sequence, $x(n)$, the forwarded difference is denoted by $\Delta x(n) \equiv x(n+1) - x(n)$. For future convenience, we define the $\tilde{}$ operation on vectors and on matrices, which means taking the absolute value of the vector or matrix componentwise, i.e., if $x = (x_1, x_2, \ldots, x_n)^T$, then by definition $\tilde{x} \equiv (|x_1|, |x_2|, \ldots, |x_n|)^T$, and similarly if $A = (a_{ij})_{n\times n}$, then $\tilde{A} \equiv (|a_{ij}|)_{n\times n}$. The relation $\leq$ between vectors means a componentwise comparison, i.e., $(x_1, x_2, \ldots, x_n)^T \leq (y_1, y_2, \ldots, y_n)^T$ if for all the components $x_i \leq y_i$.

Consider the delay difference equation

$$\Delta x(n) = \sum_{i=0}^{m} A_i x(n - k_i - \eta_i(n)), \qquad n \in \mathbb{N}, \tag{2.1}$$

with initial condition

$$x(n) = \varphi(n), \qquad n = -n_0, -n_0+1, \ldots, 0, \tag{2.2}$$

where A_i $(i = 0, \ldots, m)$ denote constant $N \times N$ matrices, $0 = k_0 \leq k_1 \leq \ldots \leq k_m$, $\varphi : [-n_0, 0] \cap \mathbb{Z} \to \mathbb{R}^N$ is a given function, and we shall assume that the delay perturbations, $\eta_i(\cdot) : \mathbb{N} \to \mathbb{Z}$ $(i = 0, \ldots, m)$, satisfy

$$n - n_0 \leq n - k_i - \eta_i(n) \leq n \qquad \text{for} \quad n \in \mathbb{N} \quad (i = 0, \ldots, m). \tag{2.3}$$

Under our assumptions initial value problem (2.1)-(2.2) is a delay difference equation and has a unique solution.

We consider the corresponding unperturbed system with constant delays, i.e.,

$$\Delta y(n) = \sum_{i=0}^{m} A_i y(n - k_i), \qquad n \in \mathbb{N}, \tag{2.4}$$

and we assume that

(H) the trivial $(y(n) = 0)$ solution of (2.4) is asymptotically stable.

For a fixed $T \in \mathbb{N}$ the fundamental matrix solution of (2.4), $V(n)$, is defined as the solution of the following system

$$\Delta V(n) = \sum_{i=0}^{m} A_i V(n - k_i), \qquad n \in \mathbb{N}, \quad n \geq T, \tag{2.5}$$

and

$$V(n) = \begin{cases} I, & n = T, \\ 0, & n < T, \end{cases} \tag{2.6}$$

where $I, 0 \in \mathbb{R}^{n \times n}$ are the identity and the zero matrix, respectively.

Remark 2.1 *To emphasize the dependence of $V(\cdot)$ on T we use the notation $V(n; T)$. Note that $V(n; T) = V(n - T; 0)$ for $t \geq T$ because* (2.4) *is autonomous, hence* (2.6) *yields that*

$$\sum_{n=0}^{\infty} V(n; T) = \sum_{n=0}^{\infty} V(n; 0).$$

We can rewrite (2.1) in the form

$$\Delta x(n) = \sum_{i=0}^{m} A_i x(n - k_i) + f(n), \qquad n \in \mathbb{N}, \tag{2.7}$$

where

$$f(n) \equiv \sum_{i=0}^{m} A_i \Big(x(n - k_i - \eta_i(n)) - x(n - k_i) \Big). \tag{2.8}$$

In this setting (2.4) can be considered as the homogeneous equation corresponding to (2.7). The variation-of-constants formula (see e.g. in [6]) gives the following expression for the solution of the initial value problem (2.1)-(2.2):

$$x(n) = y(n) + \sum_{i=T}^{n-1} V(n - i - 1) f(i), \qquad n \in \mathbb{N}, \quad n \geq T, \tag{2.9}$$

where $T > 0$ is an integer number, and y is the solution of (2.4) with initial function $y(n) = x(n)$ for $T - N \leq t \leq T$ and $V(\cdot) = V(\cdot; T)$ is the fundamental solution of (2.4).

Remark 2.2 *Hypothesis (H) implies that there exist constants* $0 \leq \lambda < 1$ *and* $K > 0$ *such that* $|v_{ij}(n)| \leq \|V(n)\| \leq K\lambda^n$ *for* $n \geq 0$, *(where* $\|\cdot\|$ *is the matrix norm induced by the vector norm* $\|(x_1, x_2, \ldots, x_n)\| \equiv max\{|x_1|, |x_2|, \ldots, |x_n|\}$*), and therefore every element of the matrix* $\sum_{n=0}^{\infty} \tilde{V}(n)$ *is finite.*

The next theorem shows, that if the perturbations of the delays in (2.1) are small enough for large t, then the equation remains asymptotically stable.

Theorem 2.3 *Assume (H) and that the matrix*

$$M \equiv \left(\sum_{n=0}^{\infty} \tilde{V}(n) \right) \left(\sum_{i=0}^{m} \overline{\lim_{n\to\infty}} |\eta_i(n)| \cdot \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \tag{2.10}$$

has spectral radius less than 1, *i.e.,* $\rho(M) < 1$. *Then the trivial solution of* (2.1) *is asymptotically stable.*

Proof: Since the proof goes analogously to that in the continuous case (see in [5]), here we show only the main steps of the proof.

(i) First, we can show, using (2.8) and (2.1), that for some $T > 0$ and $n > T$, the function, $f(n)$, satisfies

$$\tilde{f}(n) \leq \left(\sum_{i=0}^{m} |\eta_i(n)| \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0 \leq j \leq n} \tilde{x}(j), \qquad n \geq T, \tag{2.11}$$

where we use the notation

$$\max_{0 \leq j \leq n} \tilde{x}(j) \equiv \left(\max_{0 \leq j \leq n} |x_1(j)|, \max_{0 \leq j \leq n} |x_2(j)|, \ldots, \max_{0 \leq j \leq n} |x_N(j)| \right)^T.$$

(ii) Define the matrix

$$M_0 \equiv \sum_{n=0}^{\infty} \tilde{V}(n) \left(\sum_{i=0}^{m} \tilde{A}_i \right)^2. \tag{2.12}$$

(We note, that according to Remark 2.1, matrices M and M_0 are independent of the choice of T.) It is easy to see that $\rho(M) < 1$ implies that there exists $\delta > 0$ such that

$$\rho(M + \delta M_0) < 1. \tag{2.13}$$

With this δ we can choose T such that (2.11) holds and furthermore, we have the following relations

$$|\eta_i(n)| < \overline{\lim_{j \to \infty}} |\eta_i(j)| + \delta, \qquad n \geq T, \quad i = 0, \ldots, m. \tag{2.14}$$

Then (2.11) yields the following estimate

$$\tilde{f}(n) \leq \left(\sum_{i=0}^{m} (\overline{\lim_{j \to \infty}} |\eta_i(j)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0 \leq j \leq n} \tilde{x}(j), \quad n > T. \tag{2.15}$$

(iii) Next we prove that the solution of (2.1) is bounded for all initial functions. Choose $T > 0$ such that (2.15) holds. For such T, formula (2.9) and standard estimates yield the inequality

$$\tilde{x}(n) \leq \tilde{y}(n) + \sum_{i=T}^{n-1} \tilde{V}(n-i-1) \, \tilde{f}(i), \qquad n \geq T. \tag{2.16}$$

From this inequality, using the definition of M and M_0, and estimate (2.15), we can derive that

$$\max_{0\le j\le n} \tilde{x}(j) \le \max_{0\le j\le n} \tilde{y}(j) + (M+\delta M_0) \max_{0\le j\le n} \tilde{x}(j). \tag{2.17}$$

Rearranging (2.17) and using that $y(n)$ is bounded by hypothesis (H), we have that there exists a constant vector $z \ge 0$ such that

$$(I-(M+\delta M_0)) \max_{0\le j\le n} \tilde{x}(j) \le \max_{0\le j\le n} \tilde{y}(j) \le z, \qquad n \ge T. \tag{2.18}$$

Inequality (2.13) and the fact that $M+\delta M_0$ has nonnegative components imply that $I-(M+\delta M_0)$ is a nonsingular M-matrix, therefore an application of Theorem 6.2.3 in [1] yields that $I-(M+\delta M_0)$ is a monotone matrix, hence

$$\max_{0\le j\le n} \tilde{x}(j) \le (I-(M+\delta M_0))^{-1} z, \qquad n \ge T,$$

i.e., $x(n)$ is bounded for $n \ge 0$.

(iv) Next we show that $x(n)$ tends to 0 as $n \to \infty$, i.e., $\overline{\lim}_{n\to\infty} \tilde{x}(n) = 0$. Using that by step (iii) above we have that $\overline{\lim}_{n\to\infty} \tilde{x}(n)$ is finite, and from assumption (H) it follows that $\overline{\lim}_{n\to\infty} \tilde{y}(n) = 0$, we can show that (2.16) implies

$$\overline{\lim_{n\to\infty}} \tilde{x}(n) \le M \overline{\lim_{n\to\infty}} \tilde{x}(n),$$

and hence

$$(I-M) \overline{\lim_{n\to\infty}} \tilde{x}(n) \le 0. \tag{2.19}$$

By assumption $\rho(M) < 1$, M has nonnegative components, and therefore $I-M$ is a nonsingular M-matrix. Using again Theorem 6.2.3 in [1] we get that $I-M$ is monotone, hence (2.19) yields that $\overline{\lim}_{n\to\infty} \tilde{x}(n) \le 0$. On the other hand $\overline{\lim}_{n\to\infty} \tilde{x}(n) \ge 0$, therefore $\overline{\lim}_{n\to\infty} \tilde{x}(n) = 0$.
This completes the proof of the theorem.

The following corollary is an easy consequence of the theorem.

Corollary 2.4 *Let M_0 defined by* (2.12). *If*

$$\overline{\lim_{n\to\infty}} |\eta_i(n)| < \frac{1}{\rho(M_0)}, \qquad i = 0, \dots, m,$$

then the trivial solution of (2.1) *is asymptotically stable.*

If the fundamental solution $V(n)$ of (2.4) is nonnegative, (i.e., each component $v_{ij}(n)$ of $V(t)$ is nonnegative and therefore $V(n) = \tilde{V}(n)$), then it is easy to compute the integral in (2.12). In particular, we have the following result.

Proposition 2.5 *If the trivial solution of* (2.4) *is asymptotically stable, then the fundamental solution of* (2.4) *satisfies*

$$\left(\sum_{i=0}^{m} A_i\right) \sum_{n=0}^{\infty} V(n) = -I,$$

where I *is the identity matrix.*

Proof: Let $V(t)$ be the fundamental solution of (2.4) corresponding to $T = 0$. By summing (2.5) for 0 to $n > 0$ we get

$$V(n+1) - V(0) = \sum_{i=0}^{m} A_i \sum_{j=0}^{n} V(j - k_i).$$

A change of variables in the integrals and the assumed initial condition $V(n) = 0$ for $n < 0$ yield

$$\begin{aligned} V(n+1) - V(0) &= \sum_{i=0}^{m} A_i \sum_{j=-r_i}^{n-r_i} V(j) \\ &= \sum_{i=0}^{m} A_i \sum_{j=0}^{n-r_i} V(n). \end{aligned}$$

Using $V(0) = I$ and the fact $V(t) \to 0$ as $t \to \infty$ we obtain the equality

$$-I = \left(\sum_{i=0}^{m} A_i\right) \sum_{j=0}^{\infty} V(j),$$

which proves the proposition.

Remark 2.6 *In the case when* $V(t)$ *is nonnegative, and* $\sum_{i=0}^{m} A_i$ *is nonsingular, Proposition 2.5 implies that*

$$M_0 = -\left(\sum_{i=0}^{m} A_i\right)^{-1} \left(\sum_{i=0}^{m} \tilde{A}_i\right)^2, \tag{2.20}$$

therefore our stability condition in Corollary 2.4 can be evaluated using the coefficient matrices related to the difference equation.

In the rest of this section we state the scalar version of our results. Consider the scalar linear delay difference equation

$$\Delta x(n) = \sum_{i=0}^{m} a_i x(n - k_i - \eta_i(n)), \qquad n \in \mathbb{N}, \tag{2.21}$$

and the corresponding constant delay difference equation

$$\Delta y(n) = \sum_{i=0}^{m} a_i y(n - k_i), \qquad n \in \mathbb{N}. \tag{2.22}$$

Let $v(n)$ be the fundamental solution of (2.22), i.e., the solution of (2.22) corresponding to initial condition $v(0) = 1$ and $v(n) = 0$ for $n < 0$. Then the scalar version of Theorem 2.3 can be stated as follows.

Theorem 2.7 *Assume that the trivial solution of* (2.22) *is asymptotically stable. Then if the perturbations,* η_i*, satisfy*

$$\sum_{i=0}^{m} |a_i| \varlimsup_{n\to\infty} |\eta_i(n)| < \frac{1}{\sum_{i=0}^{m} |a_i| \sum_{n=0}^{\infty} |v(n)|}, \tag{2.23}$$

then the trivial solution of (2.21) *is asymptotically stable.*

Theorem 2.7 and Proposition 2.5 have the following corollary.

Corollary 2.8 *Assume that the trivial solution of* (2.22) *is asymptotically stable, and the fundamental solution of* (2.22) *is nonnegative. Then condition*

$$\sum_{i=0}^{m} |a_i| \varlimsup_{n\to\infty} |\eta_i(n)| < \frac{-\sum_{i=0}^{m} a_i}{\sum_{i=0}^{m} |a_i|}.$$

implies that the trivial solution of (2.21) *is asymptotically stable.*

3. Examples and Applications

Example 3.1 Consider the scalar delay difference equation

$$\Delta x(n) = -px(n - k - \eta(n)), \qquad n \in \mathbb{N}, \tag{3.1}$$

TABLE 1.

k	m
20	100.00
40	99.76
60	78.54
80	53.59
100	34.13
120	19.25
140	7.71

TABLE 2.

k	m
40	15.38
60	15.38
80	15.38
100	13.74
120	8.13
140	3.49
150	1.51

and the corresponding unperturbed equation

$$\Delta y(n) = -py(n-k), \qquad n \in \mathbb{N}. \tag{3.2}$$

It is known (see e.g. in [7]), that the trivial solution of (3.2) is asymptotically stable if and only if

$$0 < p < 2\cos\frac{k\pi}{2k+1}. \tag{3.3}$$

It follows from [6], that for $p > 0$ the fundamental solution is nonnegative if and only if

$$p < \frac{k^k}{(k+1)^{k+1}}. \tag{3.4}$$

Consider a specific case, let $p = 0.01$. Then (3.3) and (3.4) yield that the trivial solution of (3.2) is asymptotically stable for $k = 0, 1, \dots, 156$, and the corresponding fundamental solution is nonnegative for $k = 0, 1, \dots, 36$. By Theorem 2.7 and Corollary 2.8 we have that the trivial solution of (3.1) is asymptotically stable if

$$\overline{\lim_{n\to\infty}}|\eta(n)| < m \equiv \begin{cases} \dfrac{1}{0.01}, & k \le 36, \\ \dfrac{1}{(0.01)^2 \sum\limits_{n=0}^{\infty} |v(n)|}, & k > 36. \end{cases}$$

In Table 1 we present some numerical values of the upper bound, m, of the perturbations corresponding to several delays. We can observe, that

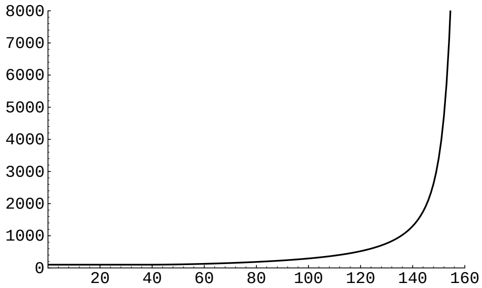

FIGURE 1.

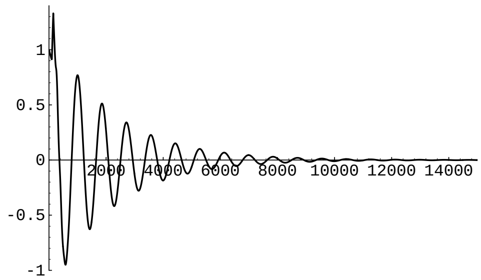

FIGURE 2.

as k increases, i.e., when there is more oscillation in the fundamental solution, m becomes smaller.

Next we examine the infinite sum of the elements of the fundamental solution of (3.2) as a function of k. Let $v_k(n)$ be the fundamental solution (3.2) corresponding to delay k, and define

$$w(k) \equiv \sum_{n=0}^{\infty} |v_k(n)|.$$

By Proposition 2.5 we have that $w(k)$ is constant, $w(k) = 100$ for $0 \leq k \leq 36$, and we have that $w(k) = \infty$ for $k > 156$. Numerical study (see on Figure 1) reveals that $w(k)$ is a monotone increasing function of k. Note, that here and later in all figures, the discrete function values are connected to a continuous graph.

Example 3.2 Consider the scalar delay difference equation with two delayed terms

$$\Delta x(n) = -0.001x(n) + 0.01x(n-100) - 0.015x(n-k-\eta(n)), \quad (3.5)$$

where, for simplicity, only the second delay is perturbed. By Theorem 2.7 we have that the trivial solution of the equation is asymptotically stable, if

$$\overline{\lim_{n\to\infty}} |\eta(n)| < m \equiv \frac{1}{0.015 \cdot 0.026 \cdot \sum_{n=0}^{\infty} |v_k(n)|},$$

where $v_k(n)$ is the fundamental solution of the corresponding unperturbed equation. Table 2 presents numerical values of m corresponding

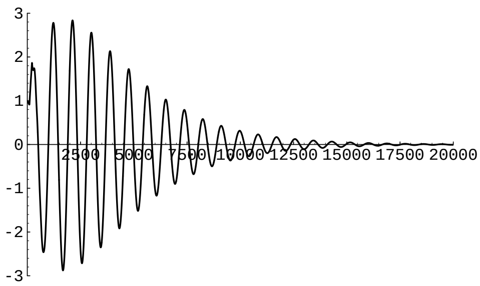

FIGURE 3.

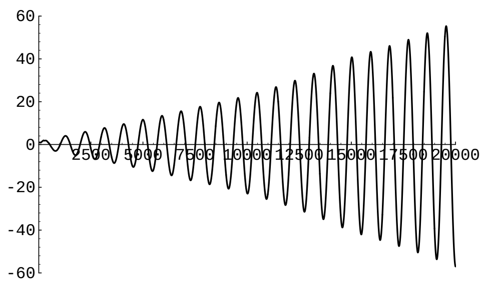

FIGURE 4.

to different k values. On Figure 2 we show the fundamental solution of the unperturbed equation with $k = 150$. We graph the numerical solution of (3.5) (with initial condition $x(0) = 1$, $x(n) = 0$ for $n < 0$) using perturbation $\eta(n) = 15000/n + 1$ on Figure 3, and with $\eta(n) = 15000/n + 10$ on Figure 4.

Example 3.3 Consider the two dimensional vector delay difference equation

$$\Delta x(n) = A_0 x(n) + A_1 x(n-100) + A_2 x(n-140-\eta(n)), \qquad (3.6)$$

where

$$A_0 = \begin{pmatrix} -0.001 & 0.002 \\ 0.000 & -0.003 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.000 & 0.001 \\ 0.000 & -0.002 \end{pmatrix} \quad \text{and}$$
$$A_2 = \begin{pmatrix} -0.002 & 0.000 \\ 0.002 & 0.000 \end{pmatrix}$$

Numerical study shows that the fundamental solution of the corresponding unperturbed equation is nonnegative (see on Figure 5 the components of the fundamental solution). Therefore by Proposition 2.5 we have that

$$\sum_{n=0}^{\infty} \tilde{V}(n) = -(A_0 + A_1 + A_2)^{-1} = \begin{pmatrix} 555.556 & 333.333 \\ 222.222 & 333.333 \end{pmatrix},$$

and hence

$$\begin{aligned} M &= -\overline{\lim_{n\to\infty}} |\eta(n)| (A_0 + A_1 + A_2)^{-1} \tilde{A}_2 (\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2) \\ &= \overline{\lim_{n\to\infty}} |\eta(n)| \begin{pmatrix} 0.0053 & 0.0053 \\ 0.0033 & 0.0033 \end{pmatrix}. \end{aligned}$$

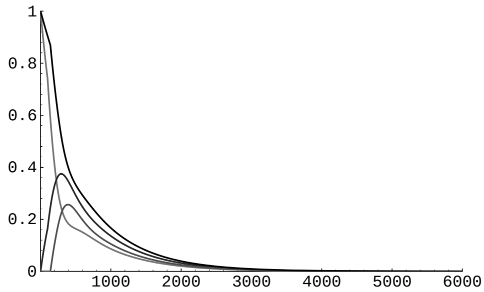

FIGURE 5.

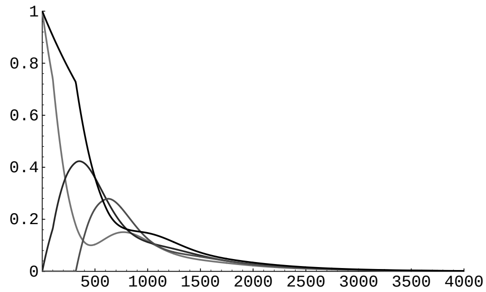

FIGURE 6.

It is easy to see that $\rho(M) < 1$ if $\overline{\lim}_{n\to\infty} |\eta(n)| < 115.385$. On Figure 6 we plot the numerical solution of (3.6) (corresponding to initial values $x(0) = 1$, $x(n) = 0$, $n < 0$) with perturbation $\eta(n) = \frac{20000}{n} + 115$.

Example 3.4 Finally, consider the vector delay difference equation

$$\Delta x(n) = A_0 x(n) + A_1 x(n-100) + A_2 x(n-150-\eta(n)), \qquad n \in \mathbb{N}, \quad (3.7)$$

with

$$A_0 = \begin{pmatrix} -0.001 & 0.003 \\ -0.005 & 0.000 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.007 & -0.004 \\ 0.005 & -0.008 \end{pmatrix} \quad \text{and}$$

$$A_2 = \begin{pmatrix} -0.01 & 0.001 \\ 0.001 & 0.004 \end{pmatrix}$$

By approximating $\sum_{n=0}^{\infty} V(n)$ numerically, and applying Theorem 2.3 we get that if $\overline{\lim}_{n\to\infty} |\eta(n)| < 7.75$ then the trivial solution of (3.7) is asymptotically stable. Figure 7 shows the components of the fundamental solution of the unperturbed equation, and Figure 8 contains the components of the solution of (3.7) (corresponding to the same initial values) with perturbation $\eta(n) = 200$ for $n < 1000$ and $\eta(n) = 7$ for $n \geq 1000$.

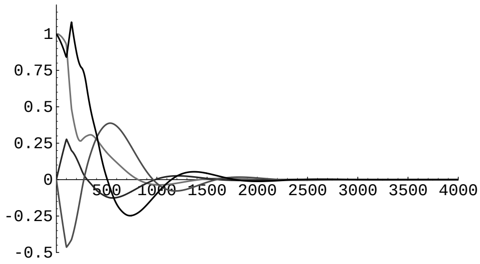

FIGURE 7.

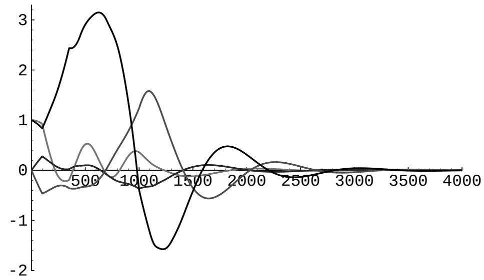

FIGURE 8.

4. Stability Results

In this section, using the perturbation result of Section 2, we obtain sufficient stability conditions for the scalar delay difference equation

$$\Delta x(n) = -\sum_{i=0}^{m} a_i x(n - \sigma_i(n)), \qquad n \in \mathbb{N}, \tag{4.1}$$

where the delay terms, $\sigma_i : \mathbb{N} \to \mathbb{Z}$, are bounded functions. We can think of $\sigma_i(n)$ in (4.1) as perturbations of zero delays, i.e., (4.1) can be considered as a perturbed equation corresponding to the unperturbed equation

$$\Delta y(n) = -\left(\sum_{i=0}^{m} a_i\right) y(n), \qquad n \in \mathbb{N}. \tag{4.2}$$

We would like to apply Theorem 2.7, and in fact, Corollary 2.8 (in order to use condition which can be checked easily), therefore we need to guarantee that the trivial solution of (4.2) is asymptotically stable, and the fundamental solution, $v(n)$, of (4.2) is nonnegative. It is easy to check, that the inequality $0 < (\sum_{i=0}^m a_i) < 1$ implies both properties. Therefore by Corollay 2.8 the following result follows immediately.

Proposition 4.1 *Assume that*

(i) $0 < \sum_{i=0}^m a_i < 1$, *and*

(ii) $\displaystyle\sum_{i=0}^{m} |a_i| \varlimsup_{n\to\infty} |\sigma_i(n)| < \frac{\sum_{i=0}^m a_i}{\sum_{i=0}^m |a_i|}$.

Then the trivial solution of (4.1) *is asymptotically stable.*

Note, that in condition (ii) of the previous proposition the right hand side of the inequality is always less or equal to 1, and equal to 1 if and only if each a_i is positive.

In the rest of this section we assume that $a_i > 0$ for $i = 0, 1, \ldots, m$. In this special case we shall improve condition (ii). Rewrite (4.1) in the form

$$\Delta x(n) = -\sum_{i=0}^{m} a_i x(n - k - (\sigma_i(n) - k)), \tag{4.3}$$

where k is a positive integer, and consider

$$\Delta y(n) = -\left(\sum_{i=0}^{m} a_i\right) y(n-k), \qquad n \in \mathbb{N}. \tag{4.4}$$

Then, again, (4.3), and hence (4.1) can be considered as an equation obtained by perturbing the constant delays of (4.4) by $\sigma_i(n) - k$. As before, if the trivial solution of (4.4) is asymptotically stable, and the fundamental solution of (4.4) is positive, then by applying Corollary 2.8, we can obtain a sufficient condition for the asymptotic stability of the trivial solution of (4.1). It is known (see [6]), that the inequality

$$0 < \sum_{i=0}^{m} a_i < \frac{k^k}{(k+1)^{k+1}},$$

or equivalently,

$$0 < k\sum_{i=0}^{m} a_i < \frac{k^{k+1}}{(k+1)^{k+1}} \tag{4.5}$$

yields both properties. To further simplify condition (4.5), using that the sequence $k^{k+1}/(k+1)^{k+1}$ is monotone increasing, and hence

$$\frac{1}{4} \le \frac{k^{k+1}}{(k+1)^{k+1}}, \qquad k = 1, 2, \ldots,$$

we get that if we can select k such that

$$0 < k\sum_{i=0}^{m} a_i \le \frac{1}{4}, \tag{4.6}$$

then the trivial solution of (4.4) is asymptotically stable, and the fundamental solution of (4.4) is nonnegative. (Note, that the equality for $k = 1$ does not follow from the previous argument, but can easily be proved directly.) Therefore, by Corollay 2.8, if in addition to (4.6)

$$\sum_{i=0}^{m} a_i \overline{\lim_{n\to\infty}} |\sigma_i(n) - k| < 1 \tag{4.7}$$

holds, then the trivial solution of (4.1) is asymptotically stable.

In two special cases, when all delays are "small", or all delays are "large", we can obtain explicit conditions.

Case 1: Assume that there exists a $T \in \mathbb{N}$ such that $\sigma_i(n) \le \frac{1}{4\sum_{j=0}^{m} a_j}$ for $n > T$ and all $i = 0, 1, \dots, m$.

In this case select $k = \left[1/\left(4\sum_{i=0}^{m} a_i\right)\right]$. With this choice of k, the following elementary estimates

$$\begin{aligned}
\sum_{i=0}^{m} a_i \overline{\lim_{n\to\infty}} |\sigma_i(n) - k| &= \left[\frac{1}{4\sum_{i=0}^{m} a_i}\right] \sum_{i=0}^{m} a_i - \sum_{i=0}^{m} a_i \underline{\lim_{n\to\infty}}\, \sigma_i(n) \\
&\le \frac{1}{4} - \sum_{i=0}^{m} a_i \underline{\lim_{n\to\infty}}\, \sigma_i(n) \\
&< 1
\end{aligned}$$

show that (4.7) is always satisfied.

Case 2: Assume that there exist constants $T \in \mathbb{N}$ and $0 < \alpha \le 1$ such that $\sigma_i(n) \ge \frac{\alpha}{4\sum_{j=0}^{m} a_j}$ for $n > T$ and all $i = 0, 1, \dots, m$, and $k \equiv \frac{\alpha}{4\sum_{i=0}^{m} a_i}$ is an integer. In this case we have that

$$\sum_{i=0}^{m} a_i \overline{\lim_{n\to\infty}} |\sigma_i(n) - k| = \sum_{i=0}^{m} a_i \overline{\lim_{n\to\infty}}\, \sigma_i(n) - \frac{\alpha}{4}.$$

We have proved the following result.

Proposition 4.2 *Assume that $a_i > 0$ for $i = 0, 1, \dots, m$. Then either one of the following two conditions implies the asymptotic stability of the trivial solution of* (4.1).

(i) There exists $T > 0$ such that $\sigma_i(n) \le \frac{1}{4\sum_{j=0}^m a_j}$ for $n > T$ and $i = 0, 1, \ldots, m$,

(ii) There exists $T > 0$ and $0 \le \alpha \le 1$ such that $\sigma_i(n) \ge \frac{\alpha}{4\sum_{j=0}^m a_j}$ for $n > T$ and all $i = 0, 1, \ldots, m$, and

$$\sum_{i=0}^m a_i \overline{\lim_{n\to\infty}} \sigma_i(n) < 1 + \frac{\alpha}{4}.$$

Propositions 4.1 and 4.2 generalize the stability condition of [2], where it was shown, that the trivial solution of (4.1) is asymptotically stable, provided that the delays are constant, the coefficients of the equation are positive, and $\sum_{i=0}^m a_i\sigma_i < 1$. Note, that similar type of stability conditions was investigated for delay differential equations by several authors. We refer to [8], [11], [12], [13], [14], and [7] and the references therein for results in this topic in the continuous, and discrete case as well.

REFERENCES

[1] A. Berman and R. J. Plemmons, "Nonnegative Matrices in the Mathematical Sciences", Academic Press, New York, 1979.

[2] K. L. Cooke and I. Győri, *Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments*, IMA Preprint Series #633, 1990.

[3] E. Cheres, Z. J. Palmor, and S. Gutman, *Qualitative measures of robustness for systems including delayed perturbations*, IEEE Trans. Automat. Contr. **34** (1989), 1203–1204.

[4] R. D. Driver, "Ordinary and Delay Differential Equations", Springer-Verlag, New York, 1977.

[5] I. Győri, F. Hartung and J. Turi, *Preservation of stability in delay equations under delay perturbations*, Preprint.

[6] I. Győri and G. Ladas, "Oscillation Theory of Delay Differential Equations", Clarendon Press, Oxford, 1991.

[7] V. L. Kocic and G. Ladas, "Global Behavior of Nonlinear Difference Equations of Higher Order with Applications", Kluwer Academic Publishers, 1993.

[8] T. Krisztin, *On stability properties for one-dimensional functional differential equations*, Funkcional Ekvacioj **34** (1991), 241–256.

[9] A. Stokes, *Stability of functional differential equations with perturbed lags*, J. Math. Anal. and Appl. **47** (1974), 604–619.

[10] Y. Z. Tsypkin and M. Fu, *Robust stability of time-delay system with an uncertain time-delay constant*, Int. J. Control **57** (1993), 865–879.

[11] T. Yoneyama and J. Sugie, *On the stability region of scalar delay-differential equation*, J. Math. Anal. Appl. **134** (1988), 408–425.

[12] T. Yoneyama and J. Sugie, *On the stability region of differential equation with two delays*, Funkcialaj Ekvacioj **31** (1988), 233–240.

[13] T. Yoneyama, *The 3/2 stability theorem for one-dimensional delay-differential equations with unbounded delay*, J. Math. Anal. Appl. **165** (1992), 133–143.

[14] J. A. Yorke, *Asymptotic stability for one dimensional differential-delay equations*, J. Differential Equations **7** (1988), 189–202.

[15] D-N. Zhang, M. Saeki, and K. Ando, *Stability margin calculation of systems with structured time-delay uncertainties*, IEEE Trans. Automat. Contr. **37** (1992), 865–868.