to appear in Applied Mathematics and Computation

BIBO stabilization of feedback control systems with time dependent delays*

Essam Awwad[a,b], István Győri[a] and Ferenc Hartung[a]
[a]Department of Mathematics, University of Pannonia, Hungary
[b]Department of Mathematics, Benha University, Benha, Egypt

Abstract

This paper investigates the bounded input bounded output (BIBO) stability in a class of control system of nonlinear differential equations with time-delay. The proofs are based on our studies on the boundedness of the solutions of a general class of nonlinear Volterra integral equations.

Keywords: boundedness, Volterra integral equations, bounded input bounded output (BIBO) stability, differential equations with delays.

1 Introduction

Differential and integral equations with time delays appear frequently as mathematical models in natural sciences, economics and engineering, and they are used to describe propagation and transport phenomena (see, e.g., [1, 6, 7, 8, 9, 11, 15, 17, 22]). In control engineering the time delay appears in the control naturally, since time is needed to sense information and to react to it. The first control applications using time delays go back to the 30s [4, 5], and since that it is an extensively studied field (see, e.g., [6, 7, 23, 25, 29]).

Stability of a differential equation is a central question in engineering applications, especially in control theory. Many different concepts of stability has been investigated in the literature, e.g., stability of an equilibrium of the system, orbital stability of the system output trajectory, or structural stability of the system. Because of its simplicity and weakness, the notion of bounded-input bounded-output (BIBO) stability was extensively studied and used in control theory for different classes of dynamical systems (see, e.g.,

*This research was partially supported by the Hungarian National Foundation for Scientific Research Grant No. K101217.

[2, 10, 16, 18, 19, 20, 26, 27, 28]). We say that a control system is BIBO stable, if any bounded input produces a bounded output. It is known (see, e.g., [2]) that a linear control system $\dot{x} = Ax + Bu$ is BIBO stable, if the trivial solution of $\dot{x} = Ax$ is asymptotically stable, i.e., all eigenvalues of A have negative real parts.

Using a novel approach, in this paper we study the BIBO stability of a feedback control system with time delays. First we rewrite the control system as a Volterra integral equation, then, motivated by the results obtained for discrete Volterra systems [12], we formulate sufficient conditions for the boundedness of nonlinear Volterra integral equations, and finally we apply our general results to obtain BIBO stability of the control system. Also we introduce a new notion of BIBO stability, which we call local BIBO stability.

The rest of the article is organized as follows. In Section 2 we give the system description, some notations and definitions which are used throughout the paper. In Section 3 we state our main results. In Section 4 we give sufficient conditions for the boundedness of the solution of Volterra integral equations. In Section 5 some special estimates are studied. In Section 6 we give the proofs of our main results.

2 System description and preliminaries

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the set of real numbers, nonnegative real numbers, n-dimensional real column vectors and $n \times n$-dimensional real matrices, respectively. The maximum norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$, i.e., $\|x\| := \max_{1\le i\le n} |x_i|$, where $x = (x_1, \dots, x_n)^T$. The matrix norm on $\mathbb{R}^{n\times n}$ generated by the maximum vector norm will be denoted by $\|\cdot\|$, as well. L^n_∞ will denote the set of bounded functions $r : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ with norm $\|r\|_\infty := \sup_{t\ge 0} \|r(t)\|$. $C(X, Y)$ denotes the set of continuous functions mapping from X to Y.

In this paper we consider the nonlinear control system

$$\dot{x}(t) = g(t, x(t - \sigma(t))) + u(t), \qquad t \ge 0, \tag{2.1}$$

where $x(t)$, $u(t)$ are the state vector, control input, respectively, $\sigma(t) \ge 0$ is a continuous function and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\|g(s, \tilde{x})\| \le b(s)\phi\left(\|\tilde{x}\|\right), \qquad s \in \mathbb{R}_+, \quad \tilde{x} \in \mathbb{R}^n, \tag{2.2}$$

where $b(s) > 0$ is a continuous function, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone nondecreasing mapping. Throughout the paper, we assume that the output of the system is the whole state $x(t)$, so we use the state in the feedback control.

We assume that the uncontrolled system, i.e., (2.1) with $u \equiv 0$ has unbounded solutions. Our goal is to find a control law of the form

$$u(t) = -Dx(t - \tau) + r(t) \tag{2.3}$$

which guarantees that the closed-loop system

$$\begin{cases} \dot{x}(t) = g(t, x(t-\sigma(t))) - Dx(t-\tau) + r(t), & t > 0, \\ x(t) = \psi(t), & -t_0 \le t \le 0 \end{cases} \tag{2.4}$$

is BIBO stable. We suppose that the feedback gain in (2.3) is a positive diagonal matrix, i.e.,

$$D = \operatorname{diag}(d_1, \dots, d_n), \quad d_i > 0, \quad i = 1, \dots, n.$$

$r \in L_\infty^n$ is the reference input, $t_0 := \max\left\{\tau, -\inf_{t\ge 0}\{t - \sigma(t)\}\right\}$, ψ is a continuous vector-valued initial function and $\|\psi\|_{t_0} := \max_{-t_0 \le t \le 0} \|\psi(t)\|$.

In applications the time delay τ in the control (2.3) appears naturally, since time is needed to sense information and react to it. Using diagonal feedback (2.3) is the simplest possible control law, and in Theorem 3.1 and 3.3 we give sufficient conditions on how to select the feedback gain and the time delay τ to guarantee the boundedness of the solutions.

Based on the definition of BIBO stability in [24], we introduce the next definition.

Definition 2.1. The control system (2.4) with reference input $r : \mathbb{R}_+ \to \mathbb{R}^n$ is BIBO stable if there exist positive constants γ_1 and $\gamma_2 = \gamma_2(\|\psi\|_{t_0})$ satisfying

$$\|x(t)\| \le \gamma_1 \|r\|_\infty + \gamma_2, \qquad t \ge 0$$

for every reference input $r \in L_\infty^n$.

In Theorem 3.3 below we need a weaker version of BIBO stability. Next we introduce this new notion, which we call local BIBO stability.

Definition 2.2. The control system (2.4) with reference input $r : \mathbb{R}_+ \to \mathbb{R}^n$ is locally BIBO stable if there exist positive constants δ_1, δ_2 and γ satisfying

$$\|x(t)\| \le \gamma, \qquad t \ge 0$$

provided that $\|\psi\|_{t_0} < \delta_1$ and $\|r\|_\infty < \delta_2$.

Our approach is the following. We associate the linear system

$$\dot{z}(t) = -Dz(t-\tau), \qquad t \ge 0 \tag{2.5}$$

with the constant delay τ and the initial condition

$$z(t) = \psi(t), \qquad -\tau \le t \le 0 \tag{2.6}$$

to (2.4). Then the state equation (2.4) can be considered as the nonlinear perturbation of (2.5), so the variation of constants formula [14] yields

$$x(t) = z(t) + \int_0^t W(t-s)\left[g(s, x(s-\sigma(s))) + r(s)\right] ds, \qquad t \ge 0. \tag{2.7}$$

Here $z(t)$ is the solution of (2.5)-(2.6) and $W(t)$ is the fundamental solution of (2.5) i.e. the solution of initial value problem

$$\begin{aligned} \dot{W}(t) &= -DW(t-\tau), \qquad t \geq 0, \\ W(t) &= \begin{cases} \mathbf{0}, & -\tau \leq t < 0, \\ \mathrm{I}, & t = 0, \end{cases} \end{aligned}$$

where I is the identity matrix and $\mathbf{0}$ is the zero matrix. Since D is a diagonal matrix, it is easy to see that $W(t)$ is diagonal matrix too, and $W(t) = \operatorname{diag}(w_1(t), \dots, w_n(t))$, where for $i = 1, \dots, n$

$$\begin{aligned} \dot{w}_i(t) &= -d_i w_i(t-\tau), \qquad t \geq 0, \\ w_i(t) &= \begin{cases} 0, & -\tau \leq t < 0, \\ 1, & t = 0. \end{cases} \end{aligned} \tag{2.8}$$

We can rewrite the equation (2.7) as a Volterra integral equation

$$x(t) = \int_0^t f(t, s, x(\cdot))ds + h(t), \qquad t \geq 0, \tag{2.9}$$

where

$$h(t) := z(t) + \int_0^t W(t-s)r(s)ds, \tag{2.10}$$

and

$$f(t, s, x(\cdot)) := W(t-s)g(s, x(s-\sigma(s))). \tag{2.11}$$

In Section 4 we will study the boundedness of the solutions of nonlinear Volterra integral equations, and apply our results for the boundedness of (2.4).

3 Main results

Our main goal in this section is to formulate sufficient conditions which grantee BIBO stability of the control system (2.4).

Our first result is given for the case when g in (2.1) has a sub-linear estimate, i.e., when $\phi(t) = t^p$, with $0 < p < 1$ in (2.2).

Theorem 3.1. *Let $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function which satisfies inequality* (2.2) *with $\phi(t) = t^p$, $0 < p < 1$, $t \geq 0$. The feedback control system* (2.4) *is BIBO stable if*

$$\|b\|_\infty := \sup_{t\geq 0} b(t) < \infty, \qquad 0 < d_i < \frac{\pi}{2\tau}, \quad t \geq 0, \quad i = 1, \dots, n \tag{3.1}$$

hold.

The following theorem gives a sufficient condition for the BIBO stability in the case of a linear estimate.

Theorem 3.2. *Let $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function which satisfies inequality* (2.2) *with $\phi(t) = t$, $t \geq 0$. The feedback control system* (2.4) *is BIBO stable if*

$$\|b\|_\infty < d_i \leq \frac{1}{e\tau}, \qquad t \geq 0, \quad i = 1, \dots, n \tag{3.2}$$

holds.

In the super-linear case, when $\phi(t) = t^p$, $p > 1$ in (2.2), we show that the control system (2.4) is locally BIBO stable.

Theorem 3.3. *Let $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function which satisfies inequality* (2.2) *with $\phi(t) = t^p$, $p > 1$, $t \geq 0$. Then the solution x of the feedback control system* (2.4) *is locally BIBO stable if (3.2) holds.*

The proofs of the main results are given in Section 6.

4 Volterra integral equation

In this section we obtain sufficient conditions for the boundedness of the solutions of nonlinear Volterra integral equations.

We consider the nonlinear Volterra integral equation

$$x(t) = \int_0^t f(t, s, x(\cdot))ds + h(t), \qquad t \geq 0, \tag{4.1}$$

with initial condition

$$x(t) = \psi(t), \qquad t \in [-\tilde{t}_0, 0], \tag{4.2}$$

where $\tilde{t}_0 \geq 0$ is fixed, and the following conditions are satisfied.

(A1) The function $f : \mathbb{R}_+ \times \mathbb{R}_+ \times C\left([-\tilde{t}_0, \infty), \mathbb{R}^n\right) \to \mathbb{R}^n$ is a Volterra-type functional, i.e., f is continuous and $f(t, s, x(\cdot)) = f(t, s, y(\cdot))$ for all $0 \leq s \leq t$ and x, $y \in C\left([-\tilde{t}_0, \infty), \mathbb{R}^n\right)$ if $x(\tilde{\mu}) = y(\tilde{\mu})$, $-\tilde{t}_0 \leq \tilde{\mu} \leq s$.

(A2) For any $0 \leq s \leq t$ and $x \in C\left([-\tilde{t}_0, \infty), \mathbb{R}^n\right)$,

$$|f_i(t, s, x(\cdot))| \leq k_i(t, s)\phi(\max_{-\tilde{t}_0 \leq \theta \leq s} ||x(\theta)||), \quad i = 1, \dots, n$$

holds, where $f(t) = (f_1(t), \dots, f_n(t))^T$. Here $k_i : \{(t, s) : 0 \leq s \leq t\} \to \mathbb{R}_+$, $i = 1, \dots, n$ is continuous, and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone nondecreasing mapping such that $\phi(\tilde{v}) > 0, \quad \tilde{v} > 0$, and $\phi(0) = 0$.

(A3) $h : \mathbb{R}_+ \to \mathbb{R}^n$ is continuous and $h(t) = (h_1(t), \dots, h_n(t))^T$, $t \geq 0$.

(A4) $\psi \in C\left([-\tilde{t}_0, 0], \mathbb{R}^n\right)$.

The next definitions will be useful in the proofs of our results on Volterra integral equations.

Definition 4.1. Let the functions ϕ and k_i be defined by assumption (A2). We say that the nonnegative constant μ has property (P_T) with $T \geq 0$ if there is $v \geq \mu$, such that

$$\phi(\mu) \int_0^T k_i(t,s)ds + \phi(v) \int_T^t k_i(t,s)ds + |h_i(t)| < v, \qquad t \geq T, \quad i = 1, \dots, n \tag{4.3}$$

holds.

Definition 4.2. We say that initial function $\psi \in C\left([-\tilde{t}_0, 0], \mathbb{R}^n\right)$ belongs to the set S if there exists $T \geq 0$ such that

$$\mu_T := \max_{-\tilde{t}_0 \leq t \leq T} \|x(t)\|$$

has property (P_T), where $x : [-\tilde{t}_0, \infty) \to \mathbb{R}^n$ is the solution of (4.1)-(4.2).

Remark 4.3. If there exist a $T \geq 0$ and two positive constants μ_T and v such that (4.3) holds, then

$$I_T : = \max_{1 \leq i \leq n} \sup_{t \geq T} \int_0^T k_i(t,s)ds < \infty, \tag{4.4}$$

$$J_T : = \max_{1 \leq i \leq n} \sup_{t \geq T} \int_T^t k_i(t,s)ds < \infty, \tag{4.5}$$

$$H_T : = \max_{1 \leq i \leq n} \sup_{t \geq T} |h_i(t)| < \infty. \tag{4.6}$$

Conditions (4.4) and (4.5) are equivalent to

$$J_0 := \max_{1 \leq i \leq n} \sup_{t \geq 0} \int_0^t k_i(t,s)ds < \infty, \tag{4.7}$$

and if $T = 0$ condition (4.6) becomes

$$H_0 := \max_{1 \leq i \leq n} \sup_{t \geq 0} |h_i(t)| < \infty. \tag{4.8}$$

The following result gives a sufficient condition for the boundedness of the solution of (4.1).

Theorem 4.4. *Let* (A1)-(A4) *be satisfied and the initial function* ψ *belongs to the set* S. *Then the solution* x *of* (4.1)-(4.2) *is bounded.*

Proof. Since the initial function ψ belongs to the set S, by Definition 4.2 there exists $T \geq 0$ such that $\mu_T := \max_{-\tilde{t}_0 \leq t \leq T} \|x(t)\|$ has property (P_T), where x is the solution of (4.1)-(4.2). From (4.1) we get

$$x(t) = \int_0^T f(t,s,x(\cdot))ds + \int_T^t f(t,s,x(\cdot))ds + h(t), \qquad t \geq T.$$

Hence (A2) yields for $i = 1, \dots, n$

$$\begin{aligned}
|x_i(t)| &\leq \int_0^T |f_i(t,s,x(\cdot))|ds + \int_T^t |f_i(t,s,x(\cdot))|ds + |h_i(t)| \\
&\leq \int_0^T k_i(t,s)\phi(\max_{-\tilde{t}_0 \leq \tilde{\mu} \leq T} \|x(\tilde{\mu})\|)ds + \int_T^t k_i(t,s)\phi(\max_{-\tilde{t}_0 \leq \tilde{\mu} \leq s} \|x(\tilde{\mu})\|)ds + |h_i(t)| \\
&\leq \phi(\mu_T)\int_0^T k_i(t,s)ds + \int_T^t k_i(t,s)\phi(\max_{-\tilde{t}_0 \leq \tilde{\mu} \leq s} \|x(\tilde{\mu})\|)ds + |h_i(t)|, \qquad t \geq T. \qquad (4.9)
\end{aligned}$$

Let $v \geq \mu_T$ be such that (4.3) holds with $\mu = \mu_T$. Then, in particular,

$$\phi(\mu_T)\int_0^T k_i(T,s)ds + |h_i(T)| < v, \qquad i = 1, \dots, n,$$

so (4.9) with $t = T$ implies $|x_i(T)| < v$ for all $i = 1, \dots, n$, hence $\|x(T)\| < v$.

Now we show that $\|x(t)\|$ is bounded for all $t > T$. For sake of contradiction, assume that $\|x(t)\|$ is an unbounded function. Since it is continuous and $\|x(T)\| < v$, there exists $t_1 > T$ such that $\|x(t_1)\| > v$. Let $\bar{t} := \inf\{t > T : \|x(t)\| > v\}$. Then the continuity of x and $\mu_T \leq v$ implies $\max_{-\tilde{t}_0 \leq \tilde{\mu} \leq \bar{t}} \|x(\tilde{\mu})\| = \|x(\bar{t})\| = v$. Therefore there exists i such that $|x_i(\bar{t})| = v$. Then from (4.9) with $t = \bar{t}$, the monotonicity of ϕ and using that μ_T has property (P_T), we obtain

$$\begin{aligned}
v = |x_i(\bar{t})| &\leq \phi(\mu_T)\int_0^T k_i(\bar{t},s)ds + \int_T^{\bar{t}} k_i(\bar{t},s)\phi(\max_{-\tilde{t}_0 \leq \tilde{\mu} \leq s} \|x(\tilde{\mu})\|)ds + |h_i(\bar{t})| \\
&\leq \phi(\mu_T)\int_0^T k_i(\bar{t},s)ds + \phi(v)\int_T^{\bar{t}} k_i(\bar{t},s)ds + |h_i(\bar{t})| < v,
\end{aligned}$$

which is a contradiction. So the solution x of (4.1) is bounded by v. $\square$

5 Some special estimates

In this section, we give some applications of our Theorem 4.4. Throughout this section we assume that the nonlinear function f in (4.1) can be estimated by the function $\phi(t) = t^p$, $t > 0$ with $p > 0$ in (A2). There are three cases:

1. Sub-linear estimate $0 < p < 1$;
2. Linear estimate $p = 1$;
3. Super-linear estimate $p > 1$.

5.1 Sub-linear estimate

Our aim in this subsection is to establish a sufficient, as well as a necessary and sufficient condition for the boundedness of all solutions of (4.1) and for the scalar case of (4.1), respectively.

The next result provides a sufficient condition for the boundedness of the solutions of (4.1) in the sub-linear case.

Theorem 5.1. *Let* (A1)-(A4) *be satisfied and* $\phi(t) = t^p$, $t \geq 0$, *with a fixed* $p \in (0,1)$. *If* (4.7) *and* (4.8) *hold, then all solutions of* (4.1) *are bounded.*

Proof. It follows from (4.7) and (4.8) that $J_0 < \infty$ and $H_0 < \infty$. Let $\psi \in C\left([-\tilde{t}_0, 0], \mathbb{R}^n\right)$, and x be the corresponding solution of (4.1)-(4.2). For $\mu_0 := \max_{-\tilde{t}_0 \leq t \leq 0} \|\psi(t)\|$, there exists $v \geq \mu_0$ large enough such that

$$v^{p-1} J_0 + \frac{1}{v} H_0 < 1. \tag{5.1}$$

Therefore

$$v^p J_0 + H_0 < v.$$

By the definitions of J_0 and H_0, we get

$$v^P \int_0^t k_i(t,s)ds + |h_i(t)| \leq v^p J_0 + H_0 < v \qquad t \geq 0, \quad i = 1, \ldots, n.$$

Hence by Definition 4.1, we obtain μ_0 has property (P_0) with $T = 0$. Since (A1)-(A4) are satisfied and μ_0 has property (P_0), then, by Theorem 4.4, the solution x of (4.1) is bounded. □

We consider the scalar Volterra integral equation

$$x(t) = \int_0^t k(t,s) x^p(s - \tau(s))ds + h(t), \qquad t \geq 0, \tag{5.2}$$

with initial condition

$$x(t) = \psi(t), \qquad t \in [-\tilde{t}_0, 0], \tag{5.3}$$

where

(B) $k : \{(t,s) : 0 \leq s \leq t\} \to \mathbb{R}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, $0 < p < 1$, $\tau \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\tilde{t}_0 := -\inf_{t \geq 0}\{t - \tau(t)\} > 0$ is finite and $\psi \in C\left([-\tilde{t}_0, 0], (0, \infty)\right)$.

The condition (B) implies that the solutions of (5.2)-(5.3) are positive.

The following result provides a necessary and sufficient condition for the boundedness of the positive solutions of (5.2). The necessary part of the next theorem was motivated by a similar result of Lipovan [21] proved for convolution-type integral equation.

Theorem 5.2. *Assume* (B),

$$\liminf_{t\to\infty}\left(\int_0^t k(t,s)ds + h(t)\right) > 0, \tag{5.4}$$

and

$$\int_0^t k(t,s)ds + h(t) > 0, \qquad t \geq 0. \tag{5.5}$$

Then the solution of (5.2)-(5.3) *is bounded for all* $\psi \in C\left([-\tilde{t}_0, 0], (0,\infty)\right)$*, if and only if* (4.7) *and* (4.8) *are satisfied.*

Proof. Suppose (4.7) and (4.8) are satisfied. Clearly, by Theorem 5.1 the solution of (5.2)-(5.3) is bounded for all $\psi \in C\left([-\tilde{t}_0, 0], (0,\infty)\right)$.

Conversely, let the solution x of (5.2)-(5.3) be bounded on $\mathbb{R}_+$. First we prove that the solution $x > 0$. Assume for the sake of contradiction that $x(t) \leq 0$ for some t. Then there exists $\lambda > 0$ such that $x(t) > 0$, $t \in [0, \lambda)$ and $x(\lambda) = 0$. From (5.5) it follows that there exists an $\epsilon > 0$ such that

$$\int_0^{\lambda-\epsilon} k(\lambda, s)ds + h(\lambda) > 0.$$

From (5.2) with $t = \lambda$ we have

$$\begin{aligned}
0 &= x(\lambda) \\
&= \int_0^{\lambda} k(\lambda,s)x^p(s-\tau(s))ds + h(\lambda) \\
&\geq \int_0^{\lambda-\epsilon} k(\lambda,s)x^p(s-\tau(s))ds + h(\lambda) \\
&\geq \min_{-\tilde{t}_0\leq\mu\leq\lambda-\epsilon} x^p(\mu) \int_0^{\lambda-\epsilon} k(\lambda,s)ds + h(\lambda) \\
&\geq \left(\int_0^{\lambda-\epsilon} k(\lambda,s)ds + h(\lambda)\right) \min\left(\min_{-\tilde{t}_0\leq\mu\leq\lambda-\epsilon} x^p(\mu), 1\right) \\
&> 0,
\end{aligned}$$

which is a contradiction.

Clearly, the positivity of x and (5.2) yield

$$x(t) \geq h(t), \qquad \text{for all} \quad t \geq 0,$$

hence condition (4.8) is satisfied.

Next we prove (4.7). For any $t \geq T^* > 0$ we get

$$x(t) \geq \int_0^{T^*} k(t,s)x^p(s-\tau(s))ds \geq \min_{-\tilde{t}_0\leq\mu\leq T^*} x^p(\mu) \int_0^{T^*} k(t,s)ds.$$

Therefore the boundedness of x implies

$$\sup_{t\geq 0}\int_0^{T^*} k(t,s)ds < \infty. \tag{5.6}$$

Define now

$$m := \liminf_{t\to\infty} x(t),$$

which is finite. We show that $m > 0$. Assume for the sake of contradiction that $m = 0$. In this case we can find a strictly increasing sequence $(t_n)_{n\geq 1}$, such that

$$x(t_n) = \inf_{-\tilde{t}_0\leq t\leq t_n} x(t) > 0, \quad \text{and} \quad x(t_n)\to 0 \quad \text{as} \quad n\to\infty.$$

From (5.2) with $t = t_n$ we obtain

$$\begin{aligned} x(t_n) &= \int_0^{t_n} k(t_n,s)\, x^p(s-\tau(s))ds + h(t_n) \\ &\geq \int_0^{t_n} k(t_n,s) \inf_{-\tilde{t}_0\leq \mu\leq t_n} x^p(\mu)ds + h(t_n) \\ &= x^p(t_n) \int_0^{t_n} k(t_n,s)ds + h(t_n). \end{aligned}$$

Since $x(t_n) > 0$, we have

$$x^{1-p}(t_n) \geq \int_0^{t_n} k(t_n,s)ds + \frac{h(t_n)}{x^p(t_n)}.$$

For n large enough such that $0 < x^p(t_n) \leq 1$, we get

$$x^{1-p}(t_n) \geq \int_0^{t_n} k(t_n,s)ds + h(t_n).$$

Taking limit of the last inequality, we obtain

$$\lim_{n\to\infty}\left(\int_0^{t_n} k(t_n,s)ds + h(t_n)\right) = 0,$$

which contradicts to (5.4). So $m > 0$. Therefore there exists $T^* \geq 0$ such that

$$x(t) \geq \frac{1}{2}\, m > 0, \qquad t\geq T^*.$$

Hence

$$\begin{aligned} x(t) &= \int_0^t k(t,s)x^p(s-\tau(s))ds + h(t) \\ &\geq \int_{T^*}^t k(t,s)x^p(s-\tau(s))ds \\ &\geq \frac{1}{2^p}\, m^p \int_{T^*}^t k(t,s)ds, \qquad t\geq T^*. \end{aligned}$$

By the boundedness of the solution x we get

$$\sup_{t\geq T^*}\int_{T^*}^{t} k(t,s)ds < \infty.$$

This and (5.6) imply condition (4.7). □

5.2 Linear estimate

Our aim in this subsection is to obtain a sufficient condition for the boundedness of the solutions of a linear Volterra integral equation.

The following result gives a sufficient condition for the boundedness.

Theorem 5.3. *Assume* (A1)-(A4) *are satisfied and* $\phi(t) = t$, $t \geq 0$. *Then all solutions of* (4.1) *are bounded, if for some* $T \geq 0$ *one of the following conditions hold:*

(i) (4.4) *and* (4.6) *hold, and*

$$J_T := \max_{1\leq i\leq n}\sup_{t\geq T}\int_{T}^{t} k_i(t,s)ds < 1, \qquad i = 1,\ldots,n. \tag{5.7}$$

(ii) *for all* $t \geq T$, $i = 1,\ldots,n$,

$$J_T := \max_{1\leq i\leq n}\sup_{t\geq T}\int_{T}^{t} k_i(t,s)ds = 1, \qquad \int_{T}^{t} k_i(t,s)ds < 1$$

$$\sup_{t\geq T}\left(1-\int_{T}^{t} k_i(t,s)ds\right)^{-1}\int_{0}^{T} k_i(t,s)ds < \infty, \tag{5.8}$$

and

$$\sup_{t\geq T}\left(1-\int_{T}^{t} k_i(t,s)ds\right)^{-1}|h_i(t)| < \infty. \tag{5.9}$$

Proof. Let $\psi \in C\left([-\tilde{t}_0,0],\mathbb{R}^n\right)$, and x be the solution of (4.1)-(4.2). We show that x is bounded if (i) or (ii) holds. Let

$$\mu_T := \max_{-\tilde{t}_0\leq t\leq T}\|x(t)\|.$$

First we prove that μ_T has property (P_T) under condition (i). By (4.4) and (4.6) we have $I_T < \infty$ and $H_T < \infty$. Therefore there exists $v \geq \mu_T$ such that

$$(1-J_T)^{-1}\left(\mu_T I_T + H_T\right) < v,$$

so

$$\mu_T I_T + H_T < (1-J_T)v.$$

Hence for all $t \geq T$, $i = 1, \dots, n$, we obtain

$$\mu_T \int_0^T k_i(t,s)ds + |h_i(t)| \leq \mu_T I_T + H_T < (1 - J_T)v \leq \left(1 - \int_T^t k_i(t,s)ds\right) v,$$

therefore

$$\mu_T \int_0^T k_i(t,s)ds + v \int_T^t k_i(t,s)ds + |h_i(t)| < v, \qquad t \geq T, \quad i = 1, \dots, n.$$

Then μ_T has property (P_T), and hence, by Theorem 4.4, the solution x of (4.1) is bounded.

Next we prove that μ_T has property (P_T) if (ii) is satisfied. For all $t \geq T$

$$\left(1 - \int_T^t k_i(t,s)ds\right)^{-1} > 0, \qquad i = 1, \dots, n,$$

(5.8) and (5.9) imply

$$\sup_{t \geq T} \left(1 - \int_T^t k_i(t,s)ds\right)^{-1} \left\{\mu_T \int_0^T k_i(t,s)ds + |h_i(t)|\right\} < \infty.$$

Then there exists $v \geq \mu_T$ large enough such that

$$\left(1 - \int_T^t k_i(t,s)ds\right)^{-1} \left\{\mu_T \int_0^T k_i(t,s)ds + |h_i(t)|\right\} < v, \qquad t \geq T,$$

which yields

$$\mu_T \int_0^T k_i(t,s)ds + v \int_T^t k_i(t,s)ds + |h_i(t)| < v, \qquad t \geq T, \quad i = 1, \dots, n.$$

Then μ_T has property (P_T), and hence, by Theorem 4.4, the solution x of (4.1) is bounded. □

The next example illustrates the applicability of Theorem 5.3 in a case when condition (ii) holds for a large enough T.

Example 5.4. We consider the scalar Volterra integral equation

$$x(t) = \int_0^t \frac{2}{1 - 0.5e^{-2|t - \ln 2|}} e^{-2(t-s)} x(s)ds + ce^{-2t}, \qquad t \geq 0.$$

Here $h(t) = ce^{-2t}$, $c \in \mathbb{R}$ and $k(t,s) = \frac{2}{1-0.5e^{-2|t-\ln 2|}} e^{-2(t-s)}$, $\tilde{t}_0 = 0$.

Clearly $\sup_{t \geq 0} |h(t)| = |c| < \infty$, and

$$\int_0^t k(t,s)ds = \int_0^t \frac{2}{1 - 0.5e^{-2|t - \ln 2|}} e^{-2(t-s)} ds = \frac{1}{1 - 0.5e^{-2|t - \ln 2|}} (1 - e^{-2t}),$$

hence

$$\lim_{t\to\infty}\int_0^t k(t,s)ds = 1.$$

Let $T_1 := \ln 2$, then

$$\int_0^{T_1} k(T_1,s)ds = \frac{1}{(1-0.5)}\left(1-e^{-2\ln 2}\right) = \frac{3}{2} > 1.$$

This means that

$$\sup_{t\geq 0}\int_0^t k(t,s)ds = M > 1,$$

therefore conditions of Theorem 5.3 do not hold with $T = 0$.
If we take $T := \ln 5$, then we obtain

$$\int_T^t k(t,s)ds = \frac{1-25e^{-2t}}{1-2e^{-2t}} < 1, \qquad t \geq T$$

and

$$\sup_{t\geq T}\int_T^t k(t,s)ds = 1.$$

Moreover

$$\left(1-\int_T^t k(t,s)ds\right)^{-1}\int_0^T k(t,s)ds = \frac{24e^{-2t}(1-2e^{-2t})}{23e^{-2t}(1-2e^{-2t})} = \frac{24}{23}, \qquad t \geq T,$$

and

$$\left(1-\int_T^t k(t,s)ds\right)^{-1}|h(t)| = \frac{(1-2e^{-2t})}{23e^{-2t}}|c|e^{-2t} < \frac{|c|}{23}, \qquad t \geq T.$$

Then condition (ii) in Theorem 5.3 is satisfied with this T, therefore x is bounded.

Here our result is applicable but the several results in the literature are not applicable (see, e.g., Proposition 1.4.2 in [3]).

5.3 Super-linear estimate

Our aim in this subsection is to obtain a sufficient condition for the boundedness in the super-linear case.

Theorem 5.5. *Assume that conditions* (A1)-(A4) *are satisfied with* $\phi(t) = t^p$, $t > 0$, *where* $p > 1$, (4.7), (4.8) *hold,* $J_0 > 0$ *and let* $\|\psi\|_{\tilde{t}_0} := \max_{-\tilde{t}_0\leq t\leq 0}\|\psi(t)\|$. *Then the solution* x *of* (4.1)-(4.2) *is bounded if one of the following conditions is satisfied*

(i) $\|\psi\|_{\tilde{t}_0} \leq \left(\frac{1}{pJ_0}\right)^{\frac{1}{p-1}}$ *and* $H_0 < \frac{p-1}{p}\left(\frac{1}{pJ_0}\right)^{\frac{1}{p-1}}$;

(ii) $H_0 < \|\psi\|_{\tilde{t}_0} - J_0 \left(\|\psi\|_{\tilde{t}_0}\right)^p$.

Proof. Assume (4.7) and (4.8) are satisfied.

(i) Since $\|\psi\|_{\tilde{t}_0} \leq \left(\frac{1}{pJ_0}\right)^{\frac{1}{p-1}}$ and

$$H_0 < \frac{p-1}{p}\left(\frac{1}{pJ_0}\right)^{\frac{1}{p-1}} = \left(\frac{1}{pJ_0}\right)^{\frac{1}{p-1}} - J_0\left(\frac{1}{pJ_0}\right)^{\frac{p}{p-1}},$$

then $v := \left(\frac{1}{pJ_0}\right)^{\frac{1}{p-1}}$ satisfies inequality

$$H_0 < v - J_0 v^p. \tag{5.10}$$

(ii) Since $H_0 < \|\psi\|_{\tilde{t}_0} - J_0 \left(\|\psi\|_{\tilde{t}_0}\right)^p$, $v = \|\psi\|_{\tilde{t}_0}$ satisfies inequality (5.10).

Hence in both cases the condition (4.3) is satisfied with $T = 0$, therefore $\mu_0 := \|\psi\|_{\tilde{t}_0}$ has property (P_0). Then the conditions of Theorem 4.4 hold, so the solution of (4.1) is bounded. □

6 The proof of the main results

In this section we give the proofs our main results (Theorems 3.1, 3.2 and 3.3) using our results obtained for Volterra integral equations.

First we give the proof of Theorem 3.1 for the sub-linear case.

Proof of Theorem 3.1. First note that the feedback control system (2.4) is equivalent to the Volterra integral equation (4.1), where h and f are defined by (2.10) and (2.11), respectively. Clearly, (A1) is satisfied with $\tilde{t}_0 = t_0$.

By [14] and under our conditions, for $i = 1, \dots, n$, the solution z_i of the initial value problem

$$\dot{z}_i(t) = -d_i z_i(t-\tau), \qquad t \geq 0 \tag{6.1}$$

with initial condition

$$z_i(t) = \psi_i(t), \qquad -\tau \leq t \leq 0 \tag{6.2}$$

satisfies the inequality

$$|z_i(t)| \leq M\|\psi_i\|_\tau e^{-\alpha_i t}, \qquad t \geq -\tau, \tag{6.3}$$

where M and α_i are positive constants. Hence every solution of (6.1) tends to zero as $t \to \infty$, and this implies that every solution of (2.5) tends to the zero vector as $t \to \infty$, and

$$\|z\|_\infty := \sup_{t\geq 0} \|z(t)\| \leq M\|\psi\|_\tau < \infty. \tag{6.4}$$

From (3.1) it follows (see, e.g., Proposition 2.1 in [13]) that the fundamental solution w_i of (6.1)-(6.2) satisfies

$$C_i := \int_0^\infty |w_i(t)|dt < \infty, \qquad i = 1, \dots, n, \tag{6.5}$$

where C_i is positive constant.

From (2.7), for all $i = 1, \dots, n$, we have

$$x_i(t) = z_i(t) + \int_0^t w_i(t-s)[g_i(s, x(s-\sigma(s))) + r_i(s)]ds, \quad t \geq 0, \quad i = 1, \dots, n,$$

where $x(t) = (x_1(t), \dots, x_n(t))$, $g(t,x) = (g_1(t,x), \dots, g_n(t,x))$, $r(t) = (r_1(t), \dots, r_n(t))$ and $z(t) = (z_1(t), \dots, z_n(t))$. Therefore (2.10) and (2.11) imply

$$f_i(t, s, x(\cdot)) = w_i(t-s)g_i(s, x(s-\sigma(s))) \quad \text{and} \quad h_i(t) = z_i(t) + \int_0^t w_i(t-s)r_i(s)ds.$$

Hence by (2.2)

$$|f_i(t, s, x(\cdot))| \leq |w_i(t-s)|\,|g_i(s, x(s-\sigma(s)))| \leq |w_i(t-s)|b(s)\phi\left(\max_{-t_0\leq\mu\leq s}\|x(\mu)\|\right),$$

so (A2) holds with $k_i(t,s) := |w_i(t-s)|b(s), \quad 0 \leq s \leq t$.

By (3.1), (6.4), (6.5) and the definition of the infinity norm, we obtain

$$\begin{aligned}
H_0 &:= \max_{1\leq i\leq n}\sup_{t\geq 0}|h_i(t)| \\
&\leq \max_{1\leq i\leq n}\sup_{t\geq 0}|z_i(t)| + \max_{1\leq i\leq n}\sup_{t\geq 0}\int_0^t |w_i(t-s)|\|r(s)\|ds \\
&\leq \max_{1\leq i\leq n}\sup_{t\geq 0}\|z_i(t)\| + \|r\|_\infty \max_{1\leq i\leq n}\int_0^\infty |w_i(t)|dt && (6.6) \\
&= \|z\|_\infty + C\|r\|_\infty \\
&\leq M\|\psi\|_\tau + C\|r\|_\infty \\
&\leq M\|\psi\|_{t_0} + C\|r\|_\infty, && (6.7)
\end{aligned}$$

where $C := \max_{i=1,\dots,n}(C_i)$ and $t_0 := \max\left\{\tau, -\inf_{t\geq 0}\{t-\sigma(t)\}\right\}$, therefore (4.8) holds.

By condition (3.1) we get

$$\int_0^t k_i(t,s)ds = \int_0^t |w_i(t-s)|b(s)ds, \qquad t \geq 0, \quad i = 1, \dots, n.$$

Hence

$$J_0 := \max_{1\leq i\leq n}\sup_{t\geq 0}\int_0^t k_i(t,s)ds \leq \|b\|_\infty \max_{1\leq i\leq n}\int_0^\infty |w_i(t)|dt \leq C\|b\|_\infty < \infty, \tag{6.8}$$

i.e., (4.7) holds with $T = 0$. Then the conditions of Theorem 5.1 for $0 < p < 1$ with $T = 0$ are satisfied, hence the solution x of (2.4) is bounded.

Now we show that the inequality (5.1) is satisfied with

$$v := \max\left(3(M\|\psi\|_{t_0} + C\|r\|_\infty), (3C\|b\|_\infty)^{\frac{1}{1-p}}\right).$$

We have $v > (2C\|b\|_\infty)^{\frac{1}{1-p}}$, and so (6.8) yields

$$v^{p-1}J_0 < \frac{J_0}{2C\|b\|_\infty} \leq \frac{1}{2}.$$

Similarly, $v > 2(M\|\psi\|_{t_0} + C\|r\|_\infty)$ and (6.7) imply

$$\frac{1}{v}H_0 < \frac{H_0}{2(M\|\psi\|_{t_0} + C\|r\|_\infty)} \leq \frac{1}{2}.$$

Therefore (5.1) is satisfied, hence by Theorem 5.1 we get

$$\|x(t)\| < v = \max\left(3(M\|\psi\|_{t_0} + C\|r\|_\infty), (3C\|b\|_\infty)^{\frac{1}{1-p}}\right), \qquad t \geq 0.$$

So

$$\begin{aligned}\|x(t)\| &< 3C\|r\|_\infty + \max\left(3M\|\psi\|_{t_0}, (3C\|b\|_\infty)^{\frac{1}{1-p}}\right) \\ &= \gamma_1\|r\|_\infty + \gamma_2, \qquad t \geq 0,\end{aligned}$$

where $\gamma_1 := 3C$ and $\gamma_2 := \max\left(3M\|\psi\|_{t_0}, (3C\|b\|_\infty)^{\frac{1}{1-p}}\right)$. Hence the feedback control system (2.4) is BIBO stable. □

The next Lemma will be useful in the proof of Theorem 3.2.

Lemma 6.1. *Assume the conditions of* Theorem 3.2 *are satisfied. The inequalities*

$$|h_i(t)| + v\int_0^t k_i(t,s)ds < v, \qquad t \geq 0, \quad i = 1, \ldots, n \tag{6.9}$$

are satisfied with

$$v := \max\left(\frac{\|r\|_\infty}{d_0\left(1 - \frac{\|b\|_\infty}{d_0}\right)} + \frac{\|z\|_\infty}{1 - \frac{\|b\|_\infty}{d_0}}, 1\right),$$

where $d_0 := \min(d_1, \ldots, d_n)$*,* $W = diag(w_1, \ldots, w_n)$ *is the fundamental solution of* (2.5)*,* z *is the solution of the initial value problem* (2.5)-(2.6)*,* $\|z\|_\infty := \sup_{t\geq 0}\|z(t)\|$ *and* $k_i(t,s) := w_i(t-s)b(s)$*,* $0 \leq s \leq t$*.*

Proof. From (3.2) it follows (see, e.g., Theorem 3.1 in [13]) that the fundamental solution w_i of (6.1)-(6.2) is positive and

$$\int_0^\infty w_i(t)dt = \frac{1}{d_i} \quad \text{and} \quad \int_0^t w_i(s)ds < \frac{1}{d_i}, \qquad i = 1, \ldots, n, \quad t \geq 0. \tag{6.10}$$

Hence

$$\begin{aligned} J_0 &:= \max_{1\leq i\leq n} \sup_{t\geq 0} \int_0^t k_i(t,s)ds \\ &= \max_{1\leq i\leq n} \sup_{t\geq 0} \int_0^t w_i(t-s)b(s)ds \\ &\leq \max_{1\leq i\leq n} \|b\|_\infty \int_0^\infty w_i(t)dt \\ &= \frac{\|b\|_\infty}{d_0}. \end{aligned} \tag{6.11}$$

By (3.2) and (6.11), we have for $t \geq 0$, $i = 1, \ldots, n$

$$\begin{aligned} v &\geq \frac{\|r\|_\infty}{d_0\left(1 - \frac{\|b\|_\infty}{d_0}\right)} + \frac{\|z\|_\infty}{1 - \frac{\|b\|_\infty}{d_0}} \\ &> \frac{\|r\|_\infty \int_0^t d_i w_i(t-s)ds}{d_0\left(1 - \int_0^t w_i(t-s)b(s)ds\right)} + \frac{\|z(t)\|}{1 - \int_0^t w_i(t-s)b(s)ds} \\ &\geq \frac{\int_0^t w_i(t-s)\|r(s)\|ds}{1 - \int_0^t w_i(t-s)b(s)ds} + \frac{\|z(t)\|}{1 - \int_0^t w_i(t-s)b(s)ds}, \qquad t \geq 0. \end{aligned}$$

Therefore

$$\left(1 - \int_0^t w_i(t-s)b(s)ds\right) v > \int_0^t w_i(t-s)\|r(s)\|ds + \|z(t)\|, \qquad t \geq 0, \quad i = 1, \ldots, n.$$

Hence

$$\begin{aligned} v &> \int_0^t w_i(t-s)\|r(s)\|ds + \|z(t)\| + v\int_0^t w_i(t-s)b(s)ds \\ &\geq |h_i(t)| + v\int_0^t k_i(t,s)ds, \qquad t \geq 0, \quad i = 1, \ldots, n, \end{aligned}$$

where $k_i(t,s) = w_i(t-s)b(s)$ and $h(t) = (h_1(t), \ldots, h_n(t))^T$ is defind by (2.10). $\square$

Proof of Theorem 3.2. According to (6.6) and (6.10) and the positivity of w_i, we have

$$H_0 := \max_{1\le i\le n}\sup_{t\ge 0}|h_i(t)| \le \|z\|_\infty + \frac{\|r\|_\infty}{d_0} \le M\|\psi\|_{t_0} + \frac{\|r\|_\infty}{d_0} < \infty, \tag{6.12}$$

where $d_0 := \min(d_1, \dots, d_n)$ and $t_0 := \max\left\{\tau, -\inf_{t\ge 0}\{t-\sigma(t)\}\right\}$. By (6.11), we obtain $J_0 < 1$, so condition (i) in Theorem 5.3 holds with $T = 0$, hence the solution x of (2.4) is bounded.

Lemma 6.1 yields that relation (6.9) is satisfied with $v := \max\left(\frac{\|r\|_\infty}{d_0-\|b\|_\infty} + \frac{\|z\|_\infty}{1-\frac{\|b\|_\infty}{d_0}}, 1\right)$. Therefore the boundedness of the solution and (6.12) gives

$$\|x(t)\| < v := \max\left(\frac{\|r\|_\infty}{d_0-\|b\|_\infty} + \frac{d_0\|z\|_\infty}{d_0-\|b\|_\infty}, 1\right), \qquad t \ge 0.$$

So (6.4) yields

$$\begin{aligned}\|x(t)\| &< \max\left(\frac{\|r\|_\infty}{d_0-\|b\|_\infty} + \frac{d_0\|z\|_\infty}{d_0-\|b\|_\infty}, 1\right)\\ &\le \frac{1}{d_0-\|b\|_\infty}\|r\|_\infty + \max\left(\frac{d_0M\|\psi\|_{t_0}}{d_0-\|b\|_\infty}, 1\right)\\ &= \gamma_1\|r\|_\infty + \gamma_2, \qquad t\ge 0,\end{aligned}$$

where

$$\gamma_1 := \frac{1}{d_0-\|b\|_\infty} \quad \text{and} \quad \gamma_2 := \max\left(\frac{d_0M\|\psi\|_{t_0}}{d_0-\|b\|_\infty}, 1\right).$$

Hence the feedback control system (2.4) is BIBO stable. □

Proof of Theorem 3.3. Suppose $\|\psi\|_{t_0} \le \delta_1$ and $\|r\|_\infty \le \delta_2$, where δ_1 and δ_2 will be specified later. According to (6.4)

$$\|z\|_\infty \le M\|\psi\|_{t_0} \le M\delta_1.$$

So (6.12) yields

$$H_0 := \max_{1\le i\le n}\sup_{t\ge 0}|h_i(t)| \le \|z\|_\infty + \frac{\|r\|_\infty}{d_0} \le M\delta_1 + \frac{\delta_2}{d_0}.$$

Relation (6.11) implies $J_0 < 1$. If the positive constants δ_1 and δ_2 are selected so that

$$M\delta_1 + \frac{\delta_2}{d_0} \le \frac{p-1}{p}\left(\frac{1}{p}\right)^{\frac{1}{p-1}},$$

and

$$\delta_1 < \left(\frac{1}{p}\right)^{\frac{1}{p-1}},$$

then condition (i) of Theorem 5.5 holds. So

$$\|x(t)\| < \gamma, \qquad t \geq -t_0$$

where

$$\gamma := \left(\frac{1}{p}\right)^{\frac{1}{p-1}}.$$

Hence by Definition 2.2 the solution of the control system (2.4) is locally BIBO stable. $\square$

References

[1] J.A.D. Appleby, I. Győri and D. Reynolds, On exact rates of decay of solutions of linear systems of Volterra equations with delay, J. Math. Anal. Appl. 320 (2006) 56-77.

[2] S, Barnett, Introduction to Mathematical Control Theory, Clarendon Press, Oxford, 1975.

[3] T. A. Burton, Volterra Integral and Differential Equations, New York, 1983.

[4] A. Callender, R. D. Hartee, A. Porter, Time lag in a control system, Phil. Trans. Roy. Soc. London, A235:756 (1936) 415-444.

[5] A. Callender, A. G. Stevenson, Time lag in a control system, Proc. Soc. Chem. Ind. 18 (1936) 108-117.

[6] J. N. Chiasson, J. J. Loiseau, Applications of Time Delay Systems, Springer, 2007.

[7] E. N. Chukwu, Stability and Time-Optimal Control of Hereditary Systems, Academic Press, 1992.

[8] C. Corduneanu, Integral Equations and Applications, Cambridge University Press, Cambridge, 1991.

[9] C. Corduneanu, Integral Equations and Stability of Feedback Systems, Academic Press New York and London 1973.

[10] J. Cui, L. Xiong, M. Ye, BIBO stability of switched uncertain neutral control system, Int. J. Control Autom. Syst., 8:4 (2010) 913-923.

[11] T. Erneux, Applied Delay Differential Equations, Springer, 2009.

[12] I. Győri, E. Awwad, On the boundedness of the solutions in nonlinear discrete Volterra difference equations, Adv. Difference Equ., (2012), 2012:2.

[13] I. Győri, F. Hartung, Fundamental solution and asymptotic stability of linear delay equations, Dyn. Contin. Discrete Impuls. Syst., 13:2 (2006) 261-288.

[14] J. K. Hale, Sjoerd M. Verduyn Lunel, Introduction to Functional Differential Equations (Applied Mathematical Sciences), Springer-Verlag, New York, 1993.

[15] T. Insperger, G. Stépán, Semi-discretization for time-delay systems – Stability and Engineering Applications, Springer, 2011.

[16] S. Kotsios and N. Kalouptsidis, BIBO stability criteria for a certain class of discrete nonlinear systems, Int. J. Contr., 58 (1993) 707-730.

[17] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.

[18] P. Li, S. M. Zhong, BIBO stabilization of time-delayed system with nonlinear perturbation, Appl. Math. Comput. 195 (2008) 264-269.

[19] P. Li, S. M. Zhong, BIBO stabilization of piecewise switched linear systems with delays and nonlinear perturbations, Appl. Math. Comput. 213 (2009) 405-410.

[20] P. Li, S. M. Zhong, J. Z. Cui, Delay-dependent robust BIBO stabilization of uncertain system via LMI approach, Chaos Solitons Fractals 40 (2009) 1021-1028.

[21] O. Lipovan, On the asymptotic behavior of solutions to some nonlinear integral equations of convolution type, Dyn. Contin. Discrete Impuls. Syst. 16 (2009) 147–154.

[22] N. MacDonald, Time-lags in Biological Models, Lect. Notes in Biomath. Vol. 27, Springer, Berlin, 1978.

[23] W. Michiels, S.-I. Niculescu, Stability and stabilization of time-delay systems, Society for Industrial and Applied Mathematics, 2007.

[24] H. Wu, K. Mizukami, Robust stabilization of uncertain linear dynamical systems, Int. J. Syst. Sci. 24 (2) (1993) 265-276.

[25] S.-I. Niculescu, Delay Effects on Stability, Spriger-Verlag, London, 2001.

[26] S. M. Shahruz, Boundedness of Outputs of a Class of Lasers, Appl. Math. Lett. 14 (2001) 1011-1015.

[27] B. Wang, J. Wang, X. Liu, S. M. Zhong, New results on BIBO stability analysis for a class of neutral delay systems, J. Franklin Inst. 348 (2011) 426-437.

[28] L. L. Xiong, S. M. Zhong, M. Ye, Delay-dependent BIBO stability analysis of switched uncertain neutral systems, Math. Comput. Modelling 53 (2011) 1607-1620.

[29] Q. C. Zhong, Robust Control of Time-delay Systems, Birkhäuser, 2006.