to appear in Journal of Difference Equations and Applications

Asymptotic behavior of nonlinear difference equations*

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Abstract

In this paper we investigate the growth/decay rate of solutions of a class of nonlinear Volterra difference equations. Our results can be applied for the case when the characteristic equation of an associated linear difference equation has complex dominant eigenvalue with higher than one multiplicity. Illustrative examples are given for describing the asymptotic behavior of solutions in a class of linear difference equations and in several discrete nonlinear population models.

Keywords: Volterra difference equation, exponential growth/decay, asymptotic behavior, discrete population model.

AMS Subject classification: 39A11, 92D25

1 Introduction

Asymptotic behavior of difference equations has been studied by many authors. One of the first result in this direction is the following theorem.

Theorem 1.1 (deBruijn (1950)) *Let $b_j \geq 0$, $j = 0, \dots, k$, $b_0 > 0$. Then for any sequence $\{y(n)\}_{n \geq -k}$ satisfying*

$$\begin{aligned} y(n+1) &= \sum_{j=0}^{k} b_j y(n-j), \qquad n \geq 0, \\ y(m) &= \varphi(m), \qquad -k \leq m \leq 0, \end{aligned}$$

the limit

$$\lim_{n\to\infty} \lambda_0^{-n} y(n) = \left(\varphi(0) + \sum_{j=1}^{k} b_j \sum_{i=-j}^{-1} \lambda_0^{-j-i-1} \varphi(i) \right) \left(1 + \sum_{j=1}^{k} j b_j \lambda_0^{-j-1} \right)^{-1} \tag{1.1}$$

*This research was partially supported by Hungarian National Foundation for Scientific Research Grant No. K73274, and F. Hartung was also partially supported by the NSF grant DMS-0705247 while visited ICDRiA at the University of Texas at Dallas, USA.

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exists, where λ_0 is the unique positive real root of

$$\lambda - b_0 - \sum_{j=1}^{k} b_j \lambda^{-j} = 0.$$

Asymptotic behavior of solutions of various classes of difference equations was studied, e.g., in [4]–[6], [8], [10], [14], [16], [26]–[30]. For example, Philos, Purnaras [27] have considered the Volterra-type equation with infinite delay

$$\begin{aligned} y(n+1) - y(n) &= \sum_{j=-\infty}^{n} a(n-j)y(j), \qquad n \geq 0 \\ y(m) &= \varphi(m), \qquad m \leq 0. \end{aligned}$$

They proved an asymptotic formula similar to (1.1). Note that in the above papers the rate of growth/decay is exponential, since the dominant eigenvalue of the associated characteristic equation is a positive real number.

Motivated by the above mentioned papers and our work done for asymptotic behavior of abstract Volterra differential equations in [15], in this paper we study the asymptotic behavior of the nonlinear difference equation

$$x(n) = y(n;\varphi) + \sum_{j=0}^{n-1} H(n, j+1) f(j, x(\cdot)), \qquad n > 0, \tag{1.2}$$

where f is a Volterra functional (see Section 2 below for the definition). Typically Eq. (1.2) is a result of a variation-of-constant formula applied for a quasilinear difference equation. In this paper, for simplicity, we assume that $y(n;\varphi)$ in (1.2) is linear in φ, e.g., it may be a solution of a linear difference equation. Our results can be generalized for the case if we omit this assumption (see [15] for related works in the continuous case). In Section 2 we will give conditions which guarantee that the growth/decay rate of the sequence $y(n;\varphi)$ is preserved for the solution of Eq. (1.2). In our main result (Theorem 2.2 below) we assume that the growth/decay rate of the sequence $y(n;\varphi)$ is not purely exponential, it may have the form

$$\lim_{n\to\infty} |\rho^{-n} n^{-k} y(n;\varphi) - d_0(\varphi)\cos\gamma n - e_0(\varphi)\sin\gamma n| = 0. \tag{1.3}$$

Therefore the novelty of our results is that it can be applied also in the case when the dominant eigenvalue of an associated characteristic equation is complex, moreover its algebraic multiplicity can be higher than 1. Therefore we can conclude that in such a case the solutions of (1.2) exhibit similar oscillatory asymptotic behavior with the same growth/decay rate. The proofs of the main results are given in Section 5. In Section 3, as an application of our main result, we study the asymptotic behavior of a class of linear difference equations which can be considered as perturbations of autonomous difference equations. In Section 4 we show that our main theorem can be applied to describe the asymptotic behavior of solutions in several classes of nonlinear discrete population models, including Nicholson's blowflies equation, Lasota-Wazewska model, Mackey-Glass equation, Ricker's equation, Pielou's equation, moreover, more general discrete models with several delays. For our model equations we obtain precise asymptotic representation not only for the case of the real but also for the case of the complex dominant eigenvalue of an associated characteristic equation. Note that for this latter case, to the best knowledge of the authors, there are no similar results in the literature.

2 Main results

We introduce the following notations: $\mathbb{Z}$ is the set of integers, $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of nonnegative and positive integers, respectively, $\mathbb{R}_+ = [0, \infty)$. The maximum norm defined on $\mathbb{R}^d$ and the corresponding induced matrix norm on $\mathbb{R}^{d\times d}$ are both denoted simply by $|\cdot|$, i.e. $|u| = \max\{|u_1|, \dots, |u_d|\}$ for $u = (u_1, \dots, u_d)^T \in \mathbb{R}^d$.

Let $r \in \mathbb{N}_0$ be a fixed nonnegative integer. $S([-r, \infty), \mathbb{R}^d)$ denotes the $\mathbb{R}^d$ valued sequences defined on $[-r, \infty) \cap \mathbb{Z}$. $S = S([-r, 0], \mathbb{R}^d)$ is the space of $\mathbb{R}^d$ valued finite sequences defined on $[-r, 0] \cap \mathbb{Z}$. Let $\varphi \in S$, then its norm is defined by $\|\varphi\|_S = \max\{|\varphi(j)| : -r \le j \le 0\}$. The space of linear operators from S to $\mathbb{R}^d$ is denoted by $\mathcal{L}(S, \mathbb{R}^d)$. We can consider $\varphi \in S$ as a column vector $(\varphi(-r)^T, \varphi(-r+1)^T, \dots, \varphi(0)^T)^T \in \mathbb{R}^{(r+1)d}$. Then we have $S \simeq \mathbb{R}^{(r+1)d}$ and $\mathcal{L}(S, \mathbb{R}^d) \simeq \mathbb{R}^{d\times(r+1)d}$, where we use the maximum vector norm on $\mathbb{R}^{(r+1)d}$, and the matrix norm induced by the maximum vector norm on $\mathbb{R}^{d\times(r+1)d}$. So $E \in \mathcal{L}(S, \mathbb{R}^d)$ and $\varphi \in S$ can be identified by their matrix and vector representations, respectively, and $E\varphi$ can be considered as matrix and vector multiplication. We note that any other p-norm ($p \ge 1$) could be used on $\mathbb{R}^d$, we could appropriately change the definition of $\|\cdot\|_S$ so that we get an isometric isomorphism between S and $\mathbb{R}^{(r+1)d}$.

In this section we study the asymptotic behavior of the Volterra-type difference equation

$$x(n) = y(n; \varphi) + \sum_{j=0}^{n-1} H(n, j+1) f(j, x(\cdot)), \qquad n > 0 \tag{2.4}$$

with initial condition

$$x(j) = \varphi(j), \qquad -r \le j \le 0 \tag{2.5}$$

for some $\varphi \in S$.

We state our hypotheses:

(H1) For all $\varphi \in S$ the sequence $y(\cdot; \varphi) \in S([-r, \infty), \mathbb{R}^d)$ satisfies

$$y(j; \varphi) = \varphi(j), \qquad -r \le j \le 0,$$

the map $\varphi \mapsto y(n; \varphi)$ is linear for all fixed $n \in \mathbb{N}_0$, and

$$|y(n; \varphi)| \le M_0 \rho^n (n+1)^k \|\varphi\|_S, \qquad n \ge 0, \quad \varphi \in S, \tag{2.6}$$

where $\rho > 0$, $k \in \mathbb{N}_0$ and $M_0 \ge 1$ are constants.

(H2) $H(n, j) \in \mathbb{R}^{d\times d}$ for $0 < j \le n < \infty$ are such that

$$c_1 := \sup_{0<j\le n} \rho^{-(n-j)} (n-j+1)^{-k} |H(n, j)| < \infty.$$

(H3) $f\colon \mathbb{N}_0 \times S([-r,\infty),\mathbb{R}^d) \to \mathbb{R}^d$ is a Volterra-type functional, i.e., for all $n \geq 0$ and $x, \widetilde{x} \in S([-r,\infty),\mathbb{R}^d)$,

$$f(n, x(\cdot)) = f(n, \widetilde{x}(\cdot)), \qquad \text{if} \quad x(j) = \widetilde{x}(j), \quad -r \leq j \leq n.$$

(H4) For all $n \geq 0$ and $x \in S([-r,\infty),\mathbb{R}^d)$,

$$|f(n, x(\cdot))| \leq \omega\Big(n, \max_{-r\leq j\leq n} \rho^{-j}(j+r+1)^{-k}|x(j)|\Big), \tag{2.7}$$

where $\omega\colon \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that

$$\omega(n, u_1) \leq \omega(n, u_2), \qquad u_1 \leq u_2 \quad \text{and} \quad n \in \mathbb{N}_0,\ u_1, u_2 \in \mathbb{R}_+,$$

and there exists $v_0 > 0$ such that

$$c_1 \sum_{j=1}^{\infty} \rho^{-j} \omega(j-1, v_0) < v_0. \tag{2.8}$$

It is clear that for any $\varphi \in S$, the IVP (2.4)-(2.5) has a unique solution, which is denoted by $x(\cdot;\varphi) \in S([-r,\infty),\mathbb{R}^d)$.

We introduce the function

$$G\colon \mathbb{R}_+ \to \mathbb{R}, \qquad G(v) = v - c_1 \sum_{j=1}^{\infty} \rho^{-j}\omega(j-1, v). \tag{2.9}$$

Assumption (2.8) yields that $G(v_0) > 0$ for some $v_0 > 0$, therefore the constant

$$R := \sup\{G(v)\colon\ v > 0\} \tag{2.10}$$

is well-defined, and it is either positive or $+\infty$. We define the constant

$$M_1 := \max\left\{ \max_{-r\leq j\leq 0} \rho^{-j}(j+r+1)^{-k},\ M_0 \right\}. \tag{2.11}$$

It is easy to see that $M_1 \geq 1$, moreover, $M_1 = M_0$ for $\rho \leq 1$. We define the set $\mathcal{U} \subset S$ by

$$\mathcal{U} := \Big\{\varphi \in S\colon\ M_1\|\varphi\|_S < R\Big\}. \tag{2.12}$$

Finally,

$$m(\|\varphi\|_S) := \inf\big\{v > 0\colon\ G(v) \geq M_1\|\varphi\|_S\big\} \tag{2.13}$$

is a well-defined nonnegative real number for all $\varphi \in \mathcal{U}$.

In Theorem 2.2 we give an exponential upper bound for the solutions of the IVP (2.4)-(2.5), and in the second part of this theorem we give a limit relation based on the following three additional hypotheses:

(H5) There exist $D_0, E_0 \in \mathcal{L}(S, \mathbb{R}^d)$ and $\gamma \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \left| \rho^{-n}(n+r+1)^{-k} y(n;\varphi) - D_0\varphi\cos\gamma n - E_0\varphi\sin\gamma n \right| = 0$$

for $\varphi \in S$.

(H6) There exist $P, Q \colon \mathbb{N} \to \mathbb{R}^{d\times d}$ such that for all $j > 0$

$$\lim_{n\to\infty} \left| \rho^{-(n-j)}(n-j+1)^{-k} H(n,j) - P(j)\cos\gamma n - Q(j)\sin\gamma n \right| = 0,$$

and

$$\|P\| := \sup_{j>0} |P(j)| < \infty \quad \text{and} \quad \|Q\| := \sup_{j>0} |Q(j)| < \infty.$$

(H7) There is an initial sequence $\varphi_0 \in \mathcal{U}$ such that

$$\max\Big\{ |D_0\varphi_0|, |E_0\varphi_0| \Big\} > (\|P\| + \|Q\|) \sum_{j=1}^{\infty} \rho^{-j}\omega(j-1, m(\|\varphi_0\|_S)). \tag{2.14}$$

Now we state our main result. Its proof is given in Section 5.

Theorem 2.2 *Assume that (H1)-(H4) are satisfied.*

(i) If $\varphi \in \mathcal{U}$, then any solution $x(\cdot;\varphi)$ of the IVP (2.4)-(2.5) satisfies

$$|x(n;\varphi)| \le \rho^n (n+r+1)^k m(\|\varphi\|_S), \qquad n \ge -r, \tag{2.15}$$

where $m(\|\varphi\|_S)$ is defined in (2.13).

(ii) If in addition (H5)-(H6) hold, then for all $\varphi \in \mathcal{U}$ there are vectors $d(\varphi), e(\varphi) \in \mathbb{R}^d$ such that

$$x(n;\varphi) = \rho^n (n+r+1)^k \Big(d(\varphi)\cos\gamma n + e(\varphi)\sin\gamma n + o(1) \Big), \tag{2.16}$$

as $n \to \infty$. Moreover, if (H7) holds, then $|d(\varphi_0)| + |e(\varphi_0)| \ne 0$, where φ_0 is given in (2.14).

We note that asymptotic formula (2.16) is equivalent to

$$x(n;\varphi) = \rho^n n^k \Big(d(\varphi)\cos\gamma n + e(\varphi)\sin\gamma n + o(1) \Big), \qquad n \to \infty.$$

In the special case when $\omega(n,u) = \rho^n a(n) u$, Theorem 2.2 yields immediately the next result.

Theorem 2.3 *Assume that (H1)-(H3) are satisfied, and*

$$|f(n, x(\cdot))| \le \rho^n a(n) \max_{-r\le j\le n} \rho^{-j}(j+r+1)^{-k}|x(j)|, \tag{2.17}$$

for all $n \ge 0$ and $x \in S([-r,\infty), \mathbb{R}^d)$, where $a \in S([0,\infty), \mathbb{R}_+)$ is such that

$$c_1 \sum_{j=0}^{\infty} a(j) < \rho. \tag{2.18}$$

Then

(i) For all $\varphi \in S$ the solution $x(\cdot;\varphi)$ of the IVP (2.4)-(2.5) satisfies

$$|x(n;\varphi)| \leq M_2 \rho^n (n+r+1)^k \|\varphi\|_S, \qquad n \geq -r, \tag{2.19}$$

where

$$M_2 = \frac{M_1}{1 - \frac{c_1}{\rho}\sum_{j=0}^{\infty} a(j)}. \tag{2.20}$$

(ii) If (H5) and (H6) also hold, then for all $\varphi \in S$, there are $d(\varphi)$, $e(\varphi) \in \mathbb{R}^d$ such that (2.16) is satisfied. Moreover, if

$$\max\Big\{|D_0\varphi_0|,\, |E_0\varphi_0|\Big\} > M_2\|\varphi_0\|_S(\|P\| + \|Q\|)\sum_{j=0}^{\infty} a(j), \tag{2.21}$$

then $|d(\varphi_0)| + |e(\varphi_0)| > 0$.

3 Asymptotic behavior of perturbed linear difference equations

In this section we consider the system of linear delay difference equations

$$\Delta x(n) = \sum_{\ell=0}^{N} A_\ell x(n-\tau_\ell) + \sum_{j=0}^{M} B_j(n) x(n-\sigma_j(n)), \qquad n \geq 0, \tag{3.22}$$

where $\Delta x(n) = x(n+1) - x(n)$ is the forward difference operator. We consider this equation as a perturbation of the associated autonomous linear difference equation

$$\Delta y(n) = \sum_{\ell=0}^{N} A_\ell y(n-\tau_\ell), \qquad n \geq 0. \tag{3.23}$$

We assume

(A1) $0 \leq \tau_0 < \tau_1 < \cdots < \tau_N$ are integers, and $A_\ell \in \mathbb{R}^{d\times d}$, $\quad 0 \leq \ell \leq N$, and

(A2) $B_j \colon \mathbb{N}_0 \to \mathbb{R}^{d\times d}$ and $\sigma_j \colon \mathbb{N}_0 \to \mathbb{N}_0$, $\lim_{n\to\infty}(n - \sigma_j(n)) = \infty$, $0 \leq j \leq M$.

Let

$$r := \max\Big\{\tau_N,\, -\min_{0\leq j\leq M}\{\min_{0\leq n}\{n - \sigma_j(n)\}\}\Big\}.$$

We associate the initial condition

$$x(n) = \varphi(n), \qquad -r \leq n \leq 0 \tag{3.24}$$

to Eq. (3.22), where $\varphi \in S$, and the initial condition to (3.23) is

$$y(n) = \varphi(n), \qquad -\tau_N \leq n \leq 0. \tag{3.25}$$

In this section we are looking for conditions which imply that the growth/decay rate of the solutions of Eq. (3.22) is equal to that of the solutions of the autonomous linear system (3.23).

By definition, the fundamental solution of (3.23) is the $d \times d$ matrix valued sequence $T(n)$ satisfying

$$\Delta T(n) = \sum_{\ell=0}^{N} A_\ell T(n-\tau_\ell), \quad n \geq 0, \quad \text{and} \quad T(0) = I,\ T(j) = 0 \quad \text{for} \ -\tau_N \leq j < 0.$$

Here I and 0 denote the $d \times d$ identity and zero matrices, respectively.

The characteristic equation associated to (3.23) is

$$\delta(\lambda) := \det\left((\lambda - 1)I - \sum_{\ell=0}^{N} A_\ell \lambda^{-\tau_\ell}\right) = 0. \tag{3.26}$$

A complex number λ is called an *eigenvalue* of Eq. (3.23) if it is a solution of Eq. (3.26). $\lambda_0 = \rho_0(\cos\gamma_0 + i\sin\gamma_0)$ is called a *dominant eigenvalue* of (3.23) if $\delta(\lambda_0) = 0$ and $|\lambda_0| > \sup\{|\lambda| : \delta(\lambda) = 0,\ \lambda \neq \lambda_0 \text{ and } \lambda \neq \overline{\lambda}_0\}$. The *ascent* of λ_0 is the order of λ_0 as a pole of $\delta^{-1}(\lambda)$ (see [7], [20]). It is known that the ascent of an eigenvalue λ is less or equal to the algebraic multiplicity of λ.

We assume

(A3) $\lambda = \rho(\cos\gamma + i\sin\gamma)$ is a dominant eigenvalue of Eq. (3.23) with $\rho > 0$ and ascent equal to $k+1$.

Some basic results which follow directly from the theory of linear autonomous systems (see, e.g., [9]) on the asymptotic representation of the solutions of Eq. (3.23) are summarized in the following lemma.

Lemma 3.4 *Assume (A1), (A3). Then the following statements hold.*

(a) There exist $D_0, E_0 \in \mathcal{L}(S, \mathbb{R}^d)$ such that for all $\varphi \in S$ the solution $y(\cdot;\varphi)$ of Eq. (3.23)-(3.25) satisfies

$$y(n;\varphi) = \rho^n n^k \Big(D_0\varphi\cos\gamma n + E_0\varphi\sin\gamma n + o(1)\Big), \qquad as \quad n \to \infty.$$

(b) There exist constant matrices $P, Q \in \mathbb{R}^{d\times d}$ for which

$$T(n) = \rho^n (n+1)^k \Big(P\cos\gamma n + Q\sin\gamma n + o(1)\Big), \qquad as \quad n \to \infty.$$

In the proof of the next theorem we will need the following estimate.

Lemma 3.5 *For any $n_0 > 0$ there exists $a > 0$ such that the solution $x(\cdot;\varphi)$ of the IVP (3.22)-(3.24) satisfies*

$$|x(n;\varphi)| \leq (1+a)^n \|\varphi\|_S, \qquad n = 0, 1, \ldots, n_0.$$

Proof Let

$$a := \sum_{\ell=0}^{N} |A_\ell| + \max_{0\le n\le n_0} \sum_{j=0}^{M} |B_j(n)|.$$

Then, it is easy to see that (3.22) yields

$$|x(n+1)| \le (1+a) \max_{-r\le j\le n} |x(j)|, \qquad n = 0, 1, \ldots, n_0,$$

which implies the statement. □

Next we state and prove the main result of this section.

Theorem 3.6 *Assume (A1)–(A3), and*

$$\sum_{n=0}^{\infty} n^k \sum_{j=0}^{M} |B_j(n)|\rho^{-\sigma_j(n)} < \infty. \tag{3.27}$$

Then there exists $K \ge 0$ such that for every $\varphi \in S$ the solution $x(\cdot;\varphi)$ of Eq. (3.22)-(3.24) satisfies

$$|x(n;\varphi)| \le K\rho^n (n+r+1)^k \|\varphi\|_S, \qquad n \ge -r. \tag{3.28}$$

Moreover, for every $\varphi \in S$ there exist vectors $\widetilde{d}(\varphi)$ and $\widetilde{e}(\varphi)$ in $\mathbb{R}^d$ such that

$$x(n;\varphi) = \rho^n n^k \Big(\widetilde{d}(\varphi)\cos\gamma_0 n + \widetilde{e}(\varphi)\sin\gamma_0 n + o(1)\Big), \qquad \text{as} \quad n \to \infty. \tag{3.29}$$

If, in addition,

$$\sigma_j(n) < \tau_N \quad \text{for } j = 0, \ldots, M,\ n \in N_0 \qquad \text{and} \qquad \det(A_N) \ne 0, \tag{3.30}$$

then there exists $\phi_0 \in S$ such that $|\widetilde{d}(\varphi_0)| + |\widetilde{e}(\varphi_0)| \ne 0$ in (3.29).

Proof From Lemma 3.4 it follows that (H1) and (H2) hold with

$$c_1 := \sup_{0\le n} \rho^{-n}(n+1)^{-k}|T(n)| < \infty.$$

Let $\psi_0 \in S$ be a fixed initial sequence such that $\max\{|D_0\psi_0|, |E_0\psi|_0\} > 0$, where $D_0\psi_0$ and $E_0\psi_0$ are defined in Lemma 3.4 (i), let $\|P\| := \sup\{|P(j)| :\ j > 0\}$ and $\|Q\| := \sup\{|Q(j)| :\ j > 0\}$, where $P(j)$ and $Q(j)$ are defined in Lemma 3.4 (ii). Let $0 < \kappa < 1$, and define the constants

$$M_2 := \frac{\max\{\rho^{-r},\, M_0\}}{1-\kappa} \qquad \text{and} \qquad L := \min\left\{\frac{\kappa\rho}{c_1},\ \frac{\kappa\max\{|D_0\psi_0|, |E_0\psi_0)|\}}{M_2\|\psi_0\|_S(|P| + |Q|)}\right\}. \tag{3.31}$$

Assumption (3.27) yields there exists $n_0 \ge r+1$ such that

$$\sum_{n=0}^{\infty}(n+n_0)^k \sum_{j=0}^{M} |B_j(n+n_0)|\rho^{-\sigma_j(n+n_0)} = \sum_{n=n_0}^{\infty} n^k \sum_{j=0}^{M} |B_j(n)|\rho^{-\sigma_j(n)} < L.$$

Let $\widetilde{B}_j(n) = B_j(n+n_0)$ and $\widetilde{\sigma}_j(n) = \sigma_j(n+n_0)$ for $n \geq 0$. Then the previous inequality implies

$$\sum_{n=0}^{\infty}(n+r+1)^k \sum_{j=0}^{M} |\widetilde{B}_j(n)|\rho^{-\widetilde{\sigma}_j(n)} < L. \tag{3.32}$$

Let $\varphi \in S$, $\varphi \neq 0$ be fixed, and $x(n;\varphi)$ be the corresponding solution of (3.22)-(3.24). Let $\psi(n) = x(n+n_0;\varphi)$ for $n = -r,\dots,0$, and consider the sequence $w(n) = w(n;\psi)$ defined by the equation

$$\Delta w(n) = \sum_{\ell=0}^{N} A_\ell w(n-\tau_\ell) + \sum_{j=0}^{M} \widetilde{B}_j(n) w(n-\widetilde{\sigma}_j(n)), \qquad n \geq 0, \tag{3.33}$$

and the initial condition

$$w(n) = \psi(n), \qquad -r \leq n \leq 0. \tag{3.34}$$

Then, clearly, $w(n) = x(n+n_0;\varphi)$ for $n \geq -r$.

Let $f\colon \mathbb{N}_0 \times S([-r,\infty),\mathbb{R}^d) \to \mathbb{R}^d$ be defined by

$$f(n,w(\cdot)) = \sum_{j=0}^{M} \widetilde{B}_j(n) w(n-\widetilde{\sigma}_j(n)).$$

We get by using the variation-of-constants formula that the solution $w(\cdot;\psi)$ of (3.33)-(3.34) satisfies

$$w(n;\psi) = y(n;\psi) + \sum_{j=0}^{n-1} T(n-j-1) f(j, w(\cdot;\psi)) \qquad n > 0,$$

where $y(\cdot;\psi)$ denotes the solution of Eq. (3.23) corresponding to the initial function ψ. Moreover, for all $(n,w) \in \mathbb{N}_0 \times S([-r,\infty),\mathbb{R}^d)$,

$$\begin{aligned} |f(n,w(\cdot))| &\leq \sum_{j=0}^{M} |\widetilde{B}_j(n)|(n-\widetilde{\sigma}_j(n)+r+1)^k \rho^{n-\widetilde{\sigma}_j(n)} \\ &\qquad \times \rho^{-(n-\widetilde{\sigma}_j(n))}(n-\widetilde{\sigma}_j(n)+r+1)^{-k}|w(n-\widetilde{\sigma}_j(n))| \\ &\leq \rho^n a(n) \max_{-r\leq j\leq n} \rho^{-j}(j+r+1)^{-k}|w(j)|, \end{aligned} \tag{3.35}$$

where

$$a(n) = (n+r+1)^k \sum_{j=0}^{M} |\widetilde{B}_j(n)|\rho^{-\widetilde{\sigma}_j(n)}, \qquad n \geq 0.$$

Thus it follows from the definition of L, (3.32) and (3.35) and Lemma 3.4 that conditions (H1)–(H6) of Theorem 2.3 hold, therefore $w(\cdot;\psi)$ satisfies (2.19), i.e.,

$$|w(n;\psi)| \leq M_2 \rho^n (n+r+1)^k \|\psi\|_S, \qquad n \geq 0, \tag{3.36}$$

since

$$M_2 \geq \frac{\max\{\rho^{-r},\, M_0\}}{1 - \frac{c_1}{\rho}\sum_{j=0}^{\infty} a(j)}.$$

It follows from Lemma 3.5 that there exists $M_3 > 0$ such that

$$|x(n;\varphi)| \le M_3\|\varphi\|_S, \qquad n = 0, 1, \ldots, n_0. \tag{3.37}$$

Then (3.36) and $x(n;\varphi) = w(n-n_0;\psi)$ yield

$$|x(n;\varphi)| \le M_2\rho^{n-n_0}(n-n_0+r+1)^k M_3\|\varphi\|_S, \qquad n \ge n_0. \tag{3.38}$$

Let

$$K = \max\left\{ \max_{0\le j\le n_0} M_3\rho^{-j}(j+r+1)^{-k}, M_2M_3\rho^{-n_0}, \rho^{-r}, 1 \right\}.$$

Then (3.37) and (3.38) yield (3.28).

Theorem 2.3 (ii) can be applied for Eq. (3.33), hence

$$\begin{aligned} x(n;\varphi) &= w(n-n_0;\psi) \\ &= \rho^{n-n_0}(n-n_0+r+1)^k\Big(d(\psi)\cos\gamma(n-n_0) + e(\psi)\sin\gamma(n-n_0) + o(1)\Big), \end{aligned}$$

as $n \to \infty$, which yields (3.29) with

$$\widetilde{d}(\varphi) = \rho^{-n_0}\Big(d(\psi)\cos\gamma n_0 - e(\psi)\sin\gamma n_0\Big) \tag{3.39}$$

$$\widetilde{e}(\varphi) = \rho^{-n_0}\Big(d(\psi)\sin\gamma n_0 + e(\psi)\cos\gamma n_0\Big). \tag{3.40}$$

It follows from (3.31), (3.32) and the definition of $a(n)$ that ψ_0 satisfies condition (2.21) in Theorem 2.3, therefore $|d(\psi_0)| + |e(\psi_0)| \ne 0$. It is easy to argue using assumption (3.30) that there exists an initial function φ_0 such that $x(n;\varphi_0) = \psi_0(n-n_0)$ for $n = n_0 - r, \ldots, n_0$. Then (3.39)-(3.40) yield $|\widetilde{d}(\varphi_0)| + |\widetilde{e}(\varphi_0)| \ne 0$.

This completes the proof of the theorem. □

Note that assumption (3.30) can be replaced in Theorem 3.6 by the property that all solutions of (3.22) can be extended to the left for $n \in \mathbb{Z}$.

The next corollary of Theorem 3.6 shows the importance of the factor $\rho^{-\sigma_j(n)}$ in condition (3.27). In the rest of this section $[\cdot]$ denotes the greatest integer function.

Corollary 3.7 *Consider the delay difference equation*

$$\Delta x(n) = \sum_{\ell=0}^{N} A_\ell x(n-\tau_\ell) + \sum_{j=0}^{M} C_j x([\gamma_j n]) \tag{3.41}$$

where (A1) holds and $\gamma_j \in (0,1)$*,* $C_j \in \mathbb{R}^{d\times d}$*,* $0 \le j \le M$*.*

If $\lambda = \rho(\cos\gamma + i\sin\gamma)$ *is a dominant eigenvalue of Eq. (3.23) with* $\rho > 1$ *and ascent equal to* $k+1$*, then the statements of Theorem 3.6 are valid for any solution* $x(\cdot;\varphi)$ *of Eq. (3.41) corresponding to initial function* $\varphi \in S$*, where* $r = \tau_N$*.*

Proof Let $\sigma_j\colon \mathbb{N}_0 \to \mathbb{N}_0$ and $B_j\colon \mathbb{N}_0 \to \mathbb{R}^{d\times d}$ be defined by

$$\sigma_j(n) = n - [\gamma_j n] \qquad \text{and} \qquad B_j(n) \equiv C_j, \qquad n \in \mathbb{N}_0,\ 0 \le j \le M.$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} n^k \sum_{j=0}^{M} |B_j(n)|\rho^{-\sigma_j(n)} &= \sum_{j=0}^{M} |C_j| \sum_{n=0}^{\infty} n^k \rho^{-(n-[\gamma_j n])} \\ &\le \sum_{j=0}^{M} |C_j| \sum_{n=0}^{\infty} n^k \rho^{-(1-\gamma_j)n} \\ &< \infty,\end{aligned}$$

and hence by Theorem 3.6 the statement of the corollary follows. □

In the special case when $M = 0$ in (3.22), Theorem 3.6 has the following corollaries.

Corollary 3.8 *Suppose (A1)–(A3) hold with $M = 0$, and*

(i) $-r \le n - \sigma_0(n) \le n, \qquad n \ge 0 \qquad$ *and* $\qquad \lim_{n\to\infty} \dfrac{\sigma_0(n)}{n} = \alpha,$

(ii) $\lim_{n\to\infty} \sqrt[n]{|B_0(n)|} < \rho^{\alpha},$

Then there exist $K > 0$ and $\widetilde{d}(\varphi), \widetilde{e}(\varphi) \in \mathbb{R}^d$ such that for all $\varphi \in S$ the solution $x(\cdot;\varphi)$ of (3.22) satisfies (3.28) and (3.29).

Proof Condition (i) and (ii) and the root test yield that

$$\sum_{n=0}^{\infty} n^k |B_0(n)|\rho^{-\sigma_0(n)} < \infty.$$

therefore Theorem 3.6 is applicable. □

We note condition (i) holds, e.g., for delays of the form $\sigma_0(n) = [\alpha n]$, $n \in \mathbb{N}_0$, where $0 < \alpha \le 1$.

The next result follows from Corollary 3.8 with $\alpha = 0$.

Corollary 3.9 *Suppose (A1)–(A3) hold with $M = 0$, and*

(i) $0 \le \sigma_0(n) \le r, \qquad n \ge 0,$

(ii) $\lim_{n\to\infty} \sqrt[n]{|B_0(n)|} < 1.$

Then there exist $K > 0$ and $\widetilde{d}(\varphi), \widetilde{e}(\varphi) \in \mathbb{R}^d$ such that for all $\varphi \in S$ the solution $x(\cdot;\varphi)$ of (3.22) satisfies (3.28) and (3.29).

4 Applications to discrete population models

4.1 Clark's model

Clark's equation is a simple but quite general discrete population model where the size of the next generation equals to the number of the survivals and the recruitments, which is a nonlinear function of the size of the population r years before (see, e.g., [5], [12], [18]). The Clark's model has the form

$$z(n+1) = \alpha z(n) + g(z(n-r)), \qquad n \geq 0,$$

where $\alpha \in (0,1]$ is the survival rate, and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the recruitment function. We rewrite this equation as

$$\Delta z(n) = -\delta z(n) + g(z(n-r)), \qquad n \geq 0, \tag{4.42}$$

where $\delta = 1-\alpha \in [0,1)$ is the death rate.

We assume

(B1) $\delta \in [0,1)$, $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable with $L := \sup_{u\geq 0} |g''(u)| < \infty$;

(B2) Equation (4.42) has a positive equilibrium $c > 0$.

One important particular case of Clark's equation is the discrete Nicholson's blowflies difference equation ([13], [18], [33]), where the recruitment function g has the form $g(u) = pue^{-au}$, and p and a are positive constants. In this case 0 is always an equilibrium of (4.42), but it has a unique positive equilibrium c if $p > \delta$, and it has no positive equilibrium if $p \leq \delta$. We refer to [18] for a short survey and comparison of conditions guaranteeing that the unique positive equilibrium is a global attractor of the positive solutions. Clearly, $g = pue^{-au}$ has a bounded second derivative on $\mathbb{R}_+$, so (B1) is satisfied.

Another important subclass of (4.42) is the discrete version of the Lasota-Wazewska red-blood model (see [18], [19], [25]), where g has the form $g(u) = pe^{-au}$, and $p, a > 0$. For this model, assuming $\delta \in (0,1)$, (4.42) always has a unique equilibrium which is positive, and (B1) also holds.

Finally, consider the discrete analogue of the Mackey-Glass equation in haematopoiesis ([11]), i.e., (4.42) with $g(u) = \frac{\beta}{1+u^p}$ with $p, \beta > 0$. It is easy to see that this equation always has a positive equilibrium, and g satisfies (B1).

We associate the initial condition

$$z(n) = \varphi(n), \qquad -r \leq n \leq 0 \tag{4.43}$$

to (4.42). Note that if we start from a nonnegative initial sequence $\varphi(-r), \ldots, \varphi(0) \geq 0$, then the corresponding solutions $z(n;\varphi)$ of (4.42) will be nonnegative for all $n > 1$. We will restrict our interest to solutions starting from nonnegative initial sequences.

In this section we examine the asymptotic behavior of solutions of (4.42) in the neighborhood of the positive equilibrium c. Note that we do not assume that (4.42) has a unique equilibrium, c denotes any fixed positive solution of

$$-\delta c + g(c) = 0.$$

We introduce the new variable $x(n) = z(n) - c$. Then the sequence x satisfies

$$\Delta x(n) = -\delta x(n) - \delta c + g(x(n-r) + c), \qquad n \geq 0,$$

and therefore

$$\Delta x(n) = -\delta x(n) + g(x(n-r) + c) - g(c), \qquad n \geq 0.$$

Hence x is the solution of

$$\Delta x(n) = -\delta x(n) + qx(n-r) + f(x(n-r)), \qquad n \geq 0, \tag{4.44}$$

where

$$q := g'(c) \qquad \text{and} \qquad f(u) := g(u+c) - g(c) - qu, \quad u \in [-c, \infty).$$

We associate the linear difference equation

$$\Delta y(n) = -\delta y(n) + qy(n-r), \qquad n \geq 0, \tag{4.45}$$

to (4.44). Although we may have $q = 0$ (e.g., for the Nicholson's blowflies equation $p = \delta e$ yields $c = \frac{1}{a}$ and $g'(c) = 0$), we are interested in the case when $q \neq 0$, i.e., when (4.45) is a delay difference equation.

Let $T(n)$ be the fundamental solution of (4.45), i.e., the solution of (4.45) corresponding to the initial condition $T(0) = 1$, $T(n) = 0$ for $n < 0$. Then the solution of (4.44) is given by the variation-of-constants formula

$$x(n) = y(n) + \sum_{j=0}^{n-1} T(n-j-1) f(x(j-r)), \qquad n > 0.$$

The characteristic equation associated to (4.45) is $h(\lambda) = 0$, where

$$h(\lambda) := \lambda^{r+1} - (1-\delta)\lambda^r - q, \qquad \lambda \in \mathbb{C}.$$

The next lemma shows that h always has a dominant root λ_0, i.e., $h(\lambda_0) = 0$, and for all other roots $\lambda \neq \overline{\lambda_0}$, it follows $|\lambda| < |\lambda_0|$.

Lemma 4.10 *Assume $q \neq 0$. Then the polynomial h has $r+1$ different roots, except for*

$$-q = \frac{(1-\delta)^{r+1} r^r}{(r+1)^{r+1}}, \tag{4.46}$$

when

$$\lambda_0 = \frac{(1-\delta)r}{r+1} \tag{4.47}$$

is a double root of h, and all other roots are simple and have modulus less than λ_0.

Moreover, in all cases if λ_j and λ_k are two different roots with $|\lambda_j| = |\lambda_k|$, then $\lambda_j = \overline{\lambda_k}$.

Proof We have
$$h'(\lambda) = (r+1)\lambda^r - (1-\delta)r\lambda^{r-1},$$
therefore its only non-zero root is λ_0 defined by (4.47). It is easy to check that λ_0 is a double root of h, if and only if (4.46) holds.

To show the second statement, consider
$$\lambda_j^{r+1} - (1-\delta)\lambda_j^r - q = 0 = \lambda_k^{r+1} - (1-\delta)\lambda_k^r - q,$$
and so
$$|\lambda_j|^r|\lambda_j - (1-\delta)| = |\lambda_k|^r|\lambda_k - (1-\delta)|.$$
This yields easily that $\lambda_j = \overline{\lambda_k}$.

In the critical case (4.46) an application of Rouché's Theorem yields easily that h has exactly $r+1$ roots inside any circle at the origin with radius ε for all $\varepsilon > \lambda_0$, which yields that λ_0 is the dominant root of h. □

Let $\lambda = \rho(\cos\gamma + i\sin\gamma)$ be the dominant root of h. Then Lemma 3.4 yields that there exist $D_0, E_0 \in \mathcal{L}(S, \mathbb{R})$ such that for every $\varphi \in S$ the solution $y(n;\varphi)$ of (4.45) corresponding to initial sequence φ satisfies
$$y(n;\varphi) = \rho^n n^k\Big(D_0\varphi\cos\gamma n + E_0\varphi\sin\gamma n + o(1)\Big), \qquad \text{as} \quad n\to\infty,$$
and there exist $P, Q \in \mathbb{R}$ that the fundamental solution $T(n)$ of (4.45) satisfies
$$T(n) = \rho^n n^k\Big(P\cos\gamma n + Q\sin\gamma n + o(1)\Big), \qquad \text{as} \quad n\to\infty.$$
Here $k=0$ if (4.46) does not hold, and $k=1$ in case of (4.46). Therefore (H1), (H2), (H5) and (H6) are satisfied with $H(n,j) = T(n-j)$.

Clearly, (H3) also holds, and to show (H4), we note first that the twice continuous differentiability of g and $|g''(u)| \le L$ for $u \ge 0$ assumed in (B1) yields that the function f satisfies
$$|f(u)| = |g(u+c) - g(c) - g'(c)u| \le Lu^2, \qquad u \in [-c,\infty).$$
Hence
$$|f(x(n-r))| \le Lx^2(n-r) \le L\rho^{2(n-r)}\Big(\max_{-r\le j\le n}\rho^{-j}|x(j)|\Big)^2, \qquad n\ge 0,$$
so (2.7) is satisfied with $\omega(n,u) = L\rho^{2(n-r)}u^2$. To check condition (2.8) consider
$$\sum_{j=1}^\infty \rho^{-j}\omega(j-1,v_0) = \sum_{j=1}^\infty \rho^{-j}L\rho^{2(j-1-r)}v_0^2 = v_0^2L\rho^{-2r-2}\sum_{j=1}^\infty \rho^j.$$
So assuming $\rho<1$ we have
$$c_1\sum_{j=1}^\infty \rho^{-j}\omega(j-1,v_0) = \frac{c_1v_0^2L\rho^{-2r-1}}{1-\rho} = Av_0^2,$$

where

$$A = \frac{c_1 L}{\rho^{2r+1}(1-\rho)}.$$

Therefore the function G defined by (2.9) is $G(v) = v - Av^2$, so $G(v) > 0$ for $0 < v < \frac{1}{A}$. This shows that (H4) holds for the case when $\rho < 1$.

For the rest of this section we will assume $\rho < 1$. Then the constant R defined by (2.10) is $R = G\left(\frac{1}{2A}\right) = \frac{1}{4A}$. It is easy to see that M_1 defined by (2.11) is equal to M_0, the set $\mathcal{U}$ defined by (2.12) is $\mathcal{U} = \{\varphi \in S\colon \|\varphi\|_S < R_0\}$, where

$$R_0 := \frac{1}{4AM_0}, \tag{4.48}$$

and finally, the constant $m(\|\varphi\|_S)$ defined by (2.13) is

$$m(\|\varphi\|_S) = \frac{1 - \sqrt{1 - 4AM_0\|\varphi\|_S}}{2A}. \tag{4.49}$$

The next two lemmas describe necessary and sufficient conditions for the oscillation and asymptotic stability of Eq. (4.45).

Lemma 4.11 *Assume $\delta \in [0,1)$. Eq. (4.45) is oscillatory, i.e., all solutions are oscillatory, if and only if*

$$-q > \frac{(1-\delta)^{r+1} r^r}{(r+1)^{r+1}}. \tag{4.50}$$

Proof It is known (see Theorem 7.1.1 in [17]) that Eq. (4.45) is oscillatory, if and only if

$$h(\lambda) \neq 0 \qquad \text{for} \quad \lambda > 0. \tag{4.51}$$

We distinguish two cases.

Case (1): $q < 0$. Then

$$\lim_{\lambda \to 0+} h(\lambda) = -q > 0 \qquad \text{and} \qquad \lim_{\lambda \to +\infty} h(\lambda) = +\infty,$$

so (4.51) holds if and only if

$$\min_{\lambda > 0} h(\lambda) > 0.$$

It is easy to check that λ_0 defined by (4.47) minimizes h, and $h(\lambda_0) > 0$ if and only if (4.50) holds.

Case (2): $q \geq 0$. Clearly, in this case there always exists a positive root of $h(\lambda)$.

□

Necessary and sufficient condition for the asymptotic stability for equations of the form (4.45) was proved first for $\delta = 0$ in [23], and later for $\delta \neq 0$ in [22]. We state this result for our equation in the following lemma.

Lemma 4.12 *Suppose $\delta \in [0,1)$. The trivial solution of (4.45) is asymptotically stable, if and only if*

$$-\delta < -q < \sqrt{(1-\delta)^2 + 1 - 2(1-\delta)\cos\theta},$$

where $\theta \in (0, \frac{\pi}{r+1})$ is the unique solution of

$$\frac{\sin r\theta}{\sin(r+1)\theta} = \frac{1}{1-\delta}. \tag{4.52}$$

Now an application of our main result, Theorem 2.2 and the above calculations and lemmas give the following precise description of the asymptotic behavior of (4.42) in a neighborhood of the positive equilibrium c. The constant sequence $\varphi(n) = c$ will be simply denoted by c in the next theorem.

Theorem 4.13 *Suppose (B1) and (B2). Let $\lambda = \rho(\cos\gamma + i\sin\gamma)$ be the dominant characteristic root of (4.45), and let the constants R_0 and $m(\|\varphi\|_S)$ be defined by (4.48) and (4.49), respectively.*

(1) If

$$-\delta < -q < \frac{(1-\delta)^{r+1} r^r}{(r+1)^{r+1}},$$

then $\lambda = \rho$ is real, and the solution $z(n;\varphi)$ of (4.42)-(4.43) satisfies

$$|z(n;\varphi) - c| \le \rho^n (n+r+1)^k m(\|\varphi - c\|_S), \qquad n \ge -r, \quad \|\varphi - c\|_S < R_0 \tag{4.53}$$

with $k = 0$, moreover, there exists $d(\varphi) \in \mathbb{R}$ such that

$$z(n;\varphi) = c + \rho^n n^k \Big(d(\varphi) + o(1)\Big), \qquad \|\varphi - c\|_S < R_0 \tag{4.54}$$

is satisfied with $k = 0$.

(2) If

$$-q = \frac{(1-\delta)^{r+1} r^r}{(r+1)^{r+1}},$$

then $\lambda = \rho$ is real and a double root, and (4.53) and (4.54) are satisfied with $k = 1$.

(3) If

$$\frac{(1-\delta)^{r+1} r^r}{(r+1)^{r+1}} < -q < \sqrt{(1-\delta)^2 + 1 - 2(1-\delta)\cos\theta},$$

where $\theta \in (0, \frac{\pi}{r+1})$ is the solution of (4.52), then λ is complex, (4.53) is satisfied with $k = 0$, and there exists $d(\varphi), e(\varphi) \in \mathbb{R}$ such that

$$z(n;\varphi) = c + \rho^n \Big(d(\varphi)\cos\gamma n + e(\varphi)\sin\gamma n + o(1)\Big), \qquad \|\varphi - c\|_S < R_0.$$

4.2 A single delay model

Next we study another general class of discrete models, the scalar equation

$$z(n+1) = z(n)g(z(n-r)), \qquad n \in \mathbb{N},$$

or in an equivalent form, consider

$$\Delta z(n) = z(n)\Big(g(z(n-r)) - 1\Big), \qquad n \in \mathbb{N}. \tag{4.55}$$

We assume conditions similar to (B1)-(B2):

(C1) $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable with $L := \sup_{u \geq 0} |g''(u)| < \infty$;

(C2) Equation (4.55) has a positive equilibrium $c > 0$.

Eq. (4.55) has many important applications in discrete population models. If we take $g(u) = e^{a-bu}$, with $a, b > 0$, then we get the delayed Ricker's equation ([24], [31]). Another important example of (4.55) is Pielou's equation ([2], [9], [28]), a discrete analogue of the delayed logistic equation, where $g(u) = \frac{a}{1+bu}$, with $a > 1$ and $b > 0$. Finally, consider (4.55) with $g(u) = e^{-a+\frac{b}{c+u^m}}$, where $a, b, c > 0$ and $m \in \mathbb{N}$, and $b > ac$. This equation is a discrete analogue of Nazarenko's equation (see, e.g., [32]). It easy to check that all three models satisfy conditions (C1)-(C2).

Let c be a positive equilibrium of (4.55), i.e., a positive solution of $g(c) = 1$. Introduce the new variable $x(n) = z(n) - c$. Then

$$\Delta x(n) = (x(n) + c)\Big(g(x(n-r) + c) - g(c)\Big)$$

Define the constant $p := cg'(c)$ and the function

$$F(u, v) := g'(c)uv + (u + c)\Big(g(v + c) - g(c) - g'(c)v\Big).$$

Then

$$\Delta x(n) = px(n-r) + F(x(n), x(n-r)), \qquad n \in \mathbb{N}. \tag{4.56}$$

We have

$$\begin{aligned} |F(x(n), x(n-r))| &\leq |g'(c)||x(n)||x(n-r)| + cLx^2(n-r) + L|x(n)|x^2(n-r) \\ &\leq \Big(|g'(c)|\rho^{2n-r} + cL\rho^{2n-2r}\Big)\Big(\max_{-r\leq j\leq n} \rho^{-j}|x(j)|\Big)^2 \\ &\quad + L\rho^{3n-2r}\Big(\max_{-r\leq j\leq n} \rho^{-j}|x(j)|\Big)^3. \end{aligned}$$

Therefore F satisfies (2.7) with

$$\omega(n, v) = \alpha_1\rho^{2n}v^2 + \alpha_2\rho^{3n}v^3, \tag{4.57}$$

where $\alpha_1 = |g'(c)|\rho^{-r} + cL\rho^{-2r}$ and $\alpha_2 = L\rho^{-2r}$. So for $\rho < 1$ the function G defined by (2.9) equals to

$$G(v) = v - \beta_1 v^2 - \beta_2 v^3,$$

where $\beta_1 = \frac{c_1\alpha_1}{\rho(1-\rho)}$ and $\beta_2 = \frac{c_1\alpha_2}{\rho(1-\rho^2)}$ are positive constants. Hence there exists $v_0 > 0$ such that $G(v) > 0$ for $v \in (0, v_0)$, and so F satisfies (H3) and (H4).

It is easy to see that a result analogous to Theorem 4.13 can be formulated showing that the asymptotic behavior of (4.55) around the positive and asymptotically stable equilibrium c is identical to that of the zero solution of the associated linear difference equation

$$\Delta y(n) = py(n-r), \qquad n \in \mathbb{N}. \tag{4.58}$$

Note that necessary and sufficient conditions guaranteeing the oscillation and asymptotic stability of (4.58) are known ([17] and [23]), which can be formulated using Lemmas 4.11 and 4.12 with $\delta = 0$.

4.3 A multiple delay model

In this subsection we mention that the method of Section 4.2 can be generalized to equations of the form

$$\Delta z(n) = z(n)\Big(g(z(n), \ldots, z(n-r)) - 1\Big), \qquad n \in \mathbb{N}. \tag{4.59}$$

We assume

(D1) $g\colon \mathbb{R}_+^{r+1} \to \mathbb{R}_+$ is twice continuously differentiable with respect to all variables with

$$\left|\frac{\partial^2 g}{\partial u_j \partial u_k}(u_0, \ldots, u_r)\right| \le L \qquad \text{for} \quad j, k = 0, \ldots, r \text{ and } u_0, \ldots, u_r \ge 0;$$

(D2) Equation (4.59) has a positive equilibrium $c > 0$.

Let $x(n) = z(n) - c$. Then

$$\Delta x(n) = c\sum_{j=0}^{r} \frac{\partial g}{\partial u_j}(c, \ldots, c)x(n-j) + F(x(n), \ldots, x(n-r)),$$

where

$$F(u_0, \ldots, u_r) = (u_0 + c)\Big(g(u_0 + c, \ldots, u_r + c) - g(c, \ldots, c)\Big) - c\sum_{j=0}^{r} \frac{\partial g}{\partial u_j}(c, \ldots, c)u_j.$$

Let $a_j = \left|\frac{\partial g}{\partial u_j}(c, \ldots, c)\right|$. Then one can show

$$|F(u_0, \ldots, u_r)| \le (|u_0| + c)L\sum_{j=0}^{r}\sum_{k=0}^{r} |u_j||u_k| + |u_0|\sum_{j=0}^{r} a_j|u_j|,$$

therefore

$$|F(x(n),\dots,x(n-r))| \le b_1\rho^{2n}\Big(\max_{-r\le j\le n}\rho^{-j}|x(j)|\Big)^2 + b_2\rho^{3n}\Big(\max_{-r\le j\le n}\rho^{-j}|x(j)|\Big)^3,$$

where b_1 and b_2 are positive constants, so F satisfies (2.7) with ω of the form (4.57) with some α_1 and α_2. Hence Theorem 2.2 is applicable to obtain asymptotic behavior of (4.59).

4.4 Case of a simple dominant eigenvalue

In Sections 4.1–4.3 we have seen that the main difficulty to apply Theorem 2.2 is to show estimate (2.7), and show the existence of v_0 satisfying inequality (2.8), i.e., check condition (H4) for function f in (2.4). In this subsection we consider again our general equation (2.4), and give a more explicit condition which implies (H4) with $k=0$. Therefore this condition is applicable, e.g., if the dominant eigenvalue of the associated linear equation has algebraic multiplicity 1.

We assume

(H4*) There exists a continuous and monotone non-decreasing function $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $b(u)>0$ for $u>0$, and the inequality

$$|f(n,x(\cdot))| \le b\Big(\rho^n \max_{-r\le j\le n}\rho^{-j}|x(j)|\Big), \qquad n\ge 0,\ x\in S([-r,\infty),\mathbb{R}^d) \tag{4.60}$$

holds.

Lemma 4.14 *Assume that (H4*) holds with*

$$\rho>1 \qquad \text{and} \qquad \int_1^\infty \frac{b(u)}{u^2}\,du<\infty. \tag{4.61}$$

Then

(i) the function $\omega(n,u)=b(\rho^n u)$ satisfies (H4) with $k=0$;

(ii) R defined by (2.10) equals to $+\infty$,

(iii) $\mathcal{U}$ defined by (2.12) equals to S, and

(iv) the equation

$$M_0\|\varphi\|_S + \frac{c_1}{\log\rho} m\int_m^\infty \frac{b(u)}{u^2}\,du = m, \qquad m\ge 0 \tag{4.62}$$

has at most two roots, and $m(\|\varphi\|_S)$ defined by (2.13) satisfies $m(\|\varphi\|_S)\le \widetilde{m}(\|\varphi\|_S)$, where $\widetilde{m}(\|\varphi\|_S)$ is the largest root of (4.62).

Proof Let $\omega(n,u)$ be defined by $\omega(n,u) = b(\rho^n u)$ for all $n \geq 0$ and $u \geq 0$. Then for $m > 0$ the monotonicity of $b(u)$ implies for $t \in [j-1, j]$ that $\rho^{-j} \leq \rho^{-t}$ and $b(\rho^{j-1}m) \leq b(\rho^t m)$. Therefore

$$\sum_{j=1}^{\infty} \rho^{-j}\omega(j-1,m) = \sum_{j=1}^{\infty} \rho^{-j} b(\rho^{j-1}m) \leq \int_0^{\infty} \rho^{-t} b(\rho^t m)\, dt.$$

Hence, using the substitution $u = \rho^t m$ and (4.61), we get for $m > 0$

$$\sum_{j=1}^{\infty} \rho^{-j}\omega(j-1,m) \leq \frac{m}{\log \rho} \int_m^{\infty} \frac{b(u)}{u^2}\, du < \infty. \tag{4.63}$$

Let G be defined by (2.9). Then

$$G(m) = m - c_1 \sum_{j=1}^{\infty} \rho^{-j}\omega(j-1,m) \geq G_2(m),$$

where

$$G_2(m) := mG_1(m), \qquad G_1(m) := 1 - \frac{c_1}{\log \rho} \int_m^{\infty} \frac{b(u)}{u^2}\, du.$$

But then R defined in (2.10) satisfies $R = \lim_{m\to\infty} G(m) = \infty$, and $\mathcal{U}$ defined in (2.12) equals to S. Note that equation (4.62) can be rewritten as

$$G_2(m) = M_0 \|\varphi\|_S, \qquad m \geq 0. \tag{4.64}$$

Therefore any solution $m = \widetilde{m}(\|\varphi\|_S)$ of (4.62) satisfies $m(\|\varphi\|_S) \leq \widetilde{m}(\|\varphi\|_S)$, where $m(\|\varphi\|_S)$ is defined by (2.13).

To show that equation (4.62) has one or two roots, we refer to the analogous proof of Theorem 2.7 in [15]. □

The next result deals with the case $0 < \rho < 1$.

Lemma 4.15 *Assume that (H4*) holds with*

$$0 < \rho < 1 \qquad \text{and} \qquad \int_0^1 \frac{b(u)}{u^2}\, du < \infty. \tag{4.65}$$

Then

(i) the function $\omega(n,u) = b(\rho^n u)$ satisfies (H4) with $k = 0$;

(ii) R defined by (2.10) equals to

$$R = \sup_{0<v} \left\{ v \left(1 - \frac{c_1}{\rho^2 \log \frac{1}{\rho}} \int_0^{v/\rho} \frac{b(u)}{u^2}\, du \right) \right\};$$

(iii) $\mathcal{U}$ *defined by (2.12) equals to*

$$\mathcal{U} = \{\varphi \in S\colon\ \widetilde{M}_1\|\varphi\|_S < R\},$$

where $\widetilde{M}_1 := \max\Big\{\max_{-r\le j\le 0}\rho^{-j},\ M_0\Big\}$*;*

(iv) $m(\|\varphi\|_S)$ *defined by (2.13) satisfies* $m(\|\varphi\|_S) \le \widetilde{m}(\|\varphi\|_S)$*, where* $\widetilde{m}(\|\varphi\|_S)$ *is the smallest of those roots of the function*

$$\widetilde{H}(m) := m - \frac{c_1 m}{\rho^2\log\frac{1}{\rho}}\int_0^{m/\rho}\frac{b(u)}{u^2}\,du - \widetilde{M}_1\|\varphi\|_S \tag{4.66}$$

where $\widetilde{H}$ *is monotone increasing.*

The proof is analogous to that of Lemma 4.14, and therefore it is omitted (see also the proof of Theorem 2.8 in [15]).

We close this section with a typical application of our results. Consider the nonlinear scalar difference equation

$$\begin{aligned} x(n+1) - x(n) &= -\delta x(n) + qx(n-r) + f(n, x(n), \dots, x(n-r)), \quad n \ge 0 &\quad (4.67)\\ x(n) &= \varphi(n), \qquad n = -r, -r+1, \dots, 0. &\quad (4.68)\end{aligned}$$

Theorem 4.16 *Suppose* $\delta \in [0,1)$*,* $r \in \mathbb{N}$*,* $q \in \mathbb{R}$*, and*

$$0 \ne q \ne -\frac{(1-\delta)^{r+1}r^r}{(r+1)^{r+1}}.$$

Let $\lambda = \rho(\cos\gamma + i\sin\gamma)$ *(*$\rho \ge 0,\ \gamma \in \mathbb{R}$*) be a dominant root of*

$$\lambda^{r+1} - (1-\delta)\lambda^r - q = 0.$$

Suppose there exists a continuous and monotone non-decreasing function $b\colon\ [0,\infty) \to [0,\infty)$ *such that* $b(u) > 0$ *for* $u > 0$*, and the inequality*

$$|f(n, x(n), \dots, x(n-r))| \le b\Big(\rho^n \max_{n-r\le j\le n}\rho^{-j}|x(j)|\Big), \qquad n \ge 0$$

holds for all sequences x*, and suppose*

$$\rho > 1 \qquad \text{and} \qquad \int_1^\infty \frac{b(u)}{u^2}\,du < \infty$$

or

$$0 < \rho < 1 \qquad \text{and} \qquad \int_0^1 \frac{b(u)}{u^2}\,du < \infty.$$

Then there exist $R > 0$ *and real numbers* $d(\varphi)$ *and* $e(\varphi)$ *that the solutions* $x(n;\varphi)$ *of (4.67)-(4.68) satisfy*

$$x(n;\varphi) = \rho^n\Big(d(\varphi)\cos\gamma n + e(\varphi)\sin\gamma n + o(1)\Big), \qquad n \to \infty,$$

for all initial sequences with $\max\{|\varphi(n)|\colon\ -r \le n \le 0\} < R$*.*

Theorem 4.16 is the consequence of our main Theorem 2.2 and Lemmas 4.10, 4.14 and 4.15.

5 Proof of the main theorem

In this section we give the proof of Theorem 2.2.

The proof of Theorem 2.2 (ii) will be based on the following lemma, which is interesting in its own right.

Lemma 5.17 *Assume that* $U(n,j) \in \mathbb{R}^{d\times d}$, $0 < j \le n < \infty$, *moreover*

$$C_1 := \sup_{0<j\le n<\infty} |U(n,j)| < \infty, \tag{5.69}$$

and there are sequences $P_1, Q_1 \colon \mathbb{N} \to \mathbb{R}^{d\times d}$ *and a constant* $\gamma \in \mathbb{R}$ *such that*

$$\lim_{n\to\infty} |U(n,j) - P_1(j)\cos\gamma(n-j) - Q_1(j)\sin\gamma(n-j)| = 0 \tag{5.70}$$

for any fixed j *satisfying* $0 < j \le n$, *and for some* $C_2 > 0$

$$\sup_{j>0} |P_1(j)| \le C_2, \qquad \sup_{j>0} |Q_1(j)| \le C_2. \tag{5.71}$$

Then for any sequence $g \colon \mathbb{N} \to \mathbb{R}^d$, *relation* $\sum_{j=1}^{\infty} |g(j)| < \infty$ *implies*

$$\lim_{n\to\infty} \left| \sum_{j=1}^{n} U(n,j)g(j) - S(n) \right| = 0, \tag{5.72}$$

where

$$S(n) := \sum_{j=1}^{\infty} \Big(P_1(j)\cos\gamma(n-j) + Q_1(j)\sin\gamma(n-j) \Big) g(j), \qquad n > 0.$$

Proof From (5.71) we find

$$\sum_{j=1}^{\infty} \Big| \Big(P_1(j)\cos\gamma(n-j) + Q_1(j)\sin\gamma(n-j) \Big) g(j) \Big| \le 2C_2 \sum_{j=1}^{\infty} |g(j)| < \infty.$$

Hence $S(n)$ is well-defined, and for all $n > n_1 \ge 0$

$$\begin{aligned} \delta(n) &:= \left| \sum_{j=1}^{n} U(n,j)g(j) - S(n) \right| \\ &\le \sum_{j=1}^{n_1} |U(n,j) - P_1(j)\cos\gamma(n-j) - Q_1(j)\sin\gamma(n-j)||g(j)| \\ &\quad + \sum_{j=n_1+1}^{n} |U(n,j)||g(j)| \\ &\quad + \sum_{j=n_1+1}^{\infty} |P_1(j)\cos\gamma(n-j) + Q_1(j)\sin\gamma(n-j)||g(j)|. \end{aligned}$$

From (5.69) and (5.71), it follows

$$\begin{aligned}\delta(n) \quad &\le \quad \sum_{j=1}^{n_1} |U(n,j) - P_1(j)\cos\gamma(n-j) - Q_1(j)\sin\gamma(n-j)||g(j)| \\ &\quad + (C_1 + 2C_2) \sum_{j=n_1+1}^{\infty} |g(j)|,\end{aligned}$$

for all $n > n_1 \ge 0$, and hence (5.70) implies

$$\limsup_{n\to\infty} \delta(n) \le (C_1 + 2C_2) \sum_{j=n_1+1}^{\infty} |g(j)|, \qquad n_1 \ge 0.$$

This yields $\lim_{n\to\infty} \delta(n) = 0$, as $n_1 \to \infty$, and the proof of the lemma is complete.

□

Proof of Theorem 2.2

(i) Let $\varphi \in \mathcal{U}$ be an arbitrarily fixed initial sequence and $x(n;\varphi)$ denote the solution of the corresponding IVP (2.4)-(2.5) for $n \ge r$. Define

$$z(n) = \rho^{-n}(n+r+1)^{-k} x(n;\varphi), \qquad n \ge -r$$

and

$$U(n,j) = \rho^{-(n-j)}(n+r+1)^{-k} H(n,j), \qquad n \ge j > 0. \tag{5.73}$$

Then it follows from (2.4)

$$z(n) = \rho^{-n}(n+r+1)^{-k} y(n;\varphi) + \sum_{j=1}^{n} U(n,j)\rho^{-j} f(j-1, x(\cdot)), \qquad n > 0. \tag{5.74}$$

We obtain from assumptions (H1) and (H2), respectively

$$|\rho^{-n}(n+r+1)^{-k} y(n;\varphi)| = |\rho^{-n}(n+1)^{-k} y(n;\varphi)| \left(\frac{n+1}{n+r+1}\right)^k \le M_0\|\varphi\|_S$$

for $n > 0$, and

$$|U(n,j)| = \rho^{-(n-j)}(n-j+1)^{-k}\,|H(n,j)| \left(\frac{n-j+1}{n+r+1}\right)^k \le c_1$$

for all $0 \le j \le n$. Thus (5.74) and (H4) imply

$$|z(n)| \le M_0\|\varphi\|_S + c_1 \sum_{j=1}^{n} \rho^{-j} \omega\Big(j-1, \max_{-r\le\tau\le j-1} |z(\tau)|\Big), \qquad n > 0. \tag{5.75}$$

The definitions of M_1 (see (2.11)) and z yield

$$|z(n)| \leq \max_{-r\leq j\leq 0} \rho^{-j}(j+r+1)^{-k}|\varphi(j)| \leq M_1\|\varphi\|_S, \qquad -r \leq n \leq 0. \tag{5.76}$$

Combining (5.75), (5.76) and (2.11) together with the monotonicity of the right-hand-side of (5.75) we get

$$w(n) \leq M_1\|\varphi\|_S + \sum_{j=1}^{n} c_1\rho^{-j}\omega(j-1, w(j-1)), \qquad n > 0,$$

where

$$w(n) := \max_{-r\leq j\leq n} |z(j)|, \qquad n \geq -r.$$

Since $m(\|\varphi\|_S) \geq M_1\|\varphi\|_S$, by (5.76) we have $w(0) \leq m(\|\varphi\|_S)$. Suppose there exists $\bar{n} > 0$ such that $w(\bar{n}) > m(\|\varphi\|_S)$, and $w(n) \leq m(\|\varphi\|_S)$ for $-r \leq n < \bar{n}$. Then the monotonicity of $v \mapsto \omega(n, v)$ and (2.13) yield

$$\begin{aligned} w(\bar{n}) &\leq M_1\|\varphi\|_S + \sum_{j=1}^{\bar{n}} c_1\rho^{-j}\omega(j-1, w(j-1)) \\ &\leq M_1\|\varphi\|_S + \sum_{j=1}^{\bar{n}} c_1\rho^{-j}\omega(j-1, m(\|\varphi\|_S)) \\ &\leq m(\|\varphi\|_S), \end{aligned} \tag{5.77}$$

which contradicts to the assumption of $\bar{n}$. Therefore

$$\rho^{-n}(n+r+1)^{-k}|x(n;\varphi)| = |z(n)| \leq w(n) \leq m(\|\varphi\|_S), \qquad n \geq -r. \tag{5.78}$$

This completes the proof of statement (i).

(ii) It follows from (2.7) and (5.78) that the function

$$g(n) := \rho^{-n} f(n-1, x(\cdot;\varphi)), \qquad n \geq 1$$

satisfies for $n \geq 1$

$$|g(n)| \leq \rho^{-n}\omega\Big(n-1, \max_{-r\leq j\leq n-1} \rho^{-j}(j+r+1)^{-k}|x(j)|\Big) \leq \rho^{-n}\omega(n-1, m(\|\varphi\|_S)). \tag{5.79}$$

The definition of $m(\|\varphi\|_S)$ yields $G(m(\|\varphi\|_S)) \geq M_1\|\varphi\|_S$, therefore

$$\sum_{j=1}^{\infty} |g(j)| \leq \sum_{j=1}^{\infty} \rho^{-j}\omega(j-1, m(\|\varphi\|_S)) \leq \frac{m(\|\varphi\|_S) - M_1\|\varphi\|_S}{c_1} < \infty. \tag{5.80}$$

On the other hand, $U(n, j)$ defined in (5.73) satisfies

$$\begin{aligned} &|U(n,j) - P(j)\cos\gamma(n-j) - Q(j)\sin\gamma(n-j)| \\ &\quad\leq \left|\rho^{-(n-j)}(n-j+1)^{-k}H(n,j)\left(\left(\frac{n-j+1}{n+r+1}\right)^k - 1\right)\right| \\ &\qquad + |\rho^{-(n-j)}(n-j+1)^{-k}H(n,j) - P(j)\cos\gamma(n-j) - Q(j)\sin\gamma(n-j)|. \end{aligned}$$

Hence (H2) and (H6) yield

$$\lim_{n\to\infty} |U(n,j) - (P(j)\cos\gamma(n-j) + Q(j)\sin\gamma(n-j))| = 0$$

for all fixed $j \geq 1$. Since $\sum_{j=1}^{\infty} |g(j)| < \infty$, the sum

$$S(n) := \sum_{j=1}^{\infty} \Big(P(j)\cos\gamma(n-j) + Q(j)\sin\gamma(n-j)\Big) g(j)$$

exists, and in virtue of Lemma 5.17, we have

$$\lim_{n\to\infty} \left| \sum_{j=1}^{n} U(n,j)g(j) - S(n) \right| = 0.$$

This, combined with (5.74), (H5) and (H6), yields

$$\begin{aligned} z(n) &= \rho^{-n}(n+r+1)^{-k} y(n;\varphi) + \sum_{j=1}^{n} U(n,j)g(j) \\ &= D_0\varphi\cos\gamma n + E_0\varphi\sin\gamma n + S(n) + o(1), \qquad \text{as} \quad n\to\infty. \end{aligned}$$

By using the definition of $g(n)$ and trigonometric identities, we find

$$S(n) = d_1(\varphi)\cos\gamma n + e_1(\varphi)\sin\gamma n, \qquad n \geq 1,$$

where

$$d_1(\varphi) = \sum_{j=1}^{\infty} \Big(P(j)\cos\gamma j - Q(j)\sin\gamma j\Big)\rho^{-j} f(j-1, x(\cdot;\varphi))$$

and

$$e_1(\varphi) = \sum_{j=1}^{\infty} \Big(P(j)\sin\gamma j + Q(j)\cos\gamma j\Big)\rho^{-j} f(j-1, x(\cdot;\varphi)).$$

Thus $x(n;\varphi)$ satisfies (2.16) with the constants

$$d(\varphi) = D_0\varphi + d_1(\varphi) \quad \text{and} \quad e(\varphi) = E_0\varphi + e_1(\varphi).$$

Now, we show that $|d(\varphi_0)| + |e(\varphi_0)| \neq 0$ for the initial function φ_0 satisfying (2.14). Indeed

$$|d(\varphi_0)| \geq |D_0\varphi_0| - |d_1(\varphi_0)| \geq |D_0\varphi_0| - (\|P\| + \|Q\|) \sum_{j=1}^{\infty} \rho^{-j} |f(j-1, x(\cdot;\varphi_0))|,$$

and

$$|e(\varphi_0)| \geq |E_0\varphi_0| - |e_1(\varphi_0)| \geq |E_0\varphi_0| - (\|P\| + \|Q\|) \sum_{j=1}^{\infty} \rho^{-j} |f(j-1, x(\cdot;\varphi_0))|.$$

Thus, using (2.14) and (5.79), we get

$$\begin{aligned} |d(\varphi_0)| + |e(\varphi_0)| &\geq \max\{|D_0\varphi_0|, |E_0\varphi_0|\} - (\|P\| + \|Q\|) \sum_{j=1}^{\infty} \rho^{-j}\omega(j-1, m(\|\varphi\|_S)) \\ &> 0. \end{aligned}$$

The proof of Theorem 2.2 is complete.

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