Asymptotically exponential solutions in nonlinear integral and differential equations ⋆

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Abstract

In this paper we investigate the growth/decay rate of solutions of an abstract integral equation which frequently arises in quasilinear differential equations applying a variation-of-constants formula. These results are applicable to some abstract equations which appear in the theory of age dependent population models and also to some quasilinear delay differential equations with bounded and unbounded delays. Examples are given to illustrate the sharpness of the results.

Key words: exponential growth/decay, abstract integral equation, quasilinear differential equations, delay equations, mathematical biology

1 Introduction

Structured population models have been studied at least from the sixties [3], and it is still an intensively studied area [1,2,12,13,16,23,29]. One important properties of age-structured population models is the so-called asynchronous exponential growth/decay property, i.e., when the age distribution tends to a limit independently of the initial age distribution (see, e.g., [10–13,15,23,30–32]). In these papers the partial differential equation population model is transformed into an equivalent abstract linear inhomogeneous differential equation, so the solution is given by the variation-of-constant formula:

$$x(t) = T(t)u + \int_0^t T(t-s)F(x(s))\,ds, \qquad t \geq 0, \tag{1.1}$$

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where $u \in X$, X is a Banach-space with norm $\|\cdot\|$, $T(t)$ is a strongly continuous semigroup of bounded operators in X. Gillenberg and Webb [15] and Webb [30] studied the asynchronous exponential growth in abstract differential equations originated from age-dependent population models, where the investigated abstract differential equation can be written in the form of (1.1). In [30] it has been shown that if $\lim_{t\to\infty} e^{-\alpha t}t^{-k}T(t)u$ exists for all $u \in X$ for some $\alpha > 0$, $k \geq 1$, and

$$\|F(x)\| \leq \theta(\|x\|), \qquad \lim_{s\to\infty} \frac{\theta(s)}{s} = 0,$$

$s \mapsto \frac{\theta(s)}{s}$ is a monotone nonincreasing function on the interval $(0, \infty)$, then $\lim_{t\to\infty} e^{-\alpha t}t^{-k}x(t)$ exists, as well. Therefore the growth rate of the solutions of the homogeneous equation determines that of the solutions of the inhomogeneous equation.

Motivated by this result, in this paper we study the asymptotic behavior of solutions of a nonlinear Volterra-type abstract integral equation

$$x(t) = y(t;\varphi) + \int_{t_0}^{t} T(t-s)f(s, x(\cdot))\, ds, \qquad t \geq t_0. \tag{1.2}$$

Here f is a Volterra-operator, i.e., $f(t, x(\cdot)) = f(t, \tilde{x}(\cdot))$, if $x(s) = \tilde{x}(s)$, $t_{-1} \leq s \leq t$, where $x, \tilde{x} : [t_{-1}, \infty) \to X$, $t_{-1} \leq t_0$, and we associate the initial condition

$$x(s) = \varphi(s), \qquad t_{-1} \leq s \leq t_0 \tag{1.3}$$

to (1.2).

The class of Volterra integral equations of the form (1.2) contains the ordinary integral equation (1.1) as a special case using $f(s, x(\cdot)) = F(x(s))$, $t_{-1} = t_0$, and in this case the initial condition (1.3) reduces to $x(t_0) = \phi(t_0)$. The class of Eq. (1.2) also contains, e.g., functional integral equations of the form

$$x(t) = T(t)\phi + \int_{t_0}^{t} T(t-s)F(x(s-\tau(s)))\, ds, \qquad t \geq t_0. \tag{1.4}$$

In this paper we study the asymptotic behavior of solutions of a nonlinear Volterra-type abstract integral equation of the form (1.2) assuming the knowledge of an asymptotic formula for $y(t;\varphi)$. The function $y(t;\varphi)$ in many applications is a solution of the linear part of a perturbed linear equation, although in our case it can be a nonlinear function of the initial function φ. In our main result (Theorem 2.2 below) we give sufficient conditions which imply that the asymptotic behavior of the "linear part" $y(t;\varphi)$ is preserved for the solution of the nonlinear equation (1.2). In the case when $y(t;\varphi)$ satisfies an exponential estimate of the form $\|y(t;\varphi)\| \leq m_0(\varphi)e^{\alpha(t-t_0)}$, we define a neighborhood of the zero initial function (see Theorems 2.7 and 2.8 below) such that solutions starting from this neighborhood satisfy a similar exponential estimate with the same exponent. If the exponential growth/decay rate of $y(t;\varphi)$ is known, then we give sufficient conditions under which the same growth/decay rate is preserved for the solutions of (1.2). In a special case (see Corollary 2.11 below) we give necessary and sufficient conditions for preserving this growth/decay rate for the solutions of (1.2). Our results applied for the "ordinary" integral equation (1.1) includes the result of Webb [30] under similar, or sometimes weaker condition (see Theorem 2.12 below) and they are applicable for the decaying case, as well.

As an application of the main result, in Section 3 first we show the asynchronous exponential growth property of solutions of a nonlinear PDE model describing an age-dependent population with delayed birth process. Then we discuss asymptotic behavior of solutions of differential equations with bounded and unbounded delays. In this example $y(t;\varphi)$ is a solution of an associated autonomous linear delay equation, where the asymptotic behavior is determined by the leading characteristic root of the equation. Our result can be applied in the case when the leading root is a complex number with multiplicity greater than 1. Illustrative examples are given for the pantograph and the sunflower equations.

The study of asymptotic properties of different classes of integral and differential equations is an active research area, see, e.g., [6–8,12,14,15,17,24–26,28] and the references therein. Most of the work in this direction has been done for linear equations, and guarantees only pure exponential growth/decay of the solutions. Our method is applicable for nonlinear equations of the form (1.2) and for the case when $y(t;\varphi) = e^{\alpha t}t^k(d_0(\varphi)\cos\beta t + e_0(\varphi)\sin\beta t) + o(1)$, as $t \to \infty$.

2 Main results

Let $-\infty < t_{-1} < t_0 < \infty$ be fixed, and $\widetilde{C} := C([t_{-1},\infty), X)$ denote the set of continuous functions mapping $[t_{-1},\infty)$ into the Banach space X. Let $C := C([t_{-1},t_0], X)$ be the Banach space of continuous functions mapping $[t_{-1},t_0]$ into X with the norm $\|\varphi\|_0 = \max_{t_{-1}\le s\le t_0}\|\varphi(s)\|$, $\varphi\in C$, where $\|\cdot\|$ denotes the norm in X. Any fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are denoted by $\|\cdot\|$, as well.

Let $B(X)$ be the space of bounded linear operators mapping X into X, and $\mathbb{R}_+ = [0,\infty)$. A family $T\colon \mathbb{R}_+ \to B(X)$ of bounded linear operators is called strongly continuous if the map $\mathbb{R}_+ \ni t \mapsto T(t)x \in X$ is continuous for any fixed $x \in X$. For any constant $u \in \mathbb{R}$ the corresponding constant function will be denoted by u, as well.

In this section we consider the Volterra-type integral equation

$$x(t) = y(t;\varphi) + \int_{t_0}^{t} T(t-s)f(s,x(\cdot))\,ds, \qquad t \ge t_0 \tag{2.1}$$

with initial condition

$$x(s) = \varphi(s), \qquad t_{-1} \le s \le t_0, \quad \varphi \in C. \tag{2.2}$$

We state the following hypotheses:

(H1) For all $\varphi \in C$ the function $y(\cdot;\varphi)\colon [t_{-1},\infty) \to X$ is continuous, $y(s;\varphi) = \varphi(s)$ for $s \in [t_{-1}, t_0]$, and

$$\|y(t;\varphi)\| \le m_0(\varphi)e^{\alpha(t-t_0)}(t-t_0+1)^k, \qquad t \ge t_0, \tag{2.3}$$

where α is a given constant, k is a nonnegative integer, and $m_0(\cdot)\colon C \to \mathbb{R}_+$ is such that $m_0(\varphi) \to 0$ as $\|\varphi\|_0 \to 0$.

(H2) $T\colon \mathbb{R}_+ \to B(X)$ is a strongly continuous family of bounded linear operators on X, and

$$c_1 := \sup_{0\le t} e^{-\alpha t}(t+1)^{-k}\|T(t)\| < \infty.$$

(H3) $f : [t_0,\infty) \times \widetilde{C} \to X$ is a Volterra-type functional, i.e., for all $x \in \widetilde{C}$, the map $[t_0,\infty) \ni t \mapsto f(t,x(\cdot)) \in X$ is continuous, and for all (t,x), $(t,\tilde{x}) \in [t_0,\infty) \times \widetilde{C}$,

$$f(t,x(\cdot)) = f(t,\tilde{x}(\cdot)), \qquad \text{if} \quad x(s) = \tilde{x}(s), \quad t_{-1} \le s \le t.$$

(H4) For all $(t,z) \in [t_0,\infty) \times \widetilde{C}$,

$$\|f(t, e^{\alpha(\cdot - t_0)}(\cdot - t_{-1}+1)^k\, z(\cdot))\| \le \omega\Big(t, \max_{\zeta(t)\le s\le t}\|z(s)\|\Big), \tag{2.4}$$

where $\zeta\colon [t_0,\infty) \to \mathbb{R}$ satisfies

$$t_{-1} \le \zeta(t) \le t, \qquad t \ge t_0, \tag{2.5}$$

and $[t_0,\infty) \times \mathbb{R}_+ \ni (t,u) \mapsto \omega(t,u) \in \mathbb{R}_+$ is a continuous function such that for any fixed $t \in [t_0,\infty)$, the map $\mathbb{R}_+ \ni u \mapsto \omega(t,u) \in \mathbb{R}_+$ is monotone nondecreasing, and for a positive constant v_0

$$c_1 \int_{t_0}^{\infty} e^{-\alpha(s-t_0)}\omega(s,v_0)\,ds < v_0. \tag{2.6}$$

We define the constant

$$m_1(\varphi) := \max\Big\{\max_{t_{-1}\le s\le t_0} \|e^{-\alpha(s-t_0)}(s-t_{-1}+1)^{-k}\varphi(s)\|,\, m_0(\varphi)\Big\} \tag{2.7}$$

and the function

$$H\colon \mathbb{R}_+ \to \mathbb{R}, \qquad H(v) = v - c_1 \int_{t_0}^{\infty} e^{-\alpha(s-t_0)} \omega(s,v)\, ds. \tag{2.8}$$

Assumption (2.6) yields $H(v_0) > 0$ for some $v_0 > 0$, and therefore the constant

$$\rho := \sup \{H(v)\colon\ v > 0\} \tag{2.9}$$

is well-defined, and it is either positive or $+\infty$. The set $\mathcal{U}$ defined by

$$\mathcal{U} := \Big\{\varphi \in C\colon\ m_1(\varphi) < \rho\Big\} \tag{2.10}$$

is not empty, since $m_0(\varphi) \to 0$, and therefore $m_1(\varphi) \to 0$ as $\|\varphi\|_0 \to 0$. Hence the set

$$\mathcal{M}(\varphi) := \Big\{m > 0\colon\ H(m) > m_1(\varphi)\Big\} \tag{2.11}$$

is also not empty, and the constant

$$m(\varphi) := \inf \mathcal{M}(\varphi) \tag{2.12}$$

is a well-defined real number for all $\varphi \in \mathcal{U}$.

Definition 2.1 *A function x is a solution of the initial value problem (IVP) (2.1)-(2.2) if $x \in \widetilde{C}$ and it satisfies Eq. (2.1) on $[t_0, \infty)$ and initial condition (2.2) on $[t_{-1}, t_0]$.*

In this paper we do not deal with the existence and uniqueness of solutions. We assume that some additional conditions are satisfied for f such that the solutions of the IVP (2.1)-(2.2) exist locally on some interval $[t_0, t_1]$. It will be shown in Theorem 2.2 that (H1)–(H4) imply that any solution will exist globally on $[t_0, \infty)$, as well. It is worth to note that the uniqueness of the solutions is not needed in our results. Any fixed solution of the IVP (2.1)-(2.2) is denoted by $x(\cdot;\varphi)$.

In the first part of our main result, Theorem 2.2, we give an exponential upper bound for the solutions of the IVP (2.1)-(2.2), and in the second part of this theorem we give a limit relation based on the following three additional hypotheses:

(H5) There exist $d_0(\cdot)$, $e_0(\cdot)\colon \mathcal{U} \to X$ and $\beta \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} \left\| e^{-\alpha(t-t_0)} (t - t_{-1} + 1)^{-k} y(t;\varphi) - d_0(\varphi) \cos \beta t - e_0(\varphi) \sin \beta t \right\| = 0,$$

for $\varphi \in \mathcal{U}$.

(H6) There exist $P, Q \in B(X)$ for which

$$\lim_{t\to+\infty} \left\| e^{-\alpha t}(t+1)^{-k}T(t) - P\cos\beta t - Q\sin\beta t \right\| = 0.$$

(H7) There is an initial function $\varphi_0 \in \mathcal{U}$ such that

$$\max\Big\{ \|d_0(\varphi_0)\|\,,\|e_0(\varphi_0)\| \Big\} > (\|P\| + \|Q\|)\int_{t_0}^{\infty} e^{-\alpha(s-t_0)}\omega(s, m(\varphi_0))\,ds. \tag{2.13}$$

Now, we are in a position to state and prove our main result.

Theorem 2.2 *Assume that (H1)-(H4) are satisfied.*

(i) If $\varphi \in \mathcal{U}$, then any solution $x(\cdot;\varphi)$ of the IVP (2.1)-(2.2) exists on $[t_{-1},\infty)$, and satisfies

$$\|x(t;\varphi)\| \le m(\varphi)e^{\alpha(t-t_0)}(t - t_{-1} + 1)^k, \qquad t \ge t_{-1}, \tag{2.14}$$

where $m(\varphi)$ is defined in (2.12).

(ii) If in addition (H5)-(H6) hold, then for all $\varphi \in \mathcal{U}$ there are vectors $d(\varphi)$ and $e(\varphi)$ in X such that

$$x(t;\varphi) = e^{\alpha(t-t_0)}(t - t_{-1} + 1)^k\Big(d(\varphi)\cos\beta t + e(\varphi)\sin\beta t + o(1)\Big), \tag{2.15}$$

as $t \to +\infty$. Moreover, if (H7) holds, then $\|d(\varphi_0)\| + \|e(\varphi_0)\| \ne 0$, where φ_0 is given in (2.13).

Relations (2.14) and (2.15) can be reformulated in several forms. For example, the next result follows immediately from Theorem 2.2.

Corollary 2.3 *Assume that (H1)-(H4) are satisfied.*

(i) If $\varphi \in \mathcal{U}$, then any solution $x(\cdot;\varphi)$ of the IVP (2.1)-(2.2) satisfies

$$\|x(t;\varphi)\| \le m(\varphi)(t_0 - t_{-1} + 1)^k e^{\alpha(t-t_0)}(t - t_0 + 1)^k, \qquad t \ge t_0, \tag{2.16}$$

where $m(\varphi)$ is defined in (2.12).

(ii) If in addition (H5)-(H6) hold, then for all $\varphi \in \mathcal{U}$ there are vectors $\tilde{d}(\varphi)$ and $\tilde{e}(\varphi)$ in X such that

$$x(t;\varphi) = e^{\alpha(t-t_0)}(t-t_0)^k\Big(\tilde{d}(\varphi)\cos\beta(t-t_0) + \tilde{e}(\varphi)\sin\beta(t-t_0) + o(1)\Big), \tag{2.17}$$

as $t \to +\infty$. Moreover, if (H7) holds, then $\|\tilde{d}(\varphi_0)\| + \|\tilde{e}(\varphi_0)\| \ne 0$, where φ_0 is given in (2.13).

To prove Theorem 2.2 we need the following lemma which is interesting in its own right.

Lemma 2.4 *Assume that $U(t,s)$, $t_0 \le s \le t < \infty$, is a family of linear bounded operators on X that is jointly strongly continuous in t and s, moreover*

$$M_1 := \sup_{t_0 \le s \le t < \infty} \|U(t,s)\| < \infty, \tag{2.18}$$

and there are strongly continuous operators $P_1, Q_1 \colon \mathbb{R}_+ \to B(X)$ and a constant $\beta \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} \|U(t,s) - P_1(s)\cos\beta(t-s) - Q_1(s)\sin\beta(t-s)\| = 0 \tag{2.19}$$

for any fixed $s \in [t_0, \infty)$, and for some $M_2 > 0$

$$\sup_{s \ge t_0} \|P_1(s)\| \le M_2, \qquad \sup_{s \ge t_0} \|Q_1(s)\| \le M_2. \tag{2.20}$$

Then for any continuous function $g \colon [t_0, \infty) \to X$, relation $\int_{t_0}^{\infty} \|g(s)\|\, ds < \infty$ implies

$$\begin{aligned} \lim_{t \to +\infty} \Bigg\| &\int_{t_0}^{t} U(t,s) g(s)\, ds \\ &- \int_{t_0}^{\infty} \Big(P_1(s)\cos\beta(t-s) + Q_1(s)\sin\beta(t-s) \Big) g(s)\, ds \Bigg\| = 0. \end{aligned} \tag{2.21}$$

Proof. From (2.20), we find

$$\int_{t_0}^{\infty} \Big\| \Big(P_1(s)\cos\beta(t-s) + Q_1(s)\sin\beta(t-s) \Big) g(s) \Big\|\, ds \le 2M_2 \int_{t_0}^{\infty} \|g(s)\|\, ds < \infty.$$

Thus

$$\begin{aligned} \delta(t) := \Bigg\| &\int_{t_0}^{t} U(t,s) g(s)\, ds \\ &- \int_{t_0}^{\infty} \Big(P_1(s)\cos\beta(t-s) + Q_1(s)\sin\beta(t-s) \Big) g(s)\, ds \Bigg\| \\ \le &\int_{t_0}^{t_1} \|U(t,s) - P_1(s)\cos\beta(t-s) - Q_1(s)\sin\beta(t-s)\| \|g(s)\|\, ds \\ &+ \int_{t_1}^{t} \|U(t,s)\| \|g(s)\|\, ds \\ &+ \int_{t_1}^{\infty} \|P_1(s)\cos\beta(t-s) + Q_1(s)\sin\beta(t-s)\| \|g(s)\|\, ds \end{aligned}$$

for all $t \ge t_1 \ge t_0$. From (2.18) and (2.20), it follows

$$\delta(t) \leq \int_{t_0}^{t_1} \|U(t,s) - P_1(s)\cos\beta(t-s) - Q_1(s)\sin\beta(t-s)\|\|g(s)\|\,ds$$
$$+(M_1 + 2M_2)\int_{t_1}^{\infty} \|g(s)\|\,ds,$$

for all $t \geq t_1 \geq t_0$, and hence (2.19) and the Lebesgue's Dominated Convergence Theorem imply

$$\limsup_{t\to+\infty} \delta(t) \leq (M_1 + 2M_2)\int_{t_1}^{\infty} \|g(s)\|\,ds, \qquad t_1 \geq t_0.$$

This yields $\lim_{t\to+\infty} \delta(t) = 0$, as $t_1 \to +\infty$, and the proof of the lemma is complete. $\square$

Remark 2.5 *Lemma 2.4 is an essential generalization of a result in [4] which has been proved when $U(t,s) = a(t-s)$, $0 \leq s \leq t$, is a scalar function, $\beta = 0$ and $P_1(s) = p_1$ is a constant.*

Proof. of Theorem 2.2

(i) Let $\varphi \in \mathcal{U}$ be an arbitrarily fixed initial function and $x(\cdot;\varphi)$ denote a noncontinuable solution of the corresponding IVP (2.1)-(2.2) on $[t_{-1}, t_1)$ for some $t_1 > t_0$. Define

$$z(t) = e^{-\alpha(t-t_0)}(t - t_{-1} + 1)^{-k} x(t;\varphi), \qquad t \in [t_{-1}, t_1)$$

and

$$U(t,s) = e^{-\alpha(t-s)}(t - t_{-1} + 1)^{-k} T(t-s), \qquad t \geq s \geq t_{-1}. \tag{2.22}$$

Then it follows from (2.1) for $t \in [t_0, t_1)$

$$z(t) = e^{-\alpha(t-t_0)}(t - t_{-1} + 1)^{-k} y(t;\varphi)$$
$$+ \int_{t_0}^{t} U(t,s) e^{-\alpha(s-t_0)} f(s, e^{\alpha(\cdot - t_0)}(\cdot - t_{-1} + 1)^k z(\cdot))\,ds. \tag{2.23}$$

We obtain from assumptions (H1) and (H2), respectively

$$\|e^{-\alpha(t-t_0)}(t - t_{-1} + 1)^{-k} y(t;\varphi)\|$$
$$= \|e^{-\alpha(t-t_0)}(t - t_0 + 1)^{-k} y(t;\varphi)\| \left(\frac{t - t_0 + 1}{t - t_{-1} + 1}\right)^k \leq m_0(\varphi)$$

for $t \in [t_0, t_1)$, and

$$\|U(t,s)\| = e^{-\alpha(t-s)}(t - s + 1)^{-k}\, \|T(t-s)\| \left(\frac{t - s + 1}{t - t_{-1} + 1}\right)^k \leq c_1$$

for all $t_0 \le s \le t < \infty$. Thus (2.23) together with (2.3) and (2.4) implies

$$\|z(t)\| \le m_0(\varphi) + c_1 \int_{t_0}^{t} e^{-\alpha(s-t_0)} \omega\Big(s, \max_{\zeta(t)\le\tau\le s} \|z(\tau)\|\Big) ds, \quad t \in [t_0, t_1). \quad (2.24)$$

Relation (2.7) and the definition of z yield

$$\|z(t)\| \le \max_{t_{-1}\le s\le t_0} \Big\| e^{-\alpha(s-t_0)}(s - t_{-1} + 1)^{-k}\varphi(s)\Big\| \le m_1(\varphi), \qquad t_{-1} \le t \le t_0. \quad (2.25)$$

Combining (2.24), (2.25) and (2.7) together with the monotonicity of the right-hand-side of (2.24) and the monotonicity of ω in its second argument we get

$$w(t) \le m_1(\varphi) + c_1 \int_{t_0}^{t} e^{-\alpha(s-t_0)} \omega(s, w(s))\, ds, \qquad t \in [t_0, t_1), \qquad (2.26)$$

where

$$w(t) := \max_{t_{-1}\le\tau\le t} \|z(\tau)\|, \qquad t \in [t_{-1}, t_1).$$

On the other hand, (2.6) and the definitions of $\mathcal{U}$ and $m(\varphi)$ yield (see (2.10) and (2.12)) that there exists a decreasing sequence v_n of nonnegative numbers such that

$$v_n > m_1(\varphi) + c_1 \int_{t_0}^{\infty} e^{-\alpha(s-t_0)} \omega(s, v_n)\, ds \quad \text{and} \quad \lim_{n\to\infty} v_n = m(\varphi).$$

Then for all n

$$v_n > m_1(\varphi) + c_1 \int_{t_0}^{t} e^{-\alpha(s-t_0)} \omega(s, v_n)\, ds, \qquad t \in [t_0, t_1). \qquad (2.27)$$

Since $v \mapsto \omega(t, v)$ is a monotone nondecreasing function on $[0, \infty)$ for any fixed $t \ge t_0$, a standard comparison result (see, e.g., [22]) yields from (2.26) and (2.27) that

$$e^{-\alpha(t-t_0)}(t - t_{-1} + 1)^{-k} \|x(t; \varphi)\| \le w(t) < v_n, \qquad t \in [t_0, t_1)$$

for all n. It follows from (2.12) that $m(\varphi) \ge m_1(\varphi)$, therefore

$$\|x(t; \varphi)\| < v_n e^{\alpha(t-t_0)}(t - t_{-1} + 1)^k, \qquad t \in [t_{-1}, t_1).$$

Then $t_1 = +\infty$, since otherwise $\lim_{t\to t_1-} \|x(t; \varphi)\| < \infty$. Statement (i) follows combining the previous inequality and (2.25), and taking the limit $n \to \infty$.

(ii) From (2.4) and (2.14) and the monotonicity of ω it follows that the function

$$g(t) := e^{-\alpha(t-t_0)} f(t, x(\cdot; \varphi)) = e^{-\alpha(t-t_0)} f(t, e^{\alpha(\cdot-t_0)}(\cdot - t_{-1} + 1)^k z(\cdot)), \qquad t \ge t_0,$$

satisfies

$$\|g(t)\| \le e^{-\alpha(t-t_0)} \omega(t, m(\varphi)), \qquad t \ge t_0, \qquad (2.28)$$

and (2.27) yields

$$\int_{t_0}^{\infty} \|g(t)\|\, dt < \infty.$$

On the other hand, $U(t,s)$ defined in (2.22) satisfies

$$\begin{aligned}
&\|U(t,s) - (P\cos\beta(t-s) + Q\sin\beta(t-s))\| \\
&\quad \le \left\| e^{-\alpha(t-s)}(t-s+1)^{-k}T(t-s)\left(\left(\frac{t-s+1}{t-t_{-1}+1}\right)^k - 1\right)\right\| \\
&\qquad + \|e^{-\alpha(t-s)}(t-s+1)^{-k}T(t-s) - P\cos\beta(t-s) - Q\sin\beta(t-s)\|.
\end{aligned}$$

Hence (H2) and (H6) yield

$$\lim_{t\to+\infty} \|U(t,s) - (P\cos\beta(t-s) + Q\sin\beta(t-s))\| = 0$$

for all fixed $s \in [t_0, \infty)$. Since $\int_{t_0}^{\infty} \|g(t)\|\, dt < \infty$, the integral

$$I(t) := \int_{t_0}^{\infty} \Big(P\cos\beta(t-s) + Q\sin\beta(t-s)\Big) g(s)\, ds$$

exists, and in virtue of Lemma 2.4, we have

$$\lim_{t\to+\infty} \left\| \int_{t_0}^{t} U(t,s)g(s)\, ds - I(t) \right\| = 0.$$

This, combined with (2.23), (H5) and (H6), yields

$$\begin{aligned}
z(t) &= e^{-\alpha(t-t_0)}(t-t_{-1}+1)^{-k}y(t;\varphi) + \int_{t_0}^{t} U(t,s)g(s)\, ds \\
&= d_0(\varphi)\cos\beta t + e_0(\varphi)\sin\beta t + I(t) + o(1), \qquad \text{as} \quad t\to+\infty.
\end{aligned}$$

Using the definition of $g(t)$ and trigonometric identities we find

$$I(t) = d_1(\varphi)\cos\beta t + e_1(\varphi)\sin\beta t, \qquad t \ge t_0,$$

where

$$d_1(\varphi) = \int_{t_0}^{\infty} (P\cos\beta s - Q\sin\beta s)e^{-\alpha(s-t_0)} f(s, x(\cdot;\varphi))\, ds$$

and

$$e_1(\varphi) = \int_{t_0}^{\infty} (P\sin\beta s + Q\cos\beta s)e^{-\alpha(s-t_0)} f(s, x(\cdot;\varphi))\, ds.$$

Thus $x(t;\varphi)$ satisfies (2.15) with the constants

$$d(\varphi) = d_0(\varphi) + d_1(\varphi) \quad \text{and} \quad e(\varphi) = e_0(\varphi) + e_1(\varphi).$$

Now, we show that $\|d(\varphi_0)\| + \|e(\varphi_0)\| \ne 0$ for the initial function φ_0 satisfying (2.13). Indeed,

$$
\begin{aligned}
\|d(\varphi_0)\| &\geq \|d_0(\varphi_0)\| - \|d_1(\varphi_0)\| \\
&\geq \|d_0(\varphi_0)\| - (\|P\| + \|Q\|) \int_{t_0}^{\infty} e^{-\alpha(s-t_0)} \|f(s, x(\cdot;\varphi_0))\| \, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\|e(\varphi_0)\| &\geq \|e_0(\varphi_0)\| - \|e_1(\varphi_0)\| \\
&\geq \|e_0(\varphi_0)\| - (\|P\| + \|Q\|) \int_{t_0}^{\infty} e^{-\alpha(s-t_0)} \|f(s, x(\cdot;\varphi_0))\| \, ds.
\end{aligned}
$$

Thus, from (2.13) and (2.28), it follows

$$
\begin{aligned}
&\|d(\varphi_0)\| + \|e(\varphi_0)\| \\
&\quad \geq \max\{\|d_0(\varphi_0)\|, \|e_0(\varphi_0)\|\} \\
&\qquad - (\|P\| + \|Q\|) \int_{t_0}^{\infty} e^{-\alpha(s-t_0)} \|f(s, x(\cdot;\varphi_0))\| \, ds \\
&\quad > 0.
\end{aligned}
$$

The proof of the theorem is complete. □

If $\omega(t, u)$ is linear in u, then Theorem 2.2 yields easily the next result.

Theorem 2.6 *Assume (H1)-(H3) are satisfied, and for all* $(t, z) \in [t_0, \infty) \times \widetilde{C}$

$$
\|f(t, e^{\alpha(\cdot - t_0)}(\cdot - t_{-1} + 1)^k z(\cdot))\| \leq e^{\alpha(t-t_0)} a(t) \max_{\zeta(t) \leq s \leq t} \|z(s)\|, \tag{2.29}
$$

where ζ *satisfies (2.5), and* $a\colon\ [t_0, \infty) \to \mathbb{R}_+$ *is a continuous function such that*

$$
c_1 \int_{t_0}^{\infty} a(s) \, ds < 1, \tag{2.30}
$$

where c_1 *is defined in (H2). Then*

(i) For all $\varphi \in C$ *the solution* $x(\cdot;\varphi)$ *of the IVP (2.1)-(2.2) satisfies*

$$
\|x(t;\varphi)\| \leq m_2(\varphi) e^{\alpha(t-t_0)} (t - t_{-1} + 1)^k, \qquad t \geq t_{-1}, \tag{2.31}
$$

where

$$
m_2(\varphi) = \frac{m_1(\varphi)}{1 - c_1 \int_{t_0}^{\infty} a(s) \, ds}. \tag{2.32}
$$

(ii) If (H5) and (H6) also hold, then for all $\varphi \in C$*, there are* $d(\varphi)$*,* $e(\varphi) \in X$ *such that (2.15) is satisfied. Moreover, if*

$$
\max\Big\{\|d_0(\varphi_0)\|,\ \|e_0(\varphi_0)\|\Big\} > m_2(\varphi_0)(\|P\| + \|Q\|) \int_{t_0}^{\infty} a(s) \, ds,
$$

for some $\varphi_0 \in C$*, then* $\|d(\varphi_0)\| + \|e(\varphi_0)\| > 0$*.*

Proof. The result is an easy consequence of Theorem 2.2 and hence its proof is omitted. $\square$

In the next results we use the following conditions:

(H8) Suppose $\alpha \neq 0$, and there exists a continuous and monotone nondecreasing function $b_\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $b(u) > 0$ for $u > 0$, and the inequality

$$\|f(t, e^{\alpha(\cdot - t_0)} z(\cdot))\| \leq b_\alpha\Big(e^{\alpha(t-t_0)} \max_{\zeta(t) \leq s \leq t} \|z(s)\|\Big), \qquad (t, z) \in [t_0, \infty) \times \widetilde{C} \tag{2.33}$$

holds, where ζ satisfies (2.5).

(H9) There exists $\varphi_1 \in C$ such that

$$\lim_{t \to \infty} e^{-\alpha(t-t_0)} y(t; \varphi_1) \neq 0,$$

and

$$y(t; \gamma\varphi_1) = \gamma y(t; \varphi_1), \qquad t \geq t_0, \quad \gamma > 0. \tag{2.34}$$

In the next result we study the case when $k = 0$ in (H1) and (H2). In this case $m_1(\varphi)$ defined in (2.7) simplifies to

$$\widetilde{m}_1(\varphi) := \max\Big\{ \|e^{-\alpha(\cdot - t_0)} \varphi(\cdot)\|_0, \, m_0(\varphi) \Big\}, \tag{2.35}$$

where $m_0(\varphi)$ is defined in (H1). Note that $\widetilde{m}_1(\varphi) = 0$, if and only if $\varphi = 0$ and $m_0(\varphi) = 0$, and hence $y(t; \varphi) = 0$, $t \geq t_{-1}$.

Theorem 2.7 *Assume that (H1), (H2) and (H3) are satisfied with $k = 0$, moreover (H8) holds with*

$$\alpha > 0 \qquad \textit{and} \qquad \int_1^\infty \frac{b_\alpha(u)}{u^2} \, du < \infty. \tag{2.36}$$

Then

(i) for every $\varphi \in C$ the equation

$$\widetilde{m}_1(\varphi) + \frac{c_1}{\alpha} m \int_m^\infty \frac{b_\alpha(u)}{u^2} \, du = m, \qquad m \geq 0 \tag{2.37}$$

has at most two roots, and any solution $x(\cdot; \varphi)$ of the IVP (2.1)-(2.2) satisfies

$$\|x(t; \varphi)\| \leq m_\alpha(\varphi) e^{\alpha(t-t_0)}, \qquad t \geq t_{-1}, \tag{2.38}$$

where $m_\alpha(\varphi)$ is the largest root of (2.37).

(ii) *If (H5) and (H6) are also satisfied with $k = 0$, then for every $\varphi \in C$ there are $d_\alpha(\varphi)$, $e_\alpha(\varphi) \in X$ such that*

$$x(t; \varphi) = e^{\alpha(t-t_0)}\Big(d_\alpha(\varphi)\cos\beta t + e_\alpha(\varphi)\sin\beta t + o(1)\Big), \quad t \to +\infty. \quad (2.39)$$

(iii) *If, in addition, (H9) holds, then there exists $\gamma_1 > 0$ such that*

$$\|d_\alpha(\gamma\varphi_1)\| + \|e_\alpha(\gamma\varphi_1)\| \neq 0, \qquad \gamma \geq \gamma_1.$$

Proof. (i) Let $\omega(t, u)$ be defined by $\omega(t, u) = b_\alpha(e^{\alpha(t-t_0)}u)$ for all $t \geq t_0$ and $u \geq 0$. Then for $m > 0$

$$\int_{t_0}^{\infty} e^{-\alpha(t-t_0)}\omega(t, m)\, dt = \int_{t_0}^{\infty} e^{-\alpha(t-t_0)}b_\alpha(e^{\alpha(t-t_0)}m)\, dt = \frac{m}{\alpha}\int_m^\infty \frac{b_\alpha(u)}{u^2}\, du < \infty,$$

where we used the substitution $u = e^{\alpha(t-t_0)}m$. Let H be defined by (2.8). Then $H(m) = mH_1(m)$, where

$$H_1(m) = 1 - \frac{c_1}{\alpha}\int_m^\infty \frac{b_\alpha(u)}{u^2}\, du,$$

and therefore ρ defined in (2.9) satisfies $\rho = \lim_{m\to+\infty} H(m) = +\infty$, and $\mathcal{U}$ defined in (2.10) equals to C.

Let $\varphi \in C$ be fixed, and consider

$$\widetilde{\mathcal{M}}(\varphi) := \Big\{m > 0\colon\ H(m) > \widetilde{m}_1(\varphi)\Big\}.$$

In view of Theorem 2.2, we have to show that $m_\alpha(\varphi) := \inf \widetilde{\mathcal{M}}(\varphi)$ satisfies the statement of the theorem. First note that H_1 is a monotone increasing function satisfying $\lim_{m\to+\infty} H_1(m) = 1$. Since $H'(m) = H_1(m) + \frac{c_1}{\alpha}\frac{b_\alpha(m)}{m}$, it follows that H is monotone increasing for large enough m, and $\lim_{m\to+\infty} H(m) = +\infty$. Consequently, $\mathcal{M}(\varphi)$ is not empty.

We consider three cases. Case 1: Suppose

$$0 \leq \int_0^\infty \frac{b_\alpha(u)}{u^2}\, du \leq \frac{\alpha}{c_1}.$$

Then $\lim_{m\to 0+} H_1(m) \in \mathbb{R}_+$, therefore $H(0) = 0$, and H is monotone increasing on $\mathbb{R}_+$. In this case $m_\alpha(\varphi)$ is the unique root of (2.37).

Case 2: Suppose

$$\frac{\alpha}{c_1} < \int_0^\infty \frac{b_\alpha(u)}{u^2}\, du < \infty.$$

Then $H_1(m)$ is negative for small m, and hence there exists $m^* > 0$ such that $H_1(m^*) = 0$, $H_1(m) < 0$ for $m \in (0, m^*)$, and $H_1(m) > 0$ for $m \in (m^*, \infty)$.

Moreover, H is monotone increasing on (m^*, ∞). Then (2.37) has a unique root on (m^*, ∞) for any $\widetilde{m}_1(\varphi) > 0$, and (2.37) has two roots, $m = 0$ and $m = m^*$ for $\widetilde{m}_1(\varphi) = 0$ (i.e., for $\varphi = 0$).

Case 3: Suppose

$$\int_0^\infty \frac{b_\alpha(u)}{u^2}\, du = \infty.$$

Then L'Hospital's Rule yields

$$\lim_{m\to 0+} m \int_m^\infty \frac{b_\alpha(u)}{u^2}\, du = \lim_{m\to 0+} \frac{\frac{d}{dm}\int_m^\infty \frac{b_\alpha(u)}{u^2}\, du}{\frac{d}{dm}\frac{1}{m}} = b_\alpha(0),$$

therefore

$$\lim_{m\to 0+} H(m) = -\frac{c_1}{\alpha} b_\alpha(0) \leq 0.$$

Then there exists $m^{**} \geq 0$ such that $H(m) < 0$ for $m \in [0, m^{**})$, $H(m^{**}) = 0$, and $H(m) > 0$ for $m > m^{**}$. Then (2.37) has a unique root on $[m^{**}, \infty)$ for any $\widetilde{m}_1(\varphi) \geq 0$.

Thus statement (i) is a consequence of statement (i) of Theorem 2.2 with $k = 0$. Note that in all cases above we get $m_\alpha(\varphi) \to \infty$ as $\widetilde{m}_1(\varphi) \to \infty$.

(ii) In virtue of Theorem 2.2, it is also clear that (2.39) is satisfied with some suitable $d_\alpha(\varphi)$ and $e_\alpha(\varphi)$ for all $\varphi \in C$.

(iii) It follows from (2.34) that $m_0(\gamma\varphi_1) = \gamma m_0(\varphi_1)$ for $\gamma > 0$, and so $\widetilde{m}_1(\gamma\varphi_1) = \gamma\widetilde{m}_1(\varphi_1)$ for $\gamma > 0$. The proof of the theorem is complete if we show that (H7) holds with $\varphi_0 = \gamma\varphi_1$ for any large enough $\gamma > 0$.

First note that $\|d_0(\gamma_1\varphi_1)\| + \|e_0(\gamma_1\varphi_1)\| = \gamma A$ for $\gamma > 0$, where $A := \|d_0(\varphi_1)\| + \|e_0(\varphi_1)\| \neq 0$. Then inequality (2.13) for $\varphi_0 = \gamma\varphi_1$ has the form

$$\begin{aligned}\gamma A &> (\|P\| + \|Q\|) \int_{t_0}^\infty e^{-\alpha(t-t_0)} b_\alpha\Big(e^{\alpha(t-t_0)} m_\alpha(\gamma\varphi_1)\Big)\, dt \\ &= \frac{\|P\| + \|Q\|}{\alpha} m_\alpha(\gamma\varphi_1) \int_{m_\alpha(\gamma\varphi_1)}^\infty \frac{b_\alpha(u)}{u^2}\, du. \end{aligned} \tag{2.40}$$

Therefore relations $\gamma\widetilde{m}_1(\varphi_1) = \widetilde{m}_1(\gamma\varphi_1) = m_\alpha(\gamma\varphi_1) H_1(m_\alpha(\gamma\varphi_1))$ yield an equivalent form of (2.40):

$$A > \frac{(\|P\| + \|Q\|)\widetilde{m}_1(\varphi_1)}{\alpha H_1(m_\alpha(\gamma\varphi_1))} \int_{m_\alpha(\gamma\varphi_1)}^\infty \frac{b_\alpha(u)}{u^2}\, du,$$

which holds for $\gamma \geq \gamma_1$ for some $\gamma_1 > 0$, since $m_\alpha(\gamma\varphi_1) \to \infty$ as $\gamma \to \infty$.

The proof of the theorem is complete. $\square$

The next result deals with the case $\alpha < 0$.

Theorem 2.8 *Assume that (H1), (H2) and (H3) are satisfied with $k = 0$, moreover (H8) holds with*

$$\alpha < 0 \qquad \text{and} \qquad \int_0^1 \frac{b_\alpha(u)}{u^2}\,du < \infty.$$

Let $\widetilde{m}_1(\varphi)$ be defined in (2.35),

$$\mathcal{U}_\alpha := \{\varphi \in C\colon \widetilde{m}_1(\varphi) < \rho_\alpha\}, \tag{2.41}$$

where

$$\rho_\alpha := \sup_{0<v}\left\{v\left(1 - \frac{c_1}{|\alpha|}\int_0^v \frac{b_\alpha(u)}{u^2}\,du\right)\right\}.$$

Then

(i) for all initial function $\varphi \in \mathcal{U}_\alpha$ any corresponding solution $x(\cdot;\varphi)$ of the IVP (2.1)-(2.2) satisfies (2.38), where $m = m_\alpha(\varphi)$ is the smallest root of the function

$$\widetilde{H}(m) := m - \frac{c_1}{|\alpha|} m \int_0^m \frac{b_\alpha(u)}{u^2}\,du - \widetilde{m}_1(\varphi), \tag{2.42}$$

where the function $\widetilde{H}$ is monotone increasing.

(ii) If (H5) and (H6) are also satisfied with $k = 0$, then for every $\varphi \in \mathcal{U}_\alpha$ there exist $d_\alpha(\varphi)$, $e_\alpha(\varphi) \in X$ such that (2.39) is satisfied.

(iii) If, in addition, (H9) holds, then there exists $\gamma_2 > 0$ such that

$$\|d_\alpha(\gamma\varphi_1)\| + \|e_\alpha(\gamma\varphi_1)\| \neq 0, \qquad 0 < \gamma \leq \gamma_2.$$

Proof. (i) Let $\omega(t,u)$ be defined by $\omega(t,u) = b_\alpha(e^{\alpha(t-t_0)}u)$ for all $t \geq t_0$ and $u \geq 0$. Then

$$\begin{aligned}\int_{t_0}^\infty e^{-\alpha(t-t_0)}\omega(t,m)\,dt &= \int_{t_0}^\infty e^{-\alpha(t-t_0)} b_\alpha\left(e^{\alpha(t-t_0)}m\right)dt \\ &= \frac{m}{|\alpha|}\int_0^m \frac{b_\alpha(u)}{u^2}\,du \\ &< \infty,\end{aligned}$$

where we used the substitution $u = e^{\alpha(t-t_0)}m$. Thus ρ defined in (2.9) is equal to ρ_α. We rewrite $\widetilde{H}$ in the form

$$\widetilde{H}(m) = mH_2(m) - \widetilde{m}_1(\varphi), \qquad H_2(m) = 1 - \frac{c_1}{|\alpha|}\int_0^m \frac{b_\alpha(u)}{u^2}\,du.$$

Now $H_2(0) = 1$, and there exists $m^{**} > 0$ such that $1/2 \leq H_2(m) \leq 1$ for $m \in [0, m^{**}]$. Therefore it is easy to see that $m_\alpha(\varphi) \to 0$ as $\|\varphi\|_0 \to 0$.

Statement (i) is an easy consequence of statement (i) of Theorem 2.2 with $k = 0$.

(ii) The asymptotic formula (2.39) is an immediate consequence of the assumptions and part (ii) of Theorem 2.2 for all $\varphi \in \mathcal{U}_\alpha$.

(iii) Similarly to the proof of Theorem 2.7 we can argue that condition (H7) with $\varphi_0 = \gamma\varphi_1$ is equivalent to the inequality

$$A > \frac{(\|P\| + \|Q\|)\widetilde{m}_1(\varphi_1)}{|\alpha| H_2(m_\alpha(\gamma\varphi_1))} \int_0^{m_\alpha(\gamma\varphi_1)} \frac{b_\alpha(u)}{u^2}\, du,$$

which holds for $0 < \gamma \le \gamma_2$ for some $\gamma_2 > 0$, since $m_\alpha(\gamma\varphi_1) \to 0$ as $\gamma \to 0$.

The proof of the theorem is complete. $\square$

Proposition 2.9 *We have $\rho_\alpha = \infty$ and $\mathcal{U}_\alpha = C$ in Theorem 2.8, if and only if*

$$\int_0^\infty \frac{b_\alpha(u)}{u^2}\, du < \frac{|\alpha|}{c_1} \tag{2.43}$$

or

$$\int_0^\infty \frac{b_\alpha(u)}{u^2}\, du = \frac{|\alpha|}{c_1} \quad \textit{and} \quad \lim_{m\to+\infty} b_\alpha(m) = \infty. \tag{2.44}$$

Proof. If (2.43) holds, then $H_2(m) \ge \varepsilon$ for $m > 0$ for some $\varepsilon > 0$, and hence $H(m) \ge \varepsilon m$, $m > 0$, therefore $\rho_\alpha = \infty$.

If

$$\int_0^\infty \frac{b_\alpha(u)}{u^2}\, du > \frac{|\alpha|}{c_1},$$

then there exists $m^* > 0$ such that $H_2(m^*) = 0$ and $H_2(m) < 0$ for $m > m^*$. This yields $\rho_\alpha < \infty$.

If

$$\int_0^\infty \frac{b_\alpha(u)}{u^2}\, du = \frac{|\alpha|}{c_1},$$

then

$$\lim_{m\to+\infty} H(m) = \lim_{m\to+\infty} \frac{H_2(m)}{\frac{1}{m}} = \lim_{m\to+\infty} \frac{H_2'(m)}{-\frac{1}{m^2}} = \lim_{m\to+\infty} \frac{c_1}{|\alpha|} b_\alpha(m),$$

which completes the proof. $\square$

The sharpness of Theorems 2.7 and 2.8 is analyzed for the following scalar equation

$$x(t) = V(t - t_0)\varphi(t_0) + \int_{t_0}^t V(t - s) g(s, x(\cdot)) ds, \qquad t \ge t_0, \tag{2.45}$$

with initial condition

$$x(s) = \varphi(s), \qquad t_{-1} \le s \le t_0, \qquad \varphi \in C([t_{-1}, t_0], \mathbb{R}).$$

Here

(A1) $V\colon \mathbb{R}_+ \to (0, \infty)$ is a continuous function, and there exists a real number α such that the limit

$$P = \lim_{t\to+\infty} e^{-\alpha t} V(t)$$

is a positive number.

(A2) $g\colon [t_0, \infty) \times C([t_{-1}, \infty), \mathbb{R}) \to \mathbb{R}$ is a continuous Volterra-type functional and there exists a continuous function $\gamma_\alpha\colon [t_0, \infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ such that the map $\mathbb{R}_+ \ni u \mapsto \gamma_\alpha(t, u) \in \mathbb{R}_+$ is monotone nonincreasing for any fixed $t \in [t_0, \infty)$, moreover for any $(t, y) \in [t_0, \infty) \times C([t_{-1}, \infty), \mathbb{R}_+)$,

$$g\Big(t, e^{\alpha(\cdot - t_0)} y(\cdot)\Big) \ge \gamma_\alpha\Big(t, \min_{\xi(t)\le s\le t} y(s)\Big), \tag{2.46}$$

where $\xi\colon [t_0, \infty) \to [t_{-1}, \infty)$ is a continuous function satisfying $t_{-1} \le \xi(t) \le t$ for $t \ge t_0$, and $\xi(t) \to +\infty$ as $t \to +\infty$.

Theorem 2.10 *Assume that (A1) and (A2) are satisfied, and there exists an initial function $\varphi_0 \in C([t_{-1}, t_0], \mathbb{R}_+)$ such that $\varphi_0(t_0) > 0$ and Eq. (2.45) has a solution $x(\cdot; \varphi_0)\colon [t_{-1}, \infty) \to \mathbb{R}$ for which the limit*

$$c_0 := \lim_{t\to+\infty} e^{-\alpha(t-t_0)} x(t; \varphi_0)$$

exists and is positive. Then there exist $v_0 > 0$ and $T_0 > 0$ satisfying the inequality

$$P \int_{t_0+T_0}^{\infty} e^{-\alpha(t-t_0)} \gamma_\alpha(t, v_0)\, dt < v_0.$$

Proof. First we show that the function $x_0 = x(\cdot; \varphi_0)$ is positive on $[t_0, \infty)$. Otherwise there exists a $t_1 > t_0$ such that

$$x_0(t) > 0, \quad t_0 \le t < t_1, \qquad \text{and} \quad x_0(t_1) = 0.$$

Then from (2.45) it follows

$$x_0(t_1) \ge V(t_1 - t_0)\varphi_0(t_0) > 0,$$

which is a contradiction. Thus $x_0(t) > 0$, $t \ge t_0$, and hence the function $y(t) = e^{-\alpha(t-t_0)} x_0(t), \quad t \ge t_{-1}$, satisfies

$$
\begin{aligned}
y(t) &= e^{-\alpha(t-t_0)} V(t-t_0)\varphi(t_0) \\
&\quad + \int_{t_0}^{t} e^{-\alpha(t-s)} V(t-s) e^{-\alpha(s-t_0)} g\Big(s, e^{\alpha(\cdot - t_0)} y(\cdot)\Big)\, ds \\
&\geq e^{-\alpha(t-t_0)} V(t-t_0)\varphi(t_0) \\
&\quad + \int_{t_0}^{t} e^{-\alpha(t-s)} V(t-s) e^{-\alpha(s-t_0)} \gamma_\alpha\Big(s, \min_{\xi(s)\leq\tau\leq s} y(\tau)\Big)\, ds
\end{aligned}
$$

for $t \geq t_0$. But $y(t) \to c_0 > 0$ and $\xi(t) \to +\infty$, as $t \to +\infty$, and hence for an arbitrarily fixed $\varepsilon \in (0,1)$, there exists a $T_0 = T_0(\varepsilon) > 0$ such that

$$
e^{-\alpha(t-t_0)} V(t-t_0) \geq (1-\varepsilon)P
$$

and

$$
(1+\varepsilon)c_0 \geq y(t) \geq \min_{\xi(t)\leq s\leq t} y(s) \geq (1-\varepsilon)c_0, \quad t \geq t_0 + T_0.
$$

This yields the inequality

$$
(1+\varepsilon)c_0 \geq (1-\varepsilon)P\varphi(t_0) + (1-\varepsilon)P \int_{t_0+T_0}^{\infty} e^{-\alpha(s-t_0)} \gamma_\alpha(s, (1-\varepsilon)c_0) ds.
$$

Since $P\varphi(t_0) > 0$, there is a constant $\varepsilon_0 \in (0,1)$ that satisfies

$$
(1+\varepsilon_0)c_0 - (1-\varepsilon_0)P\varphi(t_0) < (1-\varepsilon_0)^2 c_0,
$$

and hence

$$
v_0 > P \int_{t_0+T_0}^{\infty} e^{-\alpha(s-t_0)} \gamma_\alpha(s, v_0)\, ds, \qquad \text{where} \quad v_0 = (1-\varepsilon_0)c_0.
$$

This completes the proof of the theorem. $\square$

Next we consider a special case of Eq. (2.45).

$$
x(t) = V(t-t_0)\varphi(t_0) + \int_{t_0}^{t} V(t-s) b(x(s-\tau(s)))\, ds, \qquad t \geq t_0. \tag{2.47}
$$

with the initial condition $x(s) = \varphi(s), \quad t_0 - r \leq s \leq t_0, \quad \varphi \in C([t_0 - r, t_0], \mathbb{R})$.

In the next corollary for α defined in (A1), S_α denotes the set of all initial functions $\varphi \in C([t_0 - r, t_0], \mathbb{R})$ such that the limit

$$
c := \lim_{t\to+\infty} e^{-\alpha(t-t_0)} x(t;\varphi) \tag{2.48}
$$

exists and is not zero for a solution $x(t;\phi)$ of Eq. (2.47) corresponding to φ. It is easy to see that if b is an odd function and $\varphi \in S_\alpha$, then $-\varphi \in S_\alpha$, as well.

Now we state and prove the next sharp result.

Corollary 2.11 *Assume that $V: \mathbb{R}_+ \to (0,\infty)$ satisfies condition (A1), $b: \mathbb{R} \to \mathbb{R}$ is a continuous, monotone nondecreasing and odd function satisfying $b(u) > 0$ for $u > 0$, $\tau: [t_0,\infty) \to [0,r]$ is a continuous function, and $r > 0$. Then*

(i) If S_α is not empty, then $\alpha \neq 0$.
(ii) If $\alpha < 0$, then S_α is not empty if and only if $\int_0^1 \frac{b(u)}{u^2}\,du < \infty$.
(iii) If $\alpha > 0$, then S_α is not empty if and only if $\int_1^\infty \frac{b(u)}{u^2}\,du < \infty$.

Proof. It is easy to check that the assumptions imply (H1)–(H3) and (H5)–(H6) with $X = \mathbb{R}$, $t_{-1} = t_0 - r$, $T(t) = V(t)$, $y(t;\varphi) = V(t-t_0)\varphi(t_0)$, $f(t,x) = b(x(t-\tau(t)))$, $\zeta(t) = t - \tau(t)$, $k = 0$, $c_1 = \sup_{0\le t} e^{-\alpha t}V(t)$, $m_0(\varphi) = c_1\|\varphi\|_0$, $\beta = 0$, $Q = 0$, $d_0(\varphi) = Pe^{\alpha t_0}\varphi(t_0)$, $e_0(\varphi) = 0$, and (H8) with $b_\alpha(u) = b(\omega_\alpha u)$, where $\omega_\alpha = \sup_{0\le t} e^{-\alpha\tau(t)}$.

(i) Suppose $\varphi \in S_\alpha$, and let $x(t) = x(t;\varphi)$ be a fixed solution of (2.47) satisfying (2.48). Without the loss of generality we assume that φ is such that c defined in (2.48) is positive. For the sake of contradiction we assume that $\alpha = 0$. But in that case there exists a $t_1 > 0$ such that $x(t-\tau(t)) > c/2, \quad t \ge t_0 + t_1$, and hence

$$\int_{t_0}^t V(t-s)b(x(s-\tau(s)))\,ds \ge b\left(\frac{c}{2}\right)\int_{t_0+t_1}^t V(t-s)\,ds = b\left(\frac{c}{2}\right)\int_0^{t-t_0-t_1} V(u)\,du$$

for $t \ge t_0 + t_1$. It follows from (A1) with $\alpha = 0$ that $\lim_{t\to+\infty} V(t) = P > 0$, and hence from (2.47) we obtain $\lim_{t\to+\infty} x(t) = +\infty$. This contradicts to the definition of S_α, and consequently $\alpha \neq 0$.

(ii) In virtue of Theorem 2.8, it is clear that $\int_0^1 \frac{b(u)}{u^2}\,du < \infty$ implies that S_α is not empty. For the necessary part, assume $\varphi \in S_\alpha$.

Define the functions $g: [t_0,\infty) \times C([t_{-1},\infty),\mathbb{R}) \to \mathbb{R}$ and $\gamma_\alpha: [t_0,\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$g(t,x(\cdot)) = b(x(t-\tau(t))), \qquad (t,x) \in [t_0,\infty) \times C([t_{-1},\infty),\mathbb{R}),$$

and

$$\gamma_\alpha(t,u) = b(e^{\alpha(t-t_0)}\delta_\alpha u), \qquad (t,u) \in [t_0,\infty) \times \mathbb{R}_+,$$

where $\delta_\alpha = \inf_{0\le t} e^{-\alpha\tau(t)}$. Then for any $(t,y) \in [t_0,\infty) \times C([t_{-1},\infty),\mathbb{R})$, relation (2.46) is satisfied, where $h(t) = t - \tau(t), \quad t \ge t_0$. Thus by Theorem 2.10 we have constants $v_0 > 0$ and $T_0 > 0$ such that

$$v_0 > P\int_{t_0+T_0}^\infty e^{-\alpha(t-t_0)}\gamma_\alpha(t,v_0)\,dt = P\int_{t_0+T_0}^\infty e^{-\alpha(t-t_0)}b(\delta_\alpha e^{\alpha(t-t_0)}v_0)\,dt.$$

It can be easily seen by using the substitution $u = \delta_\alpha e^{\alpha(t-t_0)}v_0$ that the above inequality implies $\int_0^1 \frac{b(u)}{u^2}\,du < \infty$.

Part (iii) can be argued similarly to the proof of (ii). $\square$

Closing this section we consider Eq. (2.1) in the case when $t_{-1} = t_0$, i.e., consider

$$x(t) = y(t;\varphi(t_0)) + \int_{t_0}^{t} T(t-s)f(s,x(s))\,ds, \qquad t \geq t_0, \tag{2.49}$$

and the initial condition

$$x(t_0) = \varphi(t_0). \tag{2.50}$$

We will need the following assumptions.

(H8*) Suppose $\alpha \neq 0$, and there exists a continuous and monotone nondecreasing function $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\theta(u) > 0$ for $u > 0$, and

$$\|f(t, e^{\alpha(t-t_0)}(t-t_0+1)^k z)\| \leq \theta\Big(e^{\alpha(t-t_0)}(t-t_0+1)^k\|z\|\Big), \quad (t,z) \in [t_0,\infty)\times X. \tag{2.51}$$

(H9*) There exists $\varphi_1 \in C$ such that

$$\lim_{t\to\infty} e^{-\alpha(t-t_0)}(t-t_0+1)^{-k} y(t;\varphi_1(t_0)) \neq 0,$$

and (2.34) holds.

Theorem 2.12 *Suppose (H1)-(H3) hold with $t_{-1} = t_0$, there exists $\eta \geq 1$ such that the function*

$$(0,\infty) \to (0,\infty), \qquad u \to \frac{\theta(u)}{u^\eta}$$

is monotone nonincreasing, moreover (H8) holds with*

$$\alpha > 0 \qquad \text{and} \qquad \int_1^\infty \frac{\theta(u)}{u^2}(\log u)^{k\eta}\,du < \infty. \tag{2.52}$$

Then

(i) for every $\varphi \in C$, any solution $x(\cdot;\varphi)$ of the IVP (2.49)-(2.50) satisfies (2.14), where $m = m(\varphi)$ is the largest root of the equation

$$m_1(\varphi) + \frac{c_1}{\alpha^{k\eta+1}} m \int_m^\infty \frac{\theta(u)}{u^2}\left(\log\frac{u}{m}+1\right)^{k\eta} du = m. \tag{2.53}$$

(ii) If (H5) and (H6) are also satisfied with $t_{-1} = t_0$, then for every $\varphi(t_0) \in X$ there are $d_\alpha(\varphi)$, $e_\alpha(\varphi) \in X$ such that (2.15) holds.

(iii) If, in addition, (H9) holds, then there exists $\gamma_1 > 0$ such that*

$$\|d_\alpha(\gamma\varphi_1)\| + \|e_\alpha(\gamma\varphi_1)\| \neq 0, \qquad \gamma \geq \gamma_1.$$

Proof. Let $\omega(t,u)$ be defined by $\omega(t,u) = \theta(e^{\alpha(t-t_0)}(t-t_0+1)^k u)$ for all $t \geq t_0$ and $u \geq 0$. Then for $m \geq e^\alpha$

$$\begin{aligned}
&\int_{t_0}^\infty e^{-\alpha(t-t_0)}\omega(t,m)\,dt \\
&\quad = \int_{t_0}^\infty e^{-\alpha(t-t_0)} \frac{\theta(e^{\alpha(t-t_0)}(t-t_0+1)^k m)}{(e^{\alpha(t-t_0)}(t-t_0+1)^k m)^\eta} (e^{\alpha(t-t_0)}(t-t_0+1)^k m)^\eta\,dt \\
&\quad \leq \int_{t_0}^\infty e^{-\alpha(t-t_0)} \frac{\theta(e^{\alpha(t-t_0)} m)}{(e^{\alpha(t-t_0)} m)^\eta} (e^{\alpha(t-t_0)}(t-t_0+1)^k m)^\eta\,dt \\
&\quad = m\int_{t_0}^\infty \frac{\theta(e^{\alpha(t-t_0)} m)}{(e^{\alpha(t-t_0)} m)} (t-t_0+1)^{k\eta}\,dt \\
&\quad = \frac{m}{\alpha}\int_m^\infty \frac{\theta(u)}{u^2}\Big(\frac{1}{\alpha}\log\frac{u}{m}+1\Big)^{k\eta}\,du \\
&\quad \leq \frac{m}{\alpha^{k\eta+1}}\int_m^\infty \frac{\theta(u)}{u^2}(\log u)^{k\eta}\,du \\
&\quad < \infty,
\end{aligned}$$

where we used the substitution $u = e^{\alpha(t-t_0)}m$ and the fact that

$$t - t_0 + 1 = \frac{1}{\alpha}\log\frac{u}{m} + 1 = \frac{1}{\alpha}(\log u - \log m + \alpha) \leq \frac{1}{\alpha}\log u$$

for $\log m \geq \alpha$. Therefore ρ defined in (2.9) is $+\infty$ and $\mathcal{U}$ defined in (2.10) equals to C. The rest of the proof is identical to that of Theorem 2.7. $\square$

In the case $\alpha < 0$ we have

$$\int_{t_0}^\infty e^{-\alpha(t-t_0)}\omega(t,m)\,dt \leq \frac{m}{|\alpha|^{k\eta+1}}\int_0^m \frac{\theta(u)}{u^2}\Big(\log\frac{m}{u} - \alpha\Big)^{k\eta}\,du < \infty.$$

Therefore, analogously to Theorem 2.8 and Theorem 2.12, we can formulate a result for the case $\alpha < 0$ using condition

$$\int_0^1 \frac{\theta(u)}{u^2}\Big(\log\frac{1}{u} - \alpha\Big)^{k\eta}\,du < \infty$$

instead of (2.52).

We remark that Theorem 2.12 includes the result of Webb [30] using $\eta = 1$, $t_0 = 0$ and $y(t;\varphi) = T(t)\varphi(0)$.

3 Applications

In this section we apply our main results to a class of age-dependent population models with delayed birth process, to a certain class of differential equations with bounded and unbounded delays, and also to the pantograph and sunflower differential equations.

3.1 An age-dependent population with delayed birth process

In [23] the following age-structured population model has been studied:

$$\frac{\partial u}{\partial t}(t,a) = -\frac{\partial u}{\partial a}(t,a) - \mu(a)u(t,a), \qquad t \geq 0,\ a \geq 0, \tag{3.1}$$

$$u(t,0) = \int_0^\infty \int_{-\tau}^0 \beta(\sigma,a)u(t+\sigma,a)\,d\sigma\,da, \qquad t \geq 0, \tag{3.2}$$

$$u(s,a) = F_0(s,a), \qquad s \in [-\tau,0),\ a > 0, \tag{3.3}$$

$$u(0,a) = f_0(a), \qquad a \geq 0. \tag{3.4}$$

Here $u(t,a)$ denotes the density of the population at time t and age a, the death rate of the individuals is described by $\mu(a)$. $\tau > 0$ is a constant denoting the maximal delay and $\beta(\sigma,a)$ represents the probability that an individual of age a reproduces after a time lag $-\sigma$ starting from conception.

We introduce the Banach-spaces $X = L^1(\mathbb{R}_+,\mathbb{R}_+)$, $E = L^1([-\tau,0],X) \cong L^1([-\tau,0]\times\mathbb{R}_+,\mathbb{R})$ and $\mathcal{Z} = E \times X$ with the product norm

$$\left\| \begin{pmatrix} F \\ f \end{pmatrix} \right\|_{\mathcal{Z}} = \|F\|_E + \|f\|_X,$$

and define the delay operator

$$\Phi\colon E \to \mathbb{R}, \quad \Phi F = \int_0^\infty \int_{-\tau}^0 \beta(\sigma,a)F(\sigma,a)\,d\sigma\,da.$$

We assume

(B1) $\beta \in L^\infty([-\tau,0]\times\mathbb{R}_+,\mathbb{R})$ and $\mu \in L^\infty(\mathbb{R}_+,\mathbb{R})$ are bounded, nonnegative functions, and $\lim_{a\to\infty}\mu(a) > 0$, $F_0 \in E$, $f_0 \in X$.

We use the notations $u(t) = u(t,\cdot)$ and $u_t\colon [-\tau,0] \to X$, $u_t(s) = u(t+s)$, and define the function

$$z\colon \mathbb{R}_+ \to \mathcal{Z}, \qquad z(t) = \begin{pmatrix} u_t \\ u(t) \end{pmatrix}.$$

It was shown in [23] that the PDE model (3.1)-(3.4) is equivalent to the abstract Cauchy-problem

$$z'(t) = \mathcal{A}\, z(t), \qquad t \geq 0, \quad z(0) = z_0 := \begin{pmatrix} F_0 \\ f_0 \end{pmatrix}, \tag{3.5}$$

where the linear operator $\mathcal{A}$ on $D(\mathcal{A}) \subset \mathcal{Z}$ is defined by

$$\left(\mathcal{A}\begin{pmatrix} F \\ f \end{pmatrix}\right)(\sigma, a) = \begin{pmatrix} \frac{d}{d\sigma}F(\sigma) \\ -f'(a) - \mu(a)f(a) \end{pmatrix},$$

where

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} F \\ f \end{pmatrix} \in W^{1,1}([-\tau, 0], X) \times W^{1,1}(\mathbb{R}_+, \mathbb{R}) \colon \begin{pmatrix} F(0) \\ f(0) \end{pmatrix} = \begin{pmatrix} f \\ \Phi F \end{pmatrix} \right\}.$$

The following result is a consequence of Theorems 3.3, 4.6 and 4.7 of [23].

Proposition 3.1 *Assume (B1). Then $\mathcal{A}$ generates a strongly continuous positive semigroup $\mathcal{T}(t)$ on $\mathcal{Z}$, and the population model (3.1)-(3.4) is well-posed.*

Moreover, if

$$\int_0^\infty \int_{-\tau}^0 \beta(\sigma, a) e^{-\int_0^a \mu(s)\, ds} d\sigma\, da > 1, \tag{3.6}$$

then $\alpha_0 := \omega_0(\mathcal{A}) = s(\mathcal{A}) > 0$ is an eigenvalue of $\mathcal{A}$ with a one dimensional spectral projection Π and there exist constants $M, \delta > 0$ such that

$$\|e^{-\alpha_0 t}\mathcal{T}(t) - \Pi\| \leq M e^{-\delta t}, \qquad \textit{for all } t \geq 0.$$

Here $\omega_0(\mathcal{A})$ denotes the growth bound of $\mathcal{T}(t)$, i.e., $\omega_0(\mathcal{A}) = \lim_{t\to+\infty} \frac{\log\|\mathcal{T}(t)\|}{t}$, and $s(\mathcal{A})$ is the spectral bound of $\mathcal{A}$, i.e., $s(\mathcal{A}) = \sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma(\mathcal{A})\}$, where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$.

Now we consider a nonlinear version of (3.1)

$$\frac{\partial u}{\partial t}(t, a) = -\frac{\partial u}{\partial a}(t, a) - \Big(\mu(a) + G(u(t + \cdot, a))\Big)u(t, a), \quad t \geq 0, \ a \geq 0, \tag{3.7}$$

where the nonlinear functional $G \colon E \to \mathbb{R}_+$ accounts for the loss of individuals due to crowding. Similar nonlinearity was considered, e.g., in [10] and [12]. It is easy to see that (3.7) associated with boundary and initial conditions (3.2)-(3.4) is equivalent to the Cauchy-problem

$$z'(t) = \mathcal{A}\, z(t) + \mathcal{G}(z(t)), \qquad t \geq 0, \quad z(0) = z_0, \tag{3.8}$$

where

$$\left(\mathcal{G}\begin{pmatrix}F\\ f\end{pmatrix}\right)(\sigma,a)=\begin{pmatrix}0\\ -G(F(\sigma))f(a)\end{pmatrix}.$$

Then the mild solution of (3.8) is the solution of

$$z(t)=\mathcal{T}(t)z_0+\int_0^t \mathcal{T}(t-s)\mathcal{G}(z(s))\,ds,\qquad t\geq 0.$$

We assume on the nonlinearity that

(B2) $G\colon E\to\mathbb{R}_+$ is such that
 (i) there exist constants $M\geq 0$ and $-1<\eta<0$ satisfying

$$0\leq G(F)\leq M\|F\|_E^\eta,\qquad F\in E,$$

 (ii) G is locally Lipschitz-continuous, i.e., for every $K>0$ there exists $L=L(K)$ such that

$$|G(F)-G(\tilde{F})|\leq L\|F-\tilde{F}\|_E,\qquad \|F\|_E,\|\tilde{F}\|_E\leq K.$$

The next result shows that the asynchronous exponential growth of the linear equation (3.5) is preserved for the mild solution of the nonlinear equation (3.8).

Theorem 3.2 *Assume (B1), (B2) and (3.6). Then (3.8) has a unique mild solution $z(t)$ on $[0,\infty)$, and there exists a one dimensional projection Π such that the solution satisfies*

$$\lim_{t\to+\infty}\|e^{-\alpha_0 t}z(t)-\Pi z_0\|_{\mathcal{Z}}=0,\qquad z_0\in\mathcal{Z}.$$

Proof. Assumption (B2) yields easily that $\mathcal{G}$ is locally Lipschitz-continuous, therefore a standard argument (see, e.g., Section 4.3 of [5]) shows the local existence and also the uniqueness of the mild solution of (3.8).

It follows from Proposition 3.1 that (H1)–(H3), (H5), (H6) and (H9) are satisfied with $t_0=t_{-1}=0$ and $k=0$, and (3.6) yields $\alpha_0>0$.

Let $z=\begin{pmatrix}F\\ f\end{pmatrix}\in\mathcal{Z}$. Then assumption (B2) (i) yields

$$\begin{aligned}\left\|\mathcal{G}\left(e^{\alpha_0 t}z\right)\right\|_{\mathcal{Z}}&=\|G(e^{\alpha_0 t}F)e^{\alpha_0 t}f\|_X=G(e^{\alpha_0 t}F)e^{\alpha_0 t}\|f\|_X\\ &\leq Me^{(1+\eta)\alpha_0 t}\|F\|_E^\eta\|f\|_X\leq Me^{(1+\eta)\alpha_0 t}\|z\|_{\mathcal{Z}}^{1+\eta}\quad t\geq 0, z\in\mathcal{Z}.\end{aligned}$$

Therefore (H8) holds with $b_\alpha(u)=Mu^{1+\eta}$, which satisfies (2.36), as well. Therefore, Theorem 2.7 yields the statement of the theorem. □

Consider the system of linear delay equations

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t-\tau_i) + \sum_{j=0}^{M} B_j(t) x(t-\sigma_j(t)), \qquad t \geq t_0. \tag{3.9}$$

We assume

(i) $t_0 \geq 0$,
(ii) $\tau_i \in \mathbb{R}_+$ and $A_i \in \mathbb{R}^{n\times n}$, $\quad 0 \leq i \leq N$, and
(iii) $B_j \colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $\sigma_j \colon \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, $\lim_{t\to+\infty}(t-\sigma_j(t)) = +\infty$, $0 \leq j \leq M$.

Let

$$\tau = \max_{0\leq i\leq N} \tau_i \qquad \text{and} \qquad t_{-1}(t_0) := \min\Big\{t_0 - \tau, \min_{0\leq j\leq M}\{\inf_{t_0\leq t}\{t-\sigma_j(t)\}\}\Big\}.$$

We associate the initial condition

$$x(t) = \varphi(t), \qquad t_{-1}(t_0) \leq t \leq t_0 \tag{3.10}$$

to Eq. (3.9). In this section we will consider the initial time t_0 as a parameter, so the solution of the IVP (3.9)-(3.10) will be denoted by $x(\cdot; t_0, \varphi)$.

The asymptotic property of the solutions of Eq. (3.9) is given by using our main results and certain asymptotic properties of the solutions of the autonomous system

$$\dot{y}(t) = \sum_{i=0}^{N} A_i y(t-\tau_i), \qquad t \geq t_0, \tag{3.11}$$

and the associated initial condition

$$y(t) = \varphi(t), \qquad t_0 - \tau \leq t \leq t_0. \tag{3.12}$$

By definition, the fundamental solution $T(t)$ to (3.11) is the $n \times n$ matrix valued function satisfying

$$\dot{T}(t) = \sum_{i=0}^{N} A_i T(t-\tau_i), \quad t \geq 0, \quad \text{and} \quad T(0) = I,\ T(s) = 0 \quad \text{for} \quad -\tau \leq s < 0.$$

Here I and 0 denote the $n \times n$ identity and zero matrices, respectively.

The characteristic equation associated to (3.11) is

$$\Delta(\lambda) := \det\left(\lambda I - \sum_{i=0}^{N} A_i e^{-\lambda\tau_i}\right) = 0. \tag{3.13}$$

A complex number λ is called an eigenvalue of Eq. (3.11) if it is a solution of Eq. (3.13).

Definition 3.3 *$\lambda_0 = \alpha_0 + \beta_0 i$ ($i = \sqrt{-1}$), is called a dominant eigenvalue of (3.11) if $\Delta(\lambda_0) = 0$ and $\operatorname{Re}\lambda_0 > \operatorname{Re}\lambda$ for $\lambda \in \mathbb{C}$ satisfying $\Delta(\lambda) = 0,\ \lambda \neq \lambda_0$ and $\lambda \neq \overline{\lambda}_0$. The ascent of λ_0 is the order of λ_0 as a pole of $\Delta^{-1}(\lambda)$ (see [9], [18]).*

It is known ([9], [18]) that the ascent of an eigenvalue λ is less or equal to the algebraic multiplicity of λ.

We assume

(C) $\lambda_0 = \alpha_0 + \beta_0 i$ is a dominant eigenvalue of Eq. (3.11), and let $k+1$ be its ascent.

The following result follows from the general theory of the series representation of the solutions of Eq. (3.11) (see, e.g., [4], [18]).

Proposition 3.4 *Assume (C). Then the following statements hold.*

(i) For all $t_0 \geq 0$ and $\varphi \in C([t_0 - \tau, t_0], \mathbb{R}^n)$ there exist $d_0(t_0, \varphi)$, $e_0(t_0, \varphi) \in \mathbb{R}^n$ such that the solution $y(\cdot; t_0, \varphi)$ of Eq. (3.11) through (t_0, φ) satisfies

$$y(t; t_0, \varphi) = e^{\alpha_0 t} t^k \Big(d_0(t_0, \varphi) \cos \beta_0 t + e_0(t_0, \varphi) \sin \beta_0 t + o(1) \Big),$$

as $t \to \infty$.

(ii) There exist constant matrices $P, Q \in \mathbb{R}^{n \times n}$ for which

$$T(t) = e^{\alpha_0 t}(t+1)^k (P \cos \beta_0 t + Q \sin \beta_0 t + o(1)), \qquad \text{as} \quad t \to +\infty.$$

In the proof of the next theorem we will need the following estimate which can be proved using Gronwall's inequality (see, e.g., [17] for a proof of a similar result).

Lemma 3.5 *The solution $x(\cdot; t_0; \varphi)$ of the IVP (3.9)-(3.10) satisfies*

$$\|x(t; t_0, \varphi)\| \leq e^{\sum_{i=0}^{N} \|A_i\|(t-t_0) + \sum_{j=0}^{M} \int_{t_0}^{t} \|B_j(s)\|\, ds} \max_{t_{-1}(t_0) \leq s \leq t_0} \|\varphi(s)\|, \qquad t \geq t_0.$$

Now, we are in a position to state and prove the following result.

Theorem 3.6 *Assume (C), and*

$$\int_0^\infty t^k \sum_{j=0}^M \|B_j(t)\| e^{-\alpha_0\sigma_j(t)} dt < \infty. \tag{3.14}$$

Then for every $t_0 \geq 0$ there exists $M(t_0) \geq 0$ such that for every $\varphi \in C([t_{-1}(t_0), t_0], \mathbb{R}^n)$ the solution $x(\cdot; t_0, \varphi)$ of Eq. (3.9) through (t_0, φ) satisfies

$$\|x(t; t_0, \varphi)\| \leq M(t_0) e^{\alpha_0(t-t_0)} (t - t_{-1}(t_0) + 1)^k \max_{t_{-1}(t_0) \leq s \leq t_0} \|\varphi(s)\|, \qquad t \geq t_{-1}(t_0). \tag{3.15}$$

Moreover, there are vectors $\widetilde{d}(t_0, \varphi)$ and $\widetilde{e}(t_0, \varphi)$ in $\mathbb{R}^n$ such that

$$x(t; t_0, \varphi) = e^{\alpha_0 t} t^k \Big(\widetilde{d}(t_0, \varphi) \cos \beta_0 t + \widetilde{e}(t_0, \varphi) \sin \beta_0 t + o(1) \Big), \qquad \text{as} \quad t \to +\infty, \tag{3.16}$$

where $\|\widetilde{d}(t_0, \varphi_0)\| + \|\widetilde{e}(t_0, \varphi_0)\| \neq 0$ for some $\varphi_0 \in C([t_{-1}(t_0), t_0], \mathbb{R}^n)$.

Proof. From part (c) of Proposition 3.4, it follows that

$$c_1 := \sup_{0 \leq t} (t+1)^{-k} e^{-\alpha_0 t} \|T(t)\| < \infty.$$

Assumption (3.14) yields there exists $S > 0$ such that

$$A := \max\Big\{c_1, \|P\| + \|Q\|\Big\} \int_S^\infty (t - t_{-1}(0) + 1)^k \sum_{j=0}^M \|B_j(t)\| e^{-\alpha_0 \sigma_j(t)} \, dt < 1.$$

Since $t_{-1}(0) \leq t_{-1}(t_0)$ for any $t_0 \geq 0$, it follows

$$\max\Big\{c_1, \|P\| + \|Q\|\Big\} \int_{t_0}^\infty (t - t_{-1}(t_0) + 1)^k \sum_{j=0}^M \|B_j(t)\| e^{-\alpha_0 \sigma_j(t)} \, dt \leq A < 1 \tag{3.17}$$

for all $t_0 \geq S$. Let $t_0 \geq S$ be fixed, and $f\colon [t_0, \infty) \times C([t_{-1}(t_0), \infty), \mathbb{R}^n) \to \mathbb{R}^n$ be defined by

$$f(t, y(\cdot)) = \sum_{j=0}^M B_j(t) y(t - \sigma_j(t)).$$

Then, by using the variation of constants formula (see, e.g., [18]), we obtain that the solution $x(\cdot; t_0, \varphi)$ of Eq. (3.9) through (t_0, φ) satisfies

$$x(t; t_0, \varphi) = y(t; t_0, \varphi) + \int_{t_0}^t T(t-s) f(s, x(\cdot; t_0, \varphi)) \, ds, \qquad t \geq t_0,$$

where $y(\cdot; t_0, \varphi)$ denotes the solution of Eq. (3.11) through (t_0, φ). Moreover, for all $(t, z) \in [t_0, \infty) \times C([t_{-1}(t_0), t_0], \mathbb{R}^n) \to \mathbb{R}^n$

$$
\begin{aligned}
&\left\| f\left(t, e^{\alpha_0(\cdot - t_0)}(\cdot - t_{-1}(t_0)+1)^k z(\cdot)\right)\right\| \\
&\qquad = \left\| \sum_{j=0}^{M} B_j(t) e^{\alpha_0(t-\sigma_j(t)-t_0)}(t-\sigma_j(t)-t_{-1}(t_0)+1)^k z(t-\sigma_j(t)) \right\| \\
&\qquad \le e^{\alpha_0(t-t_0)} a(t) \max_{\zeta(t)\le s\le t} \|z(s)\|,
\end{aligned}
$$

where $\zeta(t) = \min\{t-\sigma_j(t) \colon\ j = 0, \dots, M\}$, and

$$
a(t) = (t - t_{-1}(t_0)+1)^k \sum_{j=0}^{M} \|B_j(t)\| e^{-\alpha_0 \sigma_j(t)}, \qquad t \ge 0.
$$

Thus it follows from (3.17) that all conditions of Theorem 2.6 hold, therefore $x(\cdot; t_0, \varphi)$ satisfies (2.31), i.e.,

$$
\|x(t;t_0,\varphi)\| \le m_2(t_0,\varphi) e^{\alpha_0(t-t_0)}(t-t_{-1}(t_0)+1)^k, \qquad t \ge t_0 \ge S, \tag{3.18}
$$

where

$$
m_2(t_0,\varphi) = \frac{\max\left\{\max_{t_{-1}(t_0)\le s\le t_0} \|e^{-\alpha_0(s-t_0)}(s-t_{-1}(t_0)+1)^{-k}\varphi(s)\|,\ m_0(\varphi)\right\}}{1 - c_1\int_{t_0}^{\infty} a(s)\, ds}.
$$

Since Eq. (3.11) is autonomous and linear, there exists a constant $K \ge 1$ such that

$$
\|y(t;t_0,\varphi)\| \le K e^{\alpha(t-t_0)} \max_{t_{-1}(t_0)\le s\le t_0} \|\varphi(s)\|, \qquad t \ge t_0 \ge 0.
$$

Therefore

$$
m_0(\varphi) = K \max_{t_{-1}(t_0)\le s\le t_0} \|\varphi(s)\|,
$$

and so

$$
m_2(t_0,\varphi) \le \hat{m}_2(t_0) \max_{t_{-1}(t_0)\le s\le t_0} \|\varphi(s)\|, \qquad t_0 \ge S, \quad \varphi \in C([t_{-1}(t_0), t_0], \mathbb{R}^n), \tag{3.19}
$$

where

$$
\hat{m}_2(t_0) = \begin{cases} \frac{\max\left\{e^{\alpha_0(t_0 - t_{-1}(t_0))}, K\right\}}{1-A}, & \alpha_0 > 0, \\ \frac{K}{1-A}, & \alpha_0 \le 0. \end{cases}
$$

Now, consider an arbitrarily fixed $\tilde{t}_0 \in [0,S)$ and an initial function $\tilde{\varphi} \in C([t_{-1}(\tilde{t}_0), \tilde{t}_0], \mathbb{R}^n)$. Lemma 3.5 yields there exists a constant $m_3 \ge 1$ such that

$$
\|x(t;\tilde{t}_0,\tilde{\varphi})\| \le m_3 \max_{t_{-1}(\tilde{t}_0)\le s\le \tilde{t}_0} \|\tilde{\varphi}(s)\|, \qquad t_0 \le t \le S. \tag{3.20}
$$

On the other hand, the solution $x(\cdot;\tilde{t}_0,\tilde{\varphi})$ of Eq. (3.9) is unique on $[t_0,\infty)$ and

$$
x(t;\tilde{t}_0,\tilde{\varphi}) = x(t;S,\varphi), \qquad t \ge S,
$$

where $\varphi(s) = x(s; \tilde{t}_0, \tilde{\varphi})$, $t_{-1}(S) \le s \le S$. Consequently, (3.18), (3.19) and (3.20) yield for $t \ge S$

$$\begin{aligned}\|x(t;\tilde{t}_0,\tilde{\varphi})\| &= \|x(t;S,\varphi)\| \\ &\le \hat{m}_2(S)e^{\alpha_0(t-S)}(t-t_{-1}(S)+1)^k \max_{t_{-1}(S)\le s\le S}\|\varphi(s)\| \\ &\le \hat{m}_2(S)e^{\alpha_0(\tilde{t}_0-S)}e^{\alpha_0(t-\tilde{t}_0)}m_3(t-t_{-1}(\tilde{t}_0)+1)^k \max_{t_{-1}(\tilde{t}_0)\le s\le\tilde{t}_0}\|\tilde{\varphi}(s)\|.\end{aligned}$$

Therefore (3.15) is satisfied with

$$M(t_0) = \begin{cases} \hat{m}_2(S)m_3e^{\alpha_0(t_0-S)}, & 0 \le t_0 < S, \\ \hat{m}_2(t_0), & t_0 \ge S. \end{cases}$$

The rest of the proof of this theorem is an easy consequence of Theorem 2.6 and the ideas used above, therefore it is omitted. □

We get immediately from the proof:

Remark 3.7 *If $\alpha_0 \le 0$ or $t_0 - t_{-1}(t_0)$ is bounded for $t_0 \ge 0$, then M in (3.16) is independent of t_0.*

Now, we give the following definition of stability:

Definition 3.8 *We say that the zero solution of Eq. (3.9) is equistable on $\mathbb{R}_+$ if for all $t_0 \ge 0$ and $\varphi \in C([t_{-1}(t_0), t_0], \mathbb{R}^n)$ the solution $x(\cdot; t_0, \varphi)$ of Eq. (3.9) through (t_0, φ) is bounded on $[t_0, \infty)$. If for all $t_0 \ge 0$ and $\varphi \in C([t_{-1}(t_0), t_0], \mathbb{R}^n)$ the solution $x(\cdot; t_0, \varphi)$ tends to zero as $t \to +\infty$, then we say that the zero solution of Eq. (3.9) is equi-asymptotically stable.*

The next stability result is an easy consequence of Theorem 3.6, therefore we state it without proof.

Theorem 3.9 *Assume (C). If (3.14) holds, then*

(i) The zero solution of Eq. (3.9) is equistable on $\mathbb{R}_+$ if and only if either $\alpha_0 < 0$ or $\alpha_0 = 0$ and $k = 0$.
(ii) The zero solution of Eq. (3.9) is equi-asymptotically stable if and only if $\alpha_0 < 0$.

We close this subsection with the following corollary of Theorem 3.6, which shows the importance of the factor $e^{-\alpha_0\sigma_j(t)}$ in condition (3.14). We apply our results to the pantograph equation (see, e.g., [19]).

Corollary 3.10 *Consider the delay differential equation*

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - \tau_i) + \sum_{j=0}^{M} \widetilde{B}_j x(\gamma_j t) \tag{3.21}$$

where $\tau_i \in \mathbb{R}_+$, $A_i \in \mathbb{R}^{n\times n}$, $0 \le i \le n$, *and* $\gamma_j \in (0,1)$, $\widetilde{B}_j \in \mathbb{R}^{n\times n}$, $0 \le j \le M$.

If (C) holds and $\alpha_0 > 0$, *then the statements of Theorem 3.6 are valid for any solution* $x(\cdot; t_0, \varphi)$ *of Eq. (3.21) through* (t_0, φ) *where* $t_0 \ge 0$, $t_{-1}(t_0) = \min\{t_0 - \tau, (\min_{0\le j\le M} \gamma_j)t_0\}$, *and* $\varphi \in C([t_{-1}(t_0), t_0], \mathbb{R}^n)$.

Proof. Let $\sigma_j \colon \mathbb{R}_+ \to \mathbb{R}_+$ and $B_j \colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be defined by

$$\sigma_j(t) = (1 - \gamma_j)t \qquad \text{and} \qquad B_j(t) \equiv \widetilde{B}_j, \qquad t \ge 0,\ 0 \le j \le M,$$

and $\zeta(t) = \min\{\gamma_j t \colon j = 0, \ldots, M\}$. Then

$$\int_0^\infty t^k \sum_{j=0}^{M} \|B_j(t)\| e^{-\alpha_0 \sigma_j(t)}\, dt = \sum_{j=0}^{M} \|\widetilde{B}_j\| \int_0^\infty t^k e^{-\alpha_0 (1-\gamma_j)t}\, dt < \infty,$$

and hence by Theorem 3.6 the statement of the corollary follows. □

Remark 3.11 *Corollary 3.10 shows that if the zero solution of Eq. (3.11) is unstable then the zero solution of Eq. (3.21) is also unstable.*

3.3 The sunflower equation

In this section we consider the second-order delay differential equation

$$\ddot{x}(t) + A\dot{x}(t) + Bg(x(t - r)) = 0, \qquad t \ge 0, \tag{3.22}$$

where

$$A, B \in \mathbb{R}, \quad r \in \mathbb{R}_+, \quad g \in C(\mathbb{R}, \mathbb{R}) \quad \text{is odd satisfying} \quad g(u) > 0 \quad \text{for} \quad u > 0. \tag{3.23}$$

When $g(x) = \sin x$, Eq. (3.22) is the so-called sunflower equation, which was investigated extensively (see, e.g., [20], [21] and [27] and the references therein).

Consider an associated linear equation

$$\ddot{y}(t) + A\dot{y}(t) + By(t - r) = 0, \qquad t \ge 0, \tag{3.24}$$

and its characteristic equation

$$\lambda^2 + A\lambda + Be^{-\lambda r} = 0. \tag{3.25}$$

Theorem 3.12 *Assume (3.23), let $\lambda_0 = \alpha_0 + \beta_0 i$ be a simple dominant eigenvalue of Eq. (3.25) and*

$$\alpha_0 > 0 \qquad \text{and} \qquad \int_1^\infty \frac{1}{u^2} \max_{0 \le |s| \le u} |g(s) - s|\, du < \infty. \tag{3.26}$$

Then for all $\varphi \in C([-r,0],\mathbb{R})$ the solution $x(\cdot;0,\varphi)$ of Eq. (3.24) through $(0,\varphi)$ satisfies

$$x(t;0,\varphi) = e^{\alpha_0 t}\Big(d_1(\varphi)\cos(\beta_0 t + \gamma_1(\varphi)) + o(1)\Big), \qquad t \to +\infty, \tag{3.27}$$

where $d_1(\varphi)$, $\gamma_1(\varphi) \in \mathbb{R}$ and $d_1(\varphi_0) \neq 0$ for some $\varphi_0 \in C([-r,0],\mathbb{R})$.

Proof. Set $t_0 = 0$, $t_{-1} = -r$, and

$$f(t,x(\cdot)) = -B\Big(g(x(t-r)) - x(t-r)\Big), \qquad (t,x) \in [t_0,\infty) \times C([t_{-1},\infty),\mathbb{R}).$$

Moreover, for all $\varphi \in C([t_{-1},t_0],\mathbb{R})$ the solution $x(\cdot;0,\varphi)$ of Eq. (3.24) through $(0,\varphi)$ is the unique solution of

$$\ddot{x}(t) + A\dot{x}(t) + Bx(t-r) = f(t,x(\cdot)), \qquad t \ge 0.$$

Thus by the variation of constants formula we find

$$x(t;0,\varphi) = y(t;0,\varphi) + \int_0^t T(t-s) f(s,x(\cdot;0,\varphi))\, ds, \qquad t \ge 0, \tag{3.28}$$

where $y(\cdot;0,\varphi)$ is the solution of Eq. (3.24) through $(0,\varphi)$ and $T(t)$ is the unique function satisfying

$$\ddot{T}(t) + A\dot{T}(t) + BT(t-r) = 0, \qquad \text{a.e.} \quad t \ge 0,$$

and

$$\dot{T}(0) = 1 \qquad \text{and} \qquad T(s) = 0, \quad -r \le s \le 0.$$

Since $\lambda_0 = \alpha_0 + \beta_0 i$ is a simple dominant eigenvalue of Eq. (3.25), from the series representation of the solutions of Eq. (3.24) (see Proposition 3.4), we have

$$y(t;0,\varphi) = e^{\alpha_0 t}\Big(d_0(\varphi)\cos\beta_0 t + e_0(\varphi)\sin\beta_0 t + o(1)\Big), \qquad t \to +\infty,$$

where $d_0, e_0 \colon C([-r,0],\mathbb{R}) \to \mathbb{R}$ are such that $|d_0(\varphi_0)| + |e_0(\varphi_0)| \neq 0$ for some $\varphi_0 \in C([-r,0],\mathbb{R})$, and

$$T(t) = e^{\alpha_0 t}\Big(P\cos\beta_0 t + Q\sin\beta_0 t + o(1)\Big), \qquad t \to +\infty,$$

where $P, Q \in \mathbb{R}$ and $|P| + |Q| \neq 0$.

In that case, it can be easily seen that Eq. (3.21) satisfies all of the conditions of Theorem 2.7, whenever $b_{\alpha_0}(u) = \max_{0 \leq |s| \leq u} |g(s) - s|$ for all $u \geq 0$. Therefore in virtue of Theorem 2.7, we obtain the following representation of $x(\cdot; 0, \varphi)$:

$$x(t; 0, \varphi) = e^{\alpha_0 t}\Big(d_{\alpha_0}(\varphi) \cos \beta_0 t + e_{\alpha_0}(\varphi) \sin \beta_0 t + o(1)\Big), \qquad t \to +\infty, \tag{3.29}$$

where $d_{\alpha_0}, e_{\alpha_0} \colon C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are such that $|d_{\alpha_0}(\varphi_0)| + |e_{\alpha_0}(\varphi_0)| \neq 0$, for some $\varphi_0 \in C([-r, 0], \mathbb{R})$.

Let

$$d_1(\varphi) = \sqrt{(d_{\alpha_0}(\varphi))^2 + (e_{\alpha_0}(\varphi))^2}$$

and $\gamma_1(\varphi)$ be such that

$$d_1(\varphi) \cos \gamma_1(\varphi) = d_{\alpha_0}(\varphi) \qquad \text{and} \qquad - d_1(\varphi) \sin \gamma_1(\varphi) = e_{\alpha_0}(\varphi).$$

Then $d_1(\varphi) \neq 0$ and (3.29) yields (3.27), and this completes the proof of the theorem. $\square$

Similarly to Theorem 3.12, Theorem 2.8 yields the next result.

Theorem 3.13 *Assume (3.23), let $\lambda_0 = \alpha_0 + \beta_0 i$ be a simple dominant eigenvalue of Eq. (3.25) and*

$$\alpha_0 < 0 \qquad \text{and} \qquad \int_0^1 \frac{1}{u^2} \max_{0 \leq |s| \leq u} |g(s) - s| \, du < \infty. \tag{3.30}$$

Then there exists a $\delta = \delta(\alpha_0) > 0$, such that for all

$$\varphi \in C([-r, 0], \mathbb{R}), \qquad \|\varphi\|_0 = \max_{-r \leq s \leq 0} |\varphi(s)| < \delta,$$

the solution $x(\cdot; 0, \varphi)$ of Eq. (3.24) through $(0, \varphi)$ satisfies (3.27), where $d_1(\varphi)$, $\gamma_1(\varphi) \in \mathbb{R}$ and $d_1(\varphi_0) \neq 0$ for some $\varphi_0 \in C([-r, 0], \mathbb{R})$, and $\|\varphi\|_0 < \delta$.

When $g(x) = \sin x$, i.e., Eq. (3.24) is the so-called sunflower equation then

$$\int_0^1 \frac{1}{u^2} \max_{0 \leq |s| \leq u} |g(s) - s| \, du = \int_0^1 \frac{1}{u^2} \max_{0 \leq |s| \leq u} |\sin s - s| \, du < \infty,$$

and therefore Theorem 3.13 is applicable.

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