# On the Exponential Stability of a Nonlinear State-Dependent Delay System

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*Dedicated to Professor V. Lakshmikantham on the occasion of his 85th birthday.*

**Abstract**

In this paper we study exponential stability of solutions of a class of nonlinear differential equations including differential equations with state-dependent delays by means of linearization.

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## 1 Introduction

In this paper we consider the nonlinear functional differential equations of the form

$$\dot{x}(t) = f(t, x_t), \qquad t \geq t_0, \tag{1.1}$$

where $r > 0$ is fixed, and the solution segment function $x_t : [-r, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$. We assume that $x = 0$ is an equilibrium of the equation. This general class of equations includes differential equations with state-dependent delays (SD-DDEs), e.g., equations of the form

$$\dot{x}(t) = h(t, x(t), x(t - \tau(t, x_t))), \qquad t \geq t_0, \tag{1.2}$$

or more general classes of SD-DDEs. We refer to [11] for a survey on basic theory and applications of SD-DDEs.

One of the most frequently used qualitative technique in applications is the linearized stability principle. It has been formulated in many papers for different classes of SD-DDEs

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([1], [3], [4], [7], [8], [9], [10], [12]). The main technical difficulty to prove a linearized stability theorem in SD-DDEs is that the map $C \ni \psi \mapsto h(t, \psi(0), \psi(-\tau(t,\psi))) \in \mathbb{R}^n$ is not Fréchet-differentiable. See [11, 13] for more details and discussions on this topic.

In this paper we formulate a new sufficient condition for exponential stability for a large class of nonlinear functional differential equations assuming exponential stability of an associated a linear delay equation. The idea of the proof uses the fact that the solution of (1.1) is continuously differentiable for $t > t_0 + r$ under mild assumptions and a careful useage of the variation-of-constants formula. These tricks make the proof much simpler than the proofs of the existing linearization results for SD-DDEs.

In Section 2 we formulate our main result (see Theorem 2.3 below), and on a simplified version of (1.2) we demonstrate how easy to apply our linearization method. We present the technique to obtain exponential stability of the trivial solution, and also exponential stability of an arbitrary (e.g., periodic) solution of the equation. Note that a linearized stability theorem for periodic SD-DDEs was given in [7], but only for the case when the examined solution is continuously differentiable. In our theorem here we do not need this strong assumption. Section 3 contains the proofs of our general linearized stability theorem.

Note that a necessary and sufficient condition was formulated in [5] using a linearization method for a special class of (1.2). It is an interesting open question whether the statement in Theorem 2.3 can be reversed, possibly under more rectrictive conditions.

## 2 Main Results

Throughout this paper a fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ is denoted by $|\cdot|$. The Banach space of continuous functions $\psi\colon [-r,0] \to \mathbb{R}^n$ equipped with the norm $\|\psi\| = \sup\{|\psi(s)|\colon s \in [-r,0]\}$ is denoted by $C$. The ball in $C$ centered at 0 with radious $\rho$ is denoted by $\mathcal{B}_C(\rho)$. The Banach space of bounded linear operators mapping $C$ to $\mathbb{R}^n$ is denoted by $\mathcal{L}(C,\mathbb{R}^n)$.

Consider the delay system

$$\dot{x}(t) = f(t, x_t), \qquad t \geq t_0. \tag{2.1}$$

and the corresponding initial condition

$$x_{t_0} = \varphi, \qquad \varphi \in C, \tag{2.2}$$

where $t_0 \in \mathbb{R}$ is fixed.

We assume

(H1) $f\colon [t_0,\infty)\times C \to \mathbb{R}^n$ is continuous, and there exist $\delta_1 = \delta_1(t_0) > 0$ and $M_1 = M_1(t_0) > 0$ such that

$$|f(t,\varphi)| \leq M_1\|\varphi\|, \qquad \varphi \in \mathcal{B}_C(\delta_1)\,, \quad t \geq t_0.$$

(H2) There exists a mapping $L\colon [t_0,\infty) \to \mathcal{L}(C,\mathbb{R}^n)$ satisfying

(i) the linear operator $L(t)$ is uniformly bounded in time, i.e., $|L(t)\psi| \leq M_2\|\psi\|$ for any $t \geq t_0$ and $\psi \in C$, where $M_2 = M_2(t_0) \geq 0$ is independent of $\psi$ and $t$;

(ii) there are two continuous and monotone nondecreasing functions $\omega_1, \omega_2\colon [0,\delta_1) \to [0,\infty)$ for which $\omega_1(0) = \omega_2(0) = 0$, and

$$|f(t,\psi) - L(t)\psi| \le \|\psi\|\omega_1(\|\psi\|) + \|\dot\psi\|\omega_2(\|\psi\|)$$

for $t \ge t_0 + r$ and $\psi \in C^1 \cap \mathcal{B}_C(\delta_1)$.

Note that (H1) yields the existence, but not the uniqueness of the solutions of the IVP (2.1)-(2.2) (see, e.g., [2], [9], [11]). Any fixed solution of (2.1)-(2.2) will be denoted by $x(t; t_0, \varphi)$.

We consider the time-dependent linear equation

$$\dot y(t) = L(t) y_t, \qquad t \ge t_0. \tag{2.3}$$

The solution of (2.3) corresponding to initial condition (2.2) is denoted by $y(t; t_0, \varphi)$.

**Definition 2.1** *We say that the trivial (zero) solution of the equation* (2.1) *is exponentially stable on* $[t_0,\infty)$, *if there exist constants* $\delta = \delta(t_0) > 0$, $K_1 = K_1(t_0) \ge 1$ *and* $\alpha_1 = \alpha_1(t_0) > 0$ *such that for any* $t_0 \ge 0$

$$|x(t; t_0, \varphi)| \le K_1 e^{-\alpha_1(t-t_0)}\|\varphi\|, \qquad t \ge t_0, \quad \varphi \in \mathcal{B}_C(\delta)\,. \tag{2.4}$$

**Definition 2.2** *We say that the trivial (zero) solution of the linear equation* (2.3) *is uniformly exponentially stable on* $[t_0,\infty)$, *if there exist constants* $K_2 = K_2(t_0) \ge 1$ *and* $\alpha_2 = \alpha_2(t_0) > 0$ *such that for any* $s \ge t_0$

$$|y(t; s, \varphi)| \le K_2 e^{-\alpha_2(t-s)}\|\varphi\|, \qquad t \ge s, \quad \varphi \in C. \tag{2.5}$$

Now we can formulate the main result of this paper.

**Theorem 2.3** *Assume (H1) and (H2), moreover, the zero solution of* (2.3) *is uniformly exponentially stable on* $[t_0,\infty)$. *Then the zero solution of* (2.1) *is exponentially stable on* $[t_0,\infty)$, *as well.*

Next consider the scalar equation with state-dependent delay

$$\dot x(t) = a(t) g(x(t - \tau(t, x_t))), \qquad t \ge t_0. \tag{2.6}$$

On this simple nonlinear equation we show the applicability of our main theorem. We assume

(A1) $a\colon [t_0,\infty) \to \mathbb{R}$ is continuous and there exists $a_0$ such that $|a(t)| \le a_0$ for $t \ge t_0$;

(A2) $g\colon (-\sigma,\sigma) \to \mathbb{R}$ is continuously differentiable, $g(0) = 0$;

(A3) $\tau : [0,\infty) \times C \to [0,r]$ is continuous, and there exists a continuous and monotone nonincreasing function $\omega_\tau\colon (-\sigma,\sigma) \to [0,\infty)$ such that $|\tau(t,\psi) - \tau(t,\mathbf{0})| \le \omega_\tau(\|\psi\|)$ for $\psi \in \mathcal{B}_C(\sigma)$, $t \ge t_0$.

Now (A1) and (A2) yield (H1) with $f(t,\psi) = a(t)g(\psi(-\tau(t,\psi)))$. Consider the time-dependent linear operator defined by

$$L(t)\psi = a(t)g'(0)\psi(-\tau(t,\mathbf{0})), \tag{2.7}$$

where $\mathbf{0}$ is the constant 0 function in $C$. Then (A1) and (A2) imply (H2) (i). To show (H2) (ii), let $\psi \in C^1 \cap \mathcal{B}_C(\sigma)$. Simple estimates, assumption (A3) and the Mean Value Theorem yield

$$\begin{aligned}
|f(t,\psi) - L(t)\psi| &= |a(t)g(\psi(-\tau(t,\psi))) - a(t)g'(0)\psi(-\tau(t,\mathbf{0}))| \\
&\le |a(t)||g(\psi(-\tau(t,\psi))) - g'(0)\psi(-\tau(t,\psi))| \\
&\qquad + |a(t)||g'(0)||\psi(-\tau(t,\psi)) - \psi(-\tau(t,\mathbf{0}))| \\
&\le a_0|\psi(-\tau(t,\psi))|\omega_g(|\psi(-\tau(t,\psi))|) + a_0|g'(0)|\|\dot\psi\||\tau(t,\psi) - \tau(t,\mathbf{0})| \\
&\le a_0\|\psi\|\omega_g(\|\psi\|) + a_0|g'(0)|\|\dot\psi\|\omega_\tau(\|\psi\|),
\end{aligned}$$

where

$$\omega_g(u) = \begin{cases} \sup\limits_{|s|\le u} \dfrac{|g(s) - g'(0)s|}{|s|}, & u > 0, \\ 0, & u = 0. \end{cases}$$

All conditions of Theorem 2.3 are satisfied, therefore we get immediately the next result.

**Theorem 2.4** *Assume (A1)–(A3), moreover, the trivial solution of*

$$\dot y(t) = a(t)g'(0)y(t - \tau(t,\mathbf{0})), \qquad t \ge t_0$$

*is uniformly exponentially stable on* $[t_0,\infty)$. *Then the trivial solution of* (2.6) *is exponentially stable, as well.*

Now suppose $\bar x\colon [t_0 - r,\infty) \to \mathbb{R}$ is a fixed solution of (2.6). Next we study the exponential stability of this solution. Consider the new variable $z(t) = x(t) - \bar x(t)$. It satisfies

$$\dot z(t) = a(t)g\Big(z(t - \tau(t, z_t + \bar x_t)) + \bar x(t - \tau(t, z_t + \bar x_t))\Big) - a(t)g(\bar x(t - \tau(t,\bar x_t))) \tag{2.8}$$

In order to show the exponential stability of solution $\bar x$ of (2.6), we apply our Theorem 2.3 to show that the trivial solution of (2.8) is exponentially stable. Let

$$f(t,\psi) = a(t)\Big[g\Big(\psi(-\tau(t,\psi + \bar x_t)) + \bar x(t - \tau(t,\psi + \bar x_t))\Big) - g(\bar x(t - \tau(t,\bar x_t)))\Big],$$

and we define the time-dependent linear operator

$$L(t)\psi = a(t)g'(\bar x(t - \tau(t,\bar x_t)))\psi(-\tau(t,\bar x_t)), \qquad t \ge t_0, \quad \psi \in C. \tag{2.9}$$

We assume $\bar x\colon [t_0 - r,\infty) \to \mathbb{R}$ is a bounded solution of (2.6), i.e., there exists $b_0 \ge 0$ such that $|\bar x(t)| \le b_0$ for $t \ge t_0 - r$. We need stronger versions of (A2) and (A3):

(A2') $g\colon (-\sigma,\sigma)\to\mathbb{R}$ is twice continuously differentiable, where $b_0<\sigma$, and $g(0)=0$;

(A3') $\tau\colon [t_0,\infty)\times C\to[0,r]$ is continuous, and also Lipschitz-continuous in its second variable, i.e., there exists $N_1>0$ such that $|\tau(t,\psi)-\tau(t,\tilde\psi)|\le N_1\|\psi-\tilde\psi\|$ for $\psi,\tilde\psi\in\mathcal{B}_C(\sigma)$, $t\ge t_0$.

Let $b_1$ be such that $b_0<b_1<\sigma$, and define $N_2=\max\{|g'(u)|:\ u\in[-b_1,b_1]\}$ and $N_3=\max\{|g''(u)|\colon u\in[-b_1,b_1]\}$. Then

$$|g(u)-g(s)|\le N_2|u-s|\quad\text{and}\quad |g(u)-g(s)-g'(s)(u-s)|\le N_3(u-s)^2$$

for $u,s\in[-b_1,b_1]$. Let $\varepsilon=b_1-b_0$. It follows from (2.6)

$$|\dot{\bar x}(t)|=|a(t)||g(\bar x(t-\tau(t,\bar x_t)))-g(0)|\le a_0N_2|\bar x(t-\tau(t,\bar x_t))|\le a_0N_2b_0,\qquad t\ge t_0,$$

therefore

$$|\bar x(u)-\bar x(s)|\le N_4|u-s|,\qquad u,s\ge t_0,$$

where $N_4=a_0N_2b_0$.

Now we can show that (H1) and (H2) are satisfied for this example. (H1) follows from the estimates

$$\begin{aligned}|f(t,\psi)| &\le a_0N_2|\psi(-\tau(t,\psi+\bar x_t))+\bar x(t-\tau(t,\psi+\bar x_t))-\bar x(t-\tau(t,\bar x_t))|\\ &\le a_0N_2(1+N_1N_4)\|\psi\|,\qquad t\ge t_0,\quad \psi\in\mathcal{B}_C(\varepsilon)\,.\end{aligned}$$

(H2) (i) can be shown easily. To prove (H2) (ii) consider

$$\begin{aligned}|f(t,\psi)-L(t)\psi| &\le |a(t)|\Big|g\Big(\psi(-\tau(t,\psi+\bar x_t))+\bar x(t-\tau(t,\psi+\bar x_t))\Big)-g\Big(\bar x(t-\tau(t,\bar x_t))\Big)\\ &\qquad -g'(\bar x(t-\tau(t,\bar x_t)))\psi(-\tau(t,\psi+\bar x_t))\Big|\\ &\quad +|a(t)||g'(\bar x(t-\tau(t,\bar x_t))||\psi(-\tau(t,\psi+\bar x_t))-\psi(-\tau(t,\bar x_t))|\\ &\le a_0N_3(\psi(-\tau(t,\psi+\bar x_t))+\bar x(t-\tau(t,\psi+\bar x_t))-\bar x(t-\tau(t,\bar x_t)))^2\\ &\quad +a_0N_2\|\dot\psi\||\tau(t,\psi+\bar x_t)-\tau(t,\bar x_t)|\\ &\le a_0N_3(1+N_1N_4)^2\|\psi\|^2+a_0N_1N_2\|\dot\psi\|\|\psi\|,\qquad t\ge t_0,\ \psi\in\mathcal{B}_C(\varepsilon)\cap C^1.\end{aligned}$$

Now the following result is the consequence of Theorem 2.3.

**Theorem 2.5** *Assume (A1), (A2'), (A3'), and let $\bar x=\bar x(\cdot;t_0,\bar\varphi):\ [t_0-r,\infty)\to\mathbb{R}$ be a bounded solution of* (2.6)*. Then if the trivial solution of*

$$\dot y(t)=a(t)g'(\bar x(t-\tau(t,\bar x_t)))y(t-\tau(t,\bar x_t)),\qquad t\ge t_0$$

*is uniformly exponentially stable on $[t_0,\infty)$, then $\bar x$ is an exponentially stable solution of* (2.6) *on $[t_0,\infty)$, i.e., there exist constants $\delta=\delta(t_0)>0$, $K_1=K_1(t_0)\ge1$ and $\alpha_1=\alpha_1(t_0)>0$ such that*

$$|x(t;t_0,\varphi)-\bar x(t;t_0,\bar\varphi)|\le K_1e^{-\alpha_1(t-t_0)}\|\varphi-\bar\varphi\|,\qquad t\ge t_0,\quad \|\varphi-\bar\varphi\|<\delta,\quad \varphi\in C.$$

## 3 Proof of Theorem 2.3

**Lemma 3.1** *Assume (H1). For any initial function* $\varphi \in \mathcal{B}_C\big(e^{-M_1 r}\delta_1\big)$ *the solution* $x(t;t_0,\varphi)$ *of the IVP* (2.1)-(2.2) *satisfies*

$$|x(t;t_0,\varphi)| \le e^{M_1 r}\|\varphi\| < \delta_1, \qquad t_0 \le t \le t_0 + r. \tag{3.1}$$

**Proof** Since $\|\varphi\| \le e^{-M_1 r}\delta_1 < \delta_1$, it follows $|x(t_0;t_0,\varphi)| < \delta_1$. Suppose there exists $t_1 \in (t_0, t_0+r)$ such that

$$|x(t;t_0,\varphi)| < e^{M_1 r}\|\varphi\|, \quad t \in [t_0,t_1), \quad \text{and} \quad |x(t_1;t_0,\varphi)| = e^{M_1 r}\|\varphi\|.$$

Integrating (2.1) we get

$$\begin{aligned} |x(t;t_0,\varphi)| &\le |\varphi(0)| + \int_{t_0}^{t} |f(s, x_s(\cdot;t_0,\varphi))|\, ds \\ &\le \|\varphi\| + M_1 \int_{t_0}^{t} \|x_s(\cdot;t_0,\varphi)\|\, ds \\ &\le \|\varphi\| + M_1 \int_{t_0}^{t} \max_{t_0 - r \le u \le s} |x(u;t_0,\varphi)|\, ds, \qquad t_0 \le t \le t_1. \end{aligned} \tag{3.2}$$

Define the function $z(t) = \max_{t_0 - r \le u \le t} |x(u;t_0,\varphi)|$. The monotonicity of the right-hand-side of (3.2) in $t$ and $z(0) \le \|\varphi\|$ imply that the function $z$ satisfies

$$z(t) \le \|\varphi\| + M_1 \int_{t_0}^{t} z(s)\, ds, \qquad t_0 \le t \le t_1.$$

Thus Gronwall's inequility yields

$$z(t) \le e^{M_1(t-t_0)}\|\varphi\|, \qquad t_0 \le t \le t_1,$$

and hence

$$|x(t_1;t_0,\varphi)| \le z(t_1) \le e^{M_1(t_1-t_0)}\|\varphi\| < e^{M_1 r}\|\varphi\|.$$

This contradicts to the definition of $t_1$, therefore (3.1) holds. □

Similarly to the proof of Lemma 3.1 one can prove the following estimate for the solutions of the linear equation (2.3).

**Lemma 3.2** *Assume (H2) (i). For any initial function* $\varphi \in C$ *the solution* $y(t;t_0,\varphi)$ *of the IVP* (2.3)-(2.2) *satisfies*

$$|y(t;t_0,\varphi)| \le e^{M_2 r}\|\varphi\|, \qquad t \ge t_0.$$

We define the fundamental solution of (2.3) as the $n \times n$ matrix solution of the IVP

$$\frac{\partial}{\partial t}V(t,s) = L(t)V_t(\cdot,s), \qquad t \ge s \ge t_0, \tag{3.3}$$

$$V(t,s) = \begin{cases} I, & t = s, \\ 0 & t < s. \end{cases} \tag{3.4}$$

Here $I$ and 0 denote the identity and the zero matrices, respectively.

If the trivial solution of (2.3) is uniformly exponentially stable on $[t_0, \infty)$ with exponent $\alpha_2$, then it is known (see, e.g., [6]), that there exists $K_3 = K_3(t_0) \geq 1$ such that

$$|V(t,s)| \leq K_3 e^{-\alpha_2(t-s)}, \qquad t \geq s \geq t_0. \tag{3.5}$$

Suppose $\varphi \in C$ is such that the solution $x(t; t_0, \varphi)$ of the IVP (2.1)-(2.2) exists on $[t_0, T)$ for some $T > t_0 + r$. We can rewrite equation (2.1) as

$$\dot{x}(t; t_0, \varphi) = L(t)x_t(\cdot; t_0, \varphi) + f(t, x_t(\cdot; t_0, \varphi)) - L(t)x_t(\cdot; t_0, \varphi), \qquad t \geq t_0 + r,$$

therefore the variation-of-constants formula (see, e.g., [6]) yields

$$\begin{aligned} x(t; t_0, \varphi) &= y(t; t_0 + r, x_{t_0+r}(\cdot; t_0, \varphi)) \\ &\quad + \int_{t_0+r}^{t} V(t,s)\Big(f(s, x_s(\cdot; t_0, \varphi)) - L(s)x_s(\cdot; t_0, \varphi)\Big)\, ds, \quad t_0 + r \leq t < T. \end{aligned} \tag{3.6}$$

Let $\delta_2 = e^{-M_1 r}\delta_1$, and suppose $\varphi \in \mathcal{B}_C(\delta_2)$. Then Lemma 3.1 yields that $|x(t; t_0, \varphi)| < \delta_1$ for $t \in [t_0, t_0 + r]$. Therefore $|x(t; t_0, \varphi)| < \delta_1$ for $t \in [t_0 - r, T)$ for some $T > t_0 + r$.

It follows from (2.5) and (3.1) for $t \geq t_0 + r$

$$|y(t; t_0 + r, x_{t_0+r}(\cdot; t_0, \varphi))| \leq K_2 e^{-\alpha_2(t-t_0-r)}\|x_{t_0+r}(\cdot; t_0, \varphi)\| \leq c_1 e^{-\alpha_2(t-t_0)}\|\varphi\|, \tag{3.7}$$

where $c_1 = K_2 e^{\alpha_2 r} e^{M_1 r}$. Note that $c_1 \geq 1$. Since $x_s(\cdot; t_0, \varphi) \in C^1$ for $s \geq t_0 + r$, assumption (H2) (ii) yields

$$\begin{aligned} |f(s, x_s(\cdot; t_0, \varphi)) - L(s)x_s(\cdot; t_0, \varphi)| &\leq \|x_s(\cdot; t_0, \varphi)\|\omega_1(\|x_s(\cdot; t_0, \varphi)\|) \\ &\quad + \|\dot{x}_s(\cdot; t_0, \varphi)\|\omega_2(\|x_s(\cdot; t_0, \varphi)\|). \end{aligned}$$

For $s \in [t_0, T)$ and $u \in [-r, 0]$ (H1) together with $\|x_{s+u}(\cdot; t_0, \varphi)\| < \delta_1$ implies

$$|\dot{x}(s+u; t_0, \varphi)| = |f(s+u, x_{s+u}(\cdot; t_0, \varphi))| \leq M_1\|x_{s+u}(\cdot; t_0, \varphi)\| \leq M_1 \max_{s-2r \leq u \leq s} |x(u; t_0, \varphi)|,$$

hence

$$|f(s, x_s(\cdot; t_0, \varphi)) - L(s)x_s(\cdot; t_0, \varphi)| \leq \max_{s-2r \leq u \leq s} |x(u; t_0, \varphi)|\omega(\|x_s(\cdot; t_0, \varphi)\|), \tag{3.8}$$

where $\omega(u) = \omega_1(u) + M_1\omega_2(u)$, $u \in [0, \delta_1)$.

It follows from (3.6) and the above estimates for $t \in [t_0 + r, T)$

$$|x(t; t_0, \varphi)| \leq c_1 e^{-\alpha_2(t-t_0)}\|\varphi\| + \int_{t_0+r}^{t} K_3 e^{-\alpha_2(t-s)} \max_{s-2r \leq u \leq s} |x(u; t_0, \varphi)|\omega(\|x_s(\cdot; t_0, \varphi)\|)\, ds. \tag{3.9}$$

Let $0 < \varepsilon_0 < \delta_1$ be such that $K_3\omega(\varepsilon_0) < \alpha_2$, and for any $0 < \varepsilon < \varepsilon_0$ let $\delta_3 = \delta_3(\varepsilon)$ be defined by

$$\delta_3 = \min\left\{\delta_2, \frac{\varepsilon(\alpha_2 - K_3\omega(\varepsilon))}{c_1\alpha_2}\right\}. \tag{3.10}$$

Fix any $\varphi \in \mathcal{B}_C(\delta_3)$, and consider the corresponding solution $x(t;t_0,\varphi)$. Since $|x(t_0;t_0,\varphi)| < \delta_3 < \varepsilon < \delta_1$, the constant $T_1 = \sup\{s \geq t_0\colon\ |x(u;t_0,\varphi)| < \varepsilon \text{ for } u \in [t_0,s)\}$ is well-defined and $T_1 > t_0$. Suppose $T_1$ is finite. Then $|x(T_1;t_0,\varphi)| = \varepsilon$, and (3.9) yields with $t = T_1$

$$\varepsilon \leq c_1 e^{-\alpha_2(T_1-t_0)}\|\varphi\| + \int_{t_0+r}^{T_1} K_3 e^{-\alpha_2(T_1-s)}\varepsilon\omega(\varepsilon)\,ds < c_1\|\varphi\| + \frac{K_3\varepsilon\omega(\varepsilon)}{\alpha_2} < c_1\delta_3 + \frac{K_3\varepsilon\omega(\varepsilon)}{\alpha_2} \leq \varepsilon,$$

which is a contradiction. Therefore $T_1 = \infty$, and consequently, $T = \infty$, as well.

Let $0 < \alpha_1 < \alpha_2$ be fixed, and $0 < \varepsilon_1 < \varepsilon_0$ be such that

$$\frac{K_3\omega(\varepsilon_1)}{\alpha_2 - \alpha_1} e^{2r\alpha_1} < \frac{1}{2},$$

and let $\delta_4 = \delta_3(\varepsilon_1)$ be defined by (3.10). Fix any $\varphi \in \mathcal{B}_C(\delta_4)$. Then $|x(t;t_0,\varphi)| < \varepsilon$ for $t \geq t_0 - r$, and multiplying (3.9) by $e^{\alpha_1(t-t_0)}$ yields for $t \geq t_0 + r$

$$\begin{aligned} e^{\alpha_1(t-t_0)}|x(t;t_0,\varphi)| &\leq c_1 e^{-(\alpha_2-\alpha_1)(t-t_0)}\|\varphi\| \\ &\quad + e^{\alpha_1(t-t_0)}\int_{t_0+r}^{t} K_3 e^{-\alpha_2(t-s)} \max_{s-2r\leq u\leq s}|x(u;t_0,\varphi)|\omega(\|x_s(\cdot;t_0,\varphi)\|)\,ds. \end{aligned}$$

Introduce the function $z(t) = e^{\alpha_1(t-t_0)}|x(t;t_0,\varphi)|$. Then

$$\begin{aligned} z(t) &\leq c_1\|\varphi\| + K_3\omega(\varepsilon_1)e^{-(\alpha_2-\alpha_1)t-\alpha_1 t_0}\int_{t_0+r}^{t} e^{\alpha_2 s}\max_{s-2r\leq u\leq s} e^{-\alpha_1(u-t_0)}z(u)\,ds \\ &\leq c_1\|\varphi\| + K_3\omega(\varepsilon_1)e^{-(\alpha_2-\alpha_1)t+2r\alpha_1}\int_{t_0+r}^{t} e^{(\alpha_2-\alpha_1)s}\max_{s-2r\leq u\leq s} z(u)\,ds \\ &\leq c_1\|\varphi\| + K_3\omega(\varepsilon_1)e^{-(\alpha_2-\alpha_1)t+2r\alpha_1}\max_{t_0-r\leq u\leq t} z(u)\int_{t_0+r}^{t} e^{(\alpha_2-\alpha_1)s}\,ds \\ &\leq c_1\|\varphi\| + \frac{K_3\omega(\varepsilon_1)}{\alpha_2-\alpha_1}e^{2r\alpha_1}\max_{t_0-r\leq u\leq t} z(u) \\ &\leq c_1\|\varphi\| + \frac{1}{2}\max_{t_0-r\leq u\leq t} z(u), \qquad t \geq t_0 + r. \end{aligned} \tag{3.11}$$

For $t \in [t_0 - r, t_0]$

$$z(t) = e^{\alpha_1(t-t_0)}|x(t;t_0,\varphi)| \leq |\varphi(t-t_0)| \leq \|\varphi\| \leq c_1\|\varphi\|,$$

and for $t \in [t_0, t_0 + r]$

$$z(t) = e^{\alpha_1(t-t_0)}|x(t;t_0,\varphi)| \leq e^{\alpha_1 r}e^{M_1 r}\|\varphi\| \leq c_1\|\varphi\|,$$

therefore (3.11) implies

$$\max_{t_0-r\leq u\leq t} z(u) \leq c_1\|\varphi\| + \frac{1}{2}\max_{t_0-r\leq u\leq t} z(u), \qquad t \geq t_0,$$

and hence

$$z(t) \leq \max_{t_0-r\leq u\leq t} z(u) \leq 2c_1\|\varphi\|, \qquad t \geq t_0.$$

Consequently,

$$|x(t;t_0,\varphi)| \le 2c_1 e^{-\alpha_1(t-t_0)}\|\varphi\|, \qquad t \ge t_0, \quad \varphi \in \mathcal{B}_C(\delta_4)\,,$$

which completes the proof of Theorem 2.3.

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