On Numerical Approximation using Differential Equations with Piecewise-Constant Arguments

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This paper is dedicated to the memory of Professor Miklós Farkas

Abstract

In this paper we give a brief overview of the application of delay differential equations with piecewise constant arguments (EPCAs) for obtaining numerical approximation of delay differential equations, and we show that this method can be used for numerical approximation in a class of age-dependent population models. We also formulate an open problem for a qualitative behaviour of a class of linear delay equations with continuous and piecewise constant arguments.

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1 Introduction

Delay differential equations (DDEs) provide a mathematical model for physical, biological systems in which the rate of change of the system depends upon their past history. The general theory of DDEs with continuous arguments by now has been thoroughly investigated, the number of the papers devoted to this area of research continues to grow very rapidly.

This paper is devoted to a generalized class of DDEs, namely delay differential equations with piecewise constant arguments (EPCAs). EPCAs include, as particular cases, impulsive DDEs and some equations of control theory, and are similar to those found in some biomedical models, hybrid control systems and numerical approximation of differential equations with discrete difference equations.

The general theory and basic results for DDEs can be found for instance in the book of Hale [16] (see also Bellman and Cooke [2], Hale and Lunel [17] and Myshkis [25]), and subsequent articles by many authors.

The study of EPCAs has been initiated by Wiener [32], [33], Cooke and Wiener [4], [5], Shah and Wiener [26]. A survey of the basic results has been given in [6], [34].

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A typical EPCA is of the form

$$\dot{x}(t) = f(t, x(h(t)), x(g(t))), \tag{1.1}$$

where the argument h is a continuous function and argument g has intervals of constancy. For example $g(t) = [t]$ or $g(t) = [t - n]$, where n is a positive integer, and $[\cdot]$ denotes the greatest-integer function. Note that if $f(t, u, v) \equiv f_1(t, u)$, then (1.1) is a classical DDE with continuous argument. If $f(t, u, v) \equiv f_2(t, v)$, and, for instance, $g(t) = [t]$, then the solution x of (1.1) is piecewise linear and at the points $t = 1, 2, \dots$ it is equal to the solution of the discrete equation

$$y(n+1) = y(n) + \int_{nh}^{(n+1)h} f_2(s, y(n))\, ds, \qquad n = 0, 1, \dots.$$

The above remark suggests that the numerical aproximation of differential equations can give rise to EPCAs in a natural way, as it has been initiated by [12].

In Section 2 we explain the use of EPCAs to get a convergent numerical approximation scheme for nonlinear delay differential equations with time- and state-dependent delays, based on our earlier results [13], [14].

In Section 3 we extend these results for certain partial differential equations. We construct a numerical approximation method for a class of nonlinear age-dependent population models, and we show the convergence of the numerical scheme.

In Section 4 we discuss a linear equation which is strongly related to the stabilization of hybrid systems with feedback delays, i.e., one with continuous plant and with discrete (sampled) controller. Since some of these systems may be described by an EPCA, we formulate an open problem for investigating stability of EPCAs at the end of the paper.

2 Numerical approximation of delay equations using EPCAs

EPCAs were first used for obtaining numerical approximation schemes for linear delay differential equations in [12]. This scheme was extended for nonlinear differential equations with time- and state-dependent delays in [14], and for neutral FDEs with time- and state-dependent delays in [19]. In this section we present the method for nolinear delay equations with time- and state-dependent delays of the form

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t, x(t)))), \qquad t \geq 0, \tag{2.1}$$

where we associate the initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.2}$$

First we introduce the simplifying notation

$$[t]_h = \left[\frac{t}{h}\right] h,$$

where $h > 0$ is a discretization parameter. The function $t \mapsto [t]_h$ is a piecewise-constant function; has jump discontinuities at the points $\{kh\colon\ k \in \mathbb{Z}\}$, where it is right-continuous; the domain of

$[t]_h$ is the whole real line, the range is the discrete values $\{kh\colon\ k \in \mathbb{Z}\}$; it satisfies $|t - [t]_h| \le h$, and therefore $\lim_{h\to 0+} [t]_h = t$ uniformly on the whole real line.

We associate the following EPCA to the initial value problem (2.1)-(2.2)

$$\begin{aligned} \dot{y}_h(t) &= f\Big([t]_h, y_h([t]_h), y_h\Big([t]_h - [\tau([t]_h, y_h([t]_h))]_h\Big)\Big), \quad t \ge 0 &(2.3)\\ y_h(t) &= \varphi(t), \qquad t \in [-r, 0]. &(2.4)\end{aligned}$$

The right-hand-side of (2.3) is constant on the intervals $[kh, (k+1)h)$, so the solution of (2.3)-(2.4) is a continuous function which is linear in between the mesh points $\{kh\colon\ k \in \mathbb{N}\}$.

We integrate both sides of (2.3) from kh to t, where $kh \le t < (k+1)h$:

$$\int_{kh}^{t} \dot{y}_h(s)\, ds = \int_{kh}^{t} f\Big([s]_h, y_h([s]_h), y_h\Big([s]_h - [\tau([s]_h, y_h([s]_h))]_h\Big)\Big)\, ds.$$

Using that the integrand on the right-hand-side is constant, we get

$$y_h(t) - y_h(kh) = f\Big(kh, y_h(kh), y_h\Big(kh - [\tau(kh, y_h(kh))]_h\Big)\Big)(t - kh).$$

So taking the limit $t \to (k+1)h-$ yields

$$y_h((k+1)h) - y_h(kh) = hf\Big(kh, y_h(kh), y_h\Big(kh - [\tau(kh, y_h(kh))]_h\Big)\Big).$$

Since y_h is linear between the mesh points, the values $a(k) = y_h(kh)$ uniquely determine the solution. The sequence $a(k)$ satisfies the difference equation

$$\begin{aligned} a(k+1) &= a(k) + f\Big(kh, a(k), a(k - d_k)\Big)\cdot h, \qquad k = 0, 1, 2, \dots,\\ a(-k) &= \varphi(-kh), \qquad k = 0, 1, 2, \dots, \quad -r \le -kh \le 0,\end{aligned}$$

where $d_k \equiv \Big[\tau(kh, a(k))/h\Big]$.

To obtain convergence of the approximation method we assume f is Lipschitz-continuous, i.e., there exists $L > 0$ such that $|f(t, u, v) - f(\tilde{t}, \tilde{u}, \tilde{v})| \le L_1(|t - \tilde{t}| + |u - \tilde{u}| + |v - \tilde{v}|)$, the delay function τ is Lipschitz-continuous, i.e., $|\tau(t, u) - \tau(\tilde{t}, \tilde{u})| \le L_2(|t - \tilde{t}| + |u - \tilde{u}|)$, and the initial function φ is also Lipschitz-continuous.

For the proof of the following theorem (under weaker conditions) we refer to [14].

Theorem 2.1 *If f, φ and τ are Lipschitz-continuous, then for any $T > 0$ there exists a constant $K > 0$ such that*

$$|x(t) - y_h(t)| \le Kh, \qquad t \in [0, T].$$

The above theorem yields that the solutions of (2.3) approximate the solution of (2.1) as $h \to 0+$, uniformly on compact time intervals. For a linear delay equation, it was proved in [3] that, if the trivial solution of the linear equation is asymptotically stable, then this convergence is uniform on the whole halfline $[0, \infty)$.

Theorem 2.2 *If the zero solution of*

$$\dot{x}(t) = \sum_{i=1}^{m} a_i x(t - \tau_i)$$

is asymptotically stable, then the solutions of

$$\dot{y}_h(t) = \sum_{i=1}^{m} a_i y_h \left([t]_h - [\tau_i]_h\right),$$

satisfy

$$\lim_{h \to 0+} \max_{t \geq 0} |x(t) - y_h(t)| = 0.$$

Note that a precise error estimate is given in [3]. This result was extended for linear neutral equations in [13].

An application of this type of approximation was given in [18], [20], [21] and [22], where numerical schemes were defined for the problem of estimating unknown parameters in several classes of differential equations. Note that these schemes were applicable and their theoretical convergences were proved for neutral equations (even with state-dependent delays), for which other approaches, like semigroup based approximation techniques, were not possible to extend.

3 Numerical approximation of nonlinear age-dependent population models

Let us denote by $u(t, a)$ the density of individuals in a population of age a at time t, where the variables t and a range in the domain $[0, T] \times [0, A]$, where A denotes the upper bound for the age of an individual. The change in the number of individuals — per unit age and time — of age a, which may be due to death or migration, at time t is denoted by $f(t, a, u(t, a))$. Thus the age-dependent population dynamics is described by the following partial differential equation

$$\frac{\partial u}{\partial t}(t, a) + \frac{\partial u}{\partial a}(t, a) = f(t, a, u(t, a)), \qquad t \in [0, T],\ a \in [0, A]. \tag{3.1}$$

This nonlinear equation includes a large class of population models, e.g., Gurtin-McCamy and McKendrick-von Foerster type models ([7], [9], [10], [32]).

We associate the initial condition

$$u(0, a) = \varphi(a), \qquad a \in [0, A], \tag{3.2}$$

where φ is the initial population distribution, and the boundary condition

$$u(t, 0) = \int_0^A \gamma(t, a) u(t, a)\, da, \qquad t \in [0, T], \tag{3.3}$$

which defines the number of newborns (births) in the population at time t.

We assume that the initial and boundary conditions are consistent at 0, i.e.,

$$\varphi(0) = \int_0^A \gamma(0,a)\varphi(a)\,da. \tag{3.4}$$

Since the original work of Sharpe and Lotka [27] and McKendrick [24] there has bee basically two equivalent approches to model the dynamics of the population: either to study the partial differential equation directly, or to rewrite it in an integral equation form. In this paper we follow this second approach and we define our approximation method through the integral equation.

By the method of characteristic lines it is easy to check (see, e.g., [11]) that the solution of (3.1) satisfies the integral equation

$$u(t,a) = \begin{cases} u(0,a-t) + \displaystyle\int_0^t f\Big(s, s+a-t, u(s,s+a-t)\Big)\,ds, & a \geq t, \\ u(t-a,0) + \displaystyle\int_0^a f\Big(s+t-a, s, u(s+t-a,s)\Big)\,ds, & a < t. \end{cases} \tag{3.5}$$

Combining this with the initial condition we get

$$u(t,a) = \begin{cases} \varphi(a-t) + \displaystyle\int_0^t f\Big(s, s+a-t, u(s,s+a-t)\Big)\,ds, & a \geq t, \\ u(t-a,0) + \displaystyle\int_0^a f\Big(s+t-a, s, u(s+t-a,s)\Big)\,ds, & a < t. \end{cases} \tag{3.6}$$

The solution of (3.6) satisfying the boundary condition (3.3) is called a weak solution of the initial boundary value problem (3.1)–(3.3). It is easy to see that if the solution u of (3.6) is differentiable, than it is the (strong) solution of (3.1).

Suppose $h > 0$ is a given discretization parameter, and consider the function v_h defined by the discretized version of (3.6):

$$v_h(t,a) = \begin{cases} \varphi([a]_h - [t]_h) + \displaystyle\int_0^t f\Big([s]_h, [s]_h + [a]_h - [t]_h, v_h([s]_h, [s]_h + [a]_h - [t]_h)\Big)ds, & a \geq t, \\ v_h([t]_h - [a]_h, 0) + \displaystyle\int_0^a f\Big([s]_h + [t]_h - [a]_h, [s]_h, v_h([s]_h + [t]_h - [a]_h, [s]_h)\Big)ds, & a < t, \end{cases} \tag{3.7}$$

where the associated discretized version of the boundary condition (3.3) is

$$v_h(t,0) = \int_0^{[A]_h} \gamma([t]_h, [s]_h) v_h([t]_h, [s]_h)\,ds. \tag{3.8}$$

Throughout this section we use the notations $N = [T/h]$, $M = [A/h]$, $w_{ij} = v_h(ih, jh)$, and $s_i = ih$. In these notations the dependence on h is omitted, but should be kept in mind.

We now show that w_{ij} is uniquely determined by (3.7) and (3.8) for $i = 0, \ldots, N$ and $j = 0, \ldots, M$. The first part of (3.7) yields

$$w_{0j} = \varphi(jh), \qquad j = 0, \ldots, M.$$

Then for $i = 1, \ldots, N$ we apply the first part of (3.7) with $t = s_i$, $a = s_j$, and we get

$$\begin{aligned} w_{ij} &= w_{0,j-i} + \sum_{k=0}^{i-1} \int_{kh}^{(k+1)h} f\Big([s]_h, [s]_h + [a]_h - [t]_h, v_h([s]_h, [s]_h + [a]_h - [t]_h)\Big)\,ds \\ &= w_{0,j-i} + h\sum_{k=0}^{i-1} f(s_k, s_{k+j-i}, w_{k,k+j-i}), \qquad j = i, i+1, \ldots, M. \end{aligned} \tag{3.9}$$

It follows from (3.8) with $t = s_i$ that

$$w_{i0} = h \sum_{k=0}^{M-1} \gamma(s_i, s_k) w_{ik},$$

and hence

$$w_{i0} = \frac{h}{1 - h\gamma(s_i, 0)} \sum_{k=1}^{M-1} \gamma(s_i, s_k) w_{ik}, \tag{3.10}$$

assuming $h\gamma(s_i, 0) < 1$.

Next for $i = 1, \ldots, N$ we do the following: applying the second part of (3.7) we get

$$\begin{aligned} w_{ij} &= w_{i-j,0} + \sum_{k=0}^{j-1} \int_{kh}^{(k+1)h} f\Big([s]_h + [t]_h - [a]_h, [s]_h, v_h([s]_h + [t]_h - [a]_h, [s]_h)\Big)\, ds \\ &= w_{i-j,0} + h \sum_{k=0}^{j-1} f(s_{k+i-j}, s_k, w_{k+i-j,k}), \qquad j = 1, \ldots, i-1, \end{aligned} \tag{3.11}$$

and then we define w_{i0} by (3.10).

The proof of the following theorem is based on the discrete Gronwall's lemma (see, e.g., [1]).

Lemma 3.1 *Let $a, b > 0$. Suppose the nonnegative sequence x_k satisfies $x_0 \le a$, and*

$$x_k \le a + \sum_{i=0}^{k-1} b x_j, \qquad k = 1, 2, \ldots.$$

Then

$$x_k \le a(1+b)^k \le a e^{bk}, \qquad k = 0, 1, 2, \ldots.$$

Now we can formulate and prove our main result on the convergence of this approximation scheme. We show that $v_h(s_i, s_j)$ approximate $u(s_i, s_j)$, as $h \to 0+$.

Theorem 3.2 *Suppose $\varphi\colon\ [0, A] \to \mathbb{R}$ is continuous, $f\colon\ [0, T] \times [0, A] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous with Lipschitz constant L, $\gamma\colon\ [0, T] \times [0, A] \to \mathbb{R}$ is continuous. Then for any continuous solution u of* (3.1)–(3.3)

$$\lim_{h \to 0+} |v_h(s_i, s_j) - u(s_i, s_j)| = 0, \tag{3.12}$$

uniformly with respect to $i = 0, \ldots, [T/h]$ and $j = 0, \ldots, [A/h]$.

Proof We introduce the following notations:

$$\omega_\gamma(s) = \max\{|\gamma(t, a) - \gamma(\bar t, \bar a)|\colon\ t, \bar t \in [0, T],\ a, \bar a \in [0, A],\ |t - \bar t| \le s,\ |a - \bar a| \le s\},$$

and

$$\omega_u(s) = \max\{|u(t, a) - u(\bar t, \bar a)|\colon\ t, \bar t \in [0, T],\ a, \bar a \in [0, A],\ |t - \bar t| \le s,\ |a - \bar a| \le s\}.$$

The uniform continuity of γ and u implies $\omega_\gamma(s) \to 0$ and $\omega_u(s) \to 0$ as $s \to 0+$.

First suppose $j \geq i$. Then substracting (3.6) and (3.7) implies

$$\begin{aligned}
&|u(s_i, s_j) - v_h(s_i, s_j)| \\
&\quad \leq \int_0^{s_i} \Big| f\Big(s, s + s_{j-i}, u(s, s+s_{j-i})\Big) - f\Big([s]_h, [s]_h + s_{j-i}, v_h([s]_h, [s]_h + s_{j-i})\Big)\Big|\, ds \\
&\quad \leq L \int_0^{s_i} \Big(2|s - [s]_h| + |u(s, s+s_{j-i}) - v_h([s]_h, [s]_h + s_{j-i})|\Big)\, ds \\
&\quad \leq LT2h + L\int_0^{s_i} |u(s, s+s_{j-i}) - u([s]_h, [s]_h + s_{j-i})|\, ds \\
&\qquad\quad + L\int_0^{s_i} |u([s]_h, [s]_h + s_{j-i}) - v_h([s]_h, [s]_h + s_{j-i})|\, ds \\
&\quad \leq LT2h + LT\omega_u(h) + \ L\int_0^{s_i} |u([s]_h, [s]_h + s_{j-i}) - v_h([s]_h, [s]_h + s_{j-i})|\, ds. \qquad (3.13)
\end{aligned}$$

Introduce the notation

$$z_{h,k}(s) = |u(s, s+s_k) - v_h(s, s+s_k)|.$$

Then (3.13) can be rewritten as

$$z_{h,j-i}(s_i) \leq LT2h + LT\omega_u(h) + \ Lh\sum_{k=0}^{i-1} z_{h,j-i}(s_k).$$

Therefore, Lemma 3.1 yields

$$|u(s_i, s_j) - v_h(s_i, s_j)| = z_{h,j-i}(s_i) \leq (LT2h + LT\omega_u(h))e^{Lhi} \leq (LT2h + LT\omega_u(h))e^{LT}, \quad 1 \leq i \leq j. \qquad (3.14)$$

This concludes the proof of (3.12) for the case when $0 \leq i \leq j$.

Introduce the constants $M_\gamma = \max\{|\gamma(t,a)| \colon t \in [0,T], a \in [0,A]\}$ and $M_u = \max\{|u(t,a)| \colon t \in [0,T], a \in [0,A]\}$. Substracting (3.3) and (3.8) yields

$$\begin{aligned}
&|u(s_i, 0) - v_h(s_i, 0)| \\
&\quad = \Big| \int_0^A \gamma(s_i, s) u(s_i, s)\, ds - \int_0^{[A]_h} \gamma([t]_h, [s]_h) v_h([t]_h, [s]_h)\, ds \Big| \\
&\quad \leq \int_{[A]_h}^A |\gamma(s_i, s) u(s_i, s)|\, ds + \int_0^{[A]_h} |\gamma(s_i, s) - \gamma(s_i, [s]_h)||u(s_i, s)|\, ds \\
&\qquad + \int_0^{[A]_h} |\gamma(s_i, [s]_h)||u(s_i, s) - u(s_i, [s]_h)|\, ds + \int_0^{[A]_h} |\gamma(s_i, [s]_h)||u(s_i, [s]_h) - v_h(s_i, [s]_h)|\, ds \\
&\quad \leq hM_\gamma M_u + A\omega_\gamma(h) M_u + AM_\gamma \omega_u(h) + \sum_{k=0}^{M-1} |\gamma(s_i, s_k)||u(s_i, s_k) - v_h(s_i, s_k)|.
\end{aligned}$$

Supposing h is such that $h|\gamma(s_i, 0)| < 1/2$ for $i = 0, \dots, N$ we get

$$|u(s_i, 0) - v_h(s_i, 0)| \leq 2\Big(hM_\gamma M_u + A\omega_\gamma(h) M_u + AM_\gamma \omega_u(h)\Big) + 2M_\gamma \sum_{k=1}^{M-1} |u(s_i, s_k) - v_h(s_i, s_k)|, \qquad (3.15)$$

for $i = 1, \ldots, N$. Suppose first that $i = 1$. Then (3.15) together with (3.12) with $i = 1$, $j \geq 1$ yields $\lim_{h\to 0+} |u(s_1, 0) - v_h(s_1, 0)| = 0$, i.e., (3.12) holds for $i = 1$ and $j = 0$, as well.

We continue with induction. Suppose we know (3.12) for $i = 0, \ldots, i^*$ and $j = 0, \ldots, M$. Let $1 \leq j < i^* + 1$. Similary to the case $j \geq i$, we get

$$\begin{aligned}
&|u(s_{i^*+1}, s_j) - v_h(s_{i^*+1}, s_j)| \\
&\quad\leq |u(s_{i^*+1-j}, 0) - v_h(s_{i^*+1-j}, 0)| + \int_0^{s_j} \Big| f\Big(s + s_{i^*+1-j}, s, u(s + s_{i^*+1-j}, s)\Big) \\
&\qquad\qquad - f\Big([s]_h + s_{i^*+1-j}, [s]_h, v_h([s]_h + s_{i^*+1-j}, [s]_h)\Big)\Big|\, ds \\
&\quad\leq |u(s_{i^*+1-j}, 0) - v_h(s_{i^*+1-j}, 0)| + LT2h + LT\omega_u(h) \\
&\qquad + L\int_0^{s_j} |u([s]_h + s_{i^*+1-j}, [s]_h) - v_h([s]_h + s_{i^*+1-j}, [s]_h)|\, ds.
\end{aligned} \tag{3.16}$$

Inequality (3.16) is equivalent to

$$\tilde{z}_{h,i^*+1-j}(s_j) \leq |u(s_{i^*+1-j}, 0) - v_h(s_{i^*+1-j}, 0)| + LT2h + LT\omega_u(h) + \; Lh\sum_{k=0}^{j-1} \tilde{z}_{h,i^*+1-j}(s_k),$$

where

$$\tilde{z}_{h,k}(s) = |u(s + s_k, s) - v_h(s + s_k, s)|.$$

Then Lemma 3.1 yields

$$\begin{aligned}
&|u(s_{i^*+1}, s_j) - v_h(s_{i^*+1}, s_j)| \\
&\quad= \tilde{z}_{h,i^*+1-j}(s_j) \\
&\quad\leq (|u(s_{i^*+1-j}, 0) - v_h(s_{i^*+1-j}, 0)| + LT2h + LT\omega_u(h))e^{Lhj} \\
&\quad\leq (|u(s_{i^*+1-j}, 0) - v_h(s_{i^*+1-j}, 0)| + LT2h + LT\omega_u(h))e^{LA}, \quad 1 \leq j < i^* + 1.
\end{aligned} \tag{3.17}$$

Therefore (3.12) holds for $i = i^* + 1$ and $j = 1, \ldots, M$. But then (3.15) combined with (3.14) and (3.17) yields (3.12) with $i = i^* + 1$ and $j = 0$, as well. Therefore (3.12) holds for all i and j. $\square$

The proof has the following corollary.

Corollary 3.3 *Suppose $\varphi\colon [0, A] \to \mathbb{R}$ is continuous, $f\colon [0, T] \times [0, A] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, $\gamma\colon [0, T] \times [0, A] \to \mathbb{R}$ is Lipschitz-continuous, and u is continuously differentiable. Then there exists $K > 0$ such that*

$$|v_h(s_i, s_j) - u(s_i, s_j)| \leq Kh,$$

for all $i = 0, \ldots, [T/h]$ and $j = 0, \ldots, [A/h]$.

Let $t = s_i + \alpha$ and $a = s_j + \beta$, where $0 \leq \alpha < h$ and $0 \leq \beta < h$. Then it is easy to show from (3.7) that

$$v_h(s_i + \alpha, s_j + \beta) = \begin{cases} v_h(s_i, s_j) + \alpha f(s_i, s_j, w_{ij}), & a \geq t, \\ v_h(s_i, s_j) + \beta f(s_i, s_j, w_{ij}), & a < t. \end{cases}$$

Therefore $t \mapsto v_h(t,a)$ is piecewise linear and $a \mapsto v_h(t,a)$ is piecewise constant for $a \geq t$, and $t \mapsto v_h(t,a)$ is piecewise constant and $a \mapsto v_h(t,a)$ is piecewise linear for $a < t$. Using this relation and the uniform continuity of u, it is easy to show the following result.

Theorem 3.4 *Suppose $\varphi\colon [0,A] \to \mathbb{R}$ is continuous, $f\colon [0,T]\times[0,A]\times\mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, $\gamma\colon [0,T]\times[0,A] \to \mathbb{R}$ is continuous. Then*

$$\lim_{h\to 0+} \max\{|v_h(t,a) - u(t,a)|\colon\ t \in [0,T],\ a \in [0,A]\} = 0.$$

4 On the qualitative analysis of a hybrid delay equation

Before we formulate some results and an open problem for a hybrid delay equation, we recall some basic facts for a simple continuous and discrete delay equation. Consider

$$\dot{x}(t) = ax(t-\tau), \qquad t \geq 0, \tag{4.1}$$

and

$$\Delta y(n) = by(n-k), \qquad n \in \mathbb{N}_0, \tag{4.2}$$

where Δ is the forward difference operator: $\Delta y(n) = y(n+1) - y(n)$, and $\mathbb{N}_0$ is the set of nonnegative integers.

It is known ([17], [15]) that the respective characteristic equations are

$$\lambda = ae^{-\lambda\tau} \tag{4.3}$$

and

$$\mu = 1 + b\mu^{-k}. \tag{4.4}$$

We say that the zero solution of (4.1) ((4.2)) is asymptotically stable, if any solution of (4.1) ((4.2)) tends to zero as $t \to \infty$ ($n \to \infty$).

We say that a (nontrivial) solution $x\colon [0,\infty) \to \mathbb{R}$ ($y\colon \mathbb{N}_0 \to \mathbb{R}$) of (4.1) ((4.2)) is oscillatory, if there exists a sequence t_k (n_k) such that $t_k \to \infty$ ($n_k \to \infty$) as $k \to \infty$, and $x(t_k)x(t_{k+1}) < 0$ ($y(n_k)y(n_{k+1}) < 0$) for $k \geq 0$.

The following result is well-known ([17], [15]).

Theorem 4.1 *Let $a \in \mathbb{R}$, $\tau \geq 0$, $b \in \mathbb{R}$, $k \in \mathbb{N}_0$.*

(A) *(i) The zero solution of* (4.1) *is asymptotically stable, if and only if* $\operatorname{Re}\lambda < 0$ *for any root $\lambda \in \mathbb{C}$ of* (4.3), *or equivalently,*

$$a < 0 \qquad \text{and} \qquad |a|\tau < \frac{\pi}{2}.$$

(ii) Every solution of (4.1) *is oscillatory, if and only if* (4.3) *has no real root, or equivalently,*

$$a < 0 \qquad \text{and} \qquad |a|\tau > \frac{1}{e}.$$

(B) (i) *The zero solution of* (4.2) *is asymptotically stable, if and only if* $|\mu| < 1$ *for any root* $\mu \in \mathbb{C}$ *of* (4.4)*, or equivalently,*

$$b < 0 \qquad \textit{and} \qquad |b| < 2\cos\frac{k\pi}{2k+1}.$$

(ii) *Every solution of* (4.2) *is oscillatory, if and only if* (4.4) *has no positive root, or equivalently,*

$$b < 0 \qquad \textit{and} \qquad |b| > \frac{k^k}{(k+1)^{k+1}}.$$

Now consider the control system

$$\dot{z}(t) = az(t-\tau) + u(t),$$

where the uncontrolled part is a continuous DDE of the form (4.1), and suppose we use the feedback law

$$u(t) = bz([t]_h - kh),$$

i.e., we assume the the contoller measures the state at discrete time values $0, h, 2h, \ldots$ and there is a delay in the control mechanism. Then we get a so-called hybrid DDE

$$\dot{z}(t) = az(t-\tau) + bz([t]_h - kh). \tag{4.5}$$

In [8] a sharp but not exact condition is given for the oscillatory behaviour of the solution of (4.5). In the book [15] the next question was raised: Does (4.5) have a characteristic equation? If yes, what is the form of it?

A partial answer was given by Wang and Yan [28] in the case when $h = 1$ and $\tau = \ell$ is a positive integer. They showed that all solutions of

$$\dot{x}(t) = ax(t-\ell) + bx([t-k]), \qquad t \geq 0$$

are oscillatory, if and only if the equation

$$\lambda = e^{a\lambda^{-\ell}}\left(1 + b\lambda^{-k}\int_0^1 e^{a\lambda^{-\ell}s}\,ds\right)$$

has no nonnegative roots. The proof is based on the so-called improved z-transform [23]. Similar results was given in [29], [30].

It is still an important open question whether it is possible to obtain a characteristic equation of (4.5) in the general case, and formulate necessary and sufficient stability and oscillation conditions for (4.5) with or without using the characteristic equation.

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