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Linearized Stability for a Class of Neutral Functional Differential Equations with State-Dependent Delays

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Abstract

In this paper we formulate a stability theorem by means of linearization around a trivial solution in a cass of autonomous neutral functional differential equations with state-dependent delay. We prove that if the trivial solution of the linearized equation is exponentially stable, then the trivial solution of the nonlinear equation is exponentially stable, as well. As an application of the main result, explicit stability conditions are given.

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1 Introduction and Formulation of the Main Results

In this paper we consider the autonomous neutral differential system

$$\frac{d}{dt}\Big(x(t) - g(x(t-\sigma(x_t)))\Big) = f\Big(x_t, x(t-\tau(x_t))\Big), \qquad t \geq 0 \tag{1.1}$$

and the associated initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0], \qquad \varphi \in C. \tag{1.2}$$

Here we assume that $r > 0$ is fixed, $g\colon \mathbb{R}^n \to \mathbb{R}^n$, $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma, \tau\colon C \to [0, r]$. A fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. C is the Banach space of continuous functions $\psi : [-r, 0] \to \mathbb{R}^n$ equipped with the norm $\|\psi\| = \sup\{|\psi(s)|\colon s \in [-r, 0]\}$. The solution segment function $x_t\colon [-r, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t + s)$.

We assume that $x = 0$ is a constant equilibrium of (1.1), and we study the exponential stability of the trivial solution by means of linearization technique.

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For retarded delay differential equations with state-dependent delays (SD-DDEs), i.e., the case when $g \equiv 0$ in (1.1), a linearized stability theorem was first proved in [5]. Later, similar results were proved for different classes of SD-DDEs in [12, 16, 20, 21, 27]. The main technical difficulty to prove a linearized stability theorem in SD-DDEs is that the map $C \ni \psi \mapsto f(\psi, \psi(-\tau(\psi))) \in \mathbb{R}^n$ is not Fréchet-differentiable. See [22, 27] for more details and discussions on this topic. We refer the interested reader also to [22] for a survey on general theory and applications of SD-DDEs. The study of SD-DDEs is an active research area (see, e.g., [1, 9, 16, 22, 24, 25] and the refences therein). Much less work is devoted to neutral functional differential equations with state-dependent delays [2, 3, 4, 6, 11, 17, 18, 19, 23, 28, 29].

We compare the exponential stability of the trivial solution of (1.1) to that of the associated linear system

$$\frac{d}{dt}\Big(y(t) - g'(0)y(t - \sigma(\mathbf{0}))\Big) = D_1 f(\mathbf{0}, 0)y_t + D_2 f(\mathbf{0}, 0)y(t - \tau(\mathbf{0})), \qquad t \geq 0, \tag{1.3}$$

where $\mathbf{0}$ is the constant 0 function in C, and we associate initial condition (1.2) to (1.3).

We assume throughout the paper

(H1) (i) the function $g\colon U_1 \to \mathbb{R}^n$ is continuously differentiable, where $U_1 \subset \mathbb{R}^n$ is open, and $0 \in U_1$;

(ii) $g(0) = 0$;

(iii) $|g'(0)| < 1$;

(H2) (i) the function $f\colon U_2 \times U_3 \to \mathbb{R}^n$ is continuously differentiable, where $U_2 \subset C$ and $U_3 \subset \mathbb{R}^n$ are open subsets, $\mathbf{0} \in U_2$ and $0 \in U_3$;

(ii) $f(\mathbf{0}, 0) = 0$;

(H3) (i) the delay functions $\sigma, \tau\colon U_4 \to [0, r]$ are continuous, where $U_4 \subset C$ is open, and $\mathbf{0} \in U_4$;

(ii) $\sigma(\mathbf{0}) \neq 0$;

(H4) $\varphi \in C$.

Note that (H1) (ii) is not a restriction on the problem, since we can always add a constant to the function g.

Assumptions (H1)–(H4) yield only the existence but not the uniqueness of solutions of the IVP (1.1)-(1.2) (see corresponding results for retarded SD-DDEs, e.g., in [7, 20, 22]).

We say that the trivial (zero) solution of the linear equation (1.3) is exponentially stable, if there exists $K_1 \geq 0$ and $\alpha > 0$ such that

$$|y(t)| \leq K_1 e^{-\alpha t}\|\varphi\|, \qquad t \geq 0. \tag{1.4}$$

In this case we say that the order of exponential stability is α.

Similarly, we say the trivial solution of the nonlinear equation (1.1) is exponentially stable, if there exist $K \geq 0$, $\theta > 0$ and $\delta > 0$ such that

$$|x(t)| \leq Ke^{-\theta t}\|\varphi\|, \qquad t \geq 0, \quad \|\varphi\| \leq \delta.$$

We formulate the main result of the paper in the next theorem.

Theorem 1.1 *Assume (H1)–(H4). If the trivial solution of* (1.3) *is exponentially stable, then the trivial solution of* (1.1) *is exponentially stable, as well.*

The proof will be given in two steps. In Section 3 we show that the trivial solution of (1.1) is stable, and in Section 4 we give the proof for its exponential stability. Section 2 contains some preliminary results and introduces notations will be used in the sequel.

We comment that the results are presented for the case of the zero equilibrium, but they are easy to generalize for the case of any constant equilibrium. Also, the proofs are easy to extend to the case when they are multiple state-dependent delay terms on the right-hand-side of (1.1), but the method we use (especially Proposition 2.3 below) relies on the fact that there is only a single delay term in the neutral part of the equation, i.e., on the left-hand-side of (1.1).

Theorem 1.1 immediately has the following corollary. Let I be the $n \times n$ identity matrix.

Corollary 1.2 *Assume (H1)–(H4). If there exists $c_0 > 0$ such that all roots of*

$$\lambda I - g'(0)\lambda e^{-\lambda\sigma(\mathbf{0})} = D_1 f(\mathbf{0}, 0)\Big(e^{\lambda \cdot} I\Big) + D_2 f(\mathbf{0}, 0)e^{-\lambda\tau(\mathbf{0})}$$

satisfiy $Re\,\lambda \leq -c_0$, then trivial solution of (1.1) *is exponentially stable.*

Combining Theorem 1.1 and known stability conditions for linear neutral equations we can formulate explicit stability conditions for equation (1.1). As an illustration, we formulate the next three theorems based on stability conditions of [8], [26] and [10], respectively. Note that for retarded SD-DDEs similar explicit conditions were given in [12] and [14].

Consider the scalar equation

$$\frac{d}{dt}\Big(x(t) - g(x(t - \sigma(x_t)))\Big) = h\Big(x(t), x(t - \tau(x_t))\Big), \qquad t \geq 0. \tag{1.5}$$

Then equation (1.5) has the form (1.1) with $f(\psi, u) = h(\psi(0), u)$. We assume

(H2*) (i) the function $h \colon \mathcal{U}_2 \times \mathcal{U}_3 \to \mathbb{R}$ is continuously differentiable, where $\mathcal{U}_2$ and $\mathcal{U}_3$ are open subsets of $\mathbb{R}$ containing 0;

(ii) $h(0, 0) = 0$.

Theorem 1.3 *Suppose (H1), (H2*), (H3), (H4), $n = 1$, and*

(i) $D_1 h(0,0) < -|D_2 h(0,0)|$,

(ii) $\sigma(\mathbf{0}) = \tau(\mathbf{0}) > 0$.

Then the trivial solution of (1.5) *is exponentially stable.*

Theorem 1.4 *Suppose (H1), (H2*), (H3), (H4), $n = 1$, and*

(i) $D_1 h(0,0) = D_2 h(0,0) < 0$,

(ii) $\sigma(\mathbf{0}) = 2\tau(\mathbf{0}) > 0$,

(iii) and either

$$-\frac{1}{3} \le g'(0) < 1, \qquad g'(0) \ne 0$$

or

$$-1 < g'(0) < -\frac{1}{3}, \quad 0 < \tau(\mathbf{0}) < \frac{1 + g'(0)}{|D_1 h(0,0)|} \sqrt{\frac{1 - g'(0)}{-3g'(0) - 1}} \arccos \frac{1 + g'(0)}{-2g'(0)}$$

hold. Then the trivial solution of (1.5) *is exponentially stable.*

Theorem 1.5 *Suppose (H1), (H2*), (H3), (H4), $n = 1$, and*

(i) $D_1 h(0,0) + D_2 h(0,0) > 0$,

(ii) $|g'(0)| + |D_2 h(0,0)|\tau(\mathbf{0}) < 1$.

Then the trivial solution of (1.5) *is exponentially stable.*

2 Preliminaries

We introduce some constants will be used throughout the paper. Let $|g'(0)| < c < 1$ be fixed, and let $\varrho > 0$ be such that

$$u \in U_1 \cap U_3 \quad \text{for } u \in \mathbb{R}^n,\ |u| \le \varrho \qquad \text{and} \quad \psi \in U_2 \quad \text{for } \psi \in C,\ \|\psi\| \le \varrho,$$

and

$$|g'(u)| \le c, \qquad |u| \le \varrho, \quad u \in \mathbb{R}^n.$$

Define

$$M = \max_{i=1,2} \Big\{ \max\{|D_i f(\psi, u)|\colon\ (\psi, u) \in C \times \mathbb{R}^n,\ \|\psi\| \le \varrho,\ |u| \le \varrho\} \Big\}.$$

It follows from the Mean Value Theorem and the definition of c that

$$|g(u)-g(\tilde{u})| \le c|u-\tilde{u}|, \qquad u,\tilde{u}\in\mathbb{R}^n,\ |u|,|\tilde{u}|\le\varrho, \tag{2.1}$$

and

$$|f(\psi,u)-f(\tilde{\psi},\tilde{u})| \le M(\|\psi-\tilde{\psi}\|+|u-\tilde{u}|), \quad (\psi,u),(\tilde{\psi},\tilde{u})\in C\times\mathbb{R}^n,\ \|\psi\|,\|\tilde{\psi}\|\le\varrho,\ |u|,|\tilde{u}|\le\varrho. \tag{2.2}$$

We introduce the following functions:

$$\begin{aligned}
\omega_1(s) &= \sup\{|g'(u)-g'(0)|:\ |u|\le s,\ u\in U_1\}\\
\omega_2(s) &= \sup\{|D_1f(\psi,u)-D_1f(\mathbf{0},0)|:\ \max\{\|\psi\|,|u|\}\le s,\ \psi\in U_2,\ u\in U_3\}\\
\omega_3(s) &= \sup\{|D_2f(\psi,u)-D_2f(\mathbf{0},0)|:\ \max\{\|\psi\|,|u|\}\le s,\ \psi\in U_2,\ u\in U_3\}\\
\omega_4(s) &= \sup\{|\sigma(\psi)-\sigma(\mathbf{0})|:\ \|\psi\|\le s,\ \psi\in U_4\}\\
\omega_5(s) &= \sup\{|\tau(\psi)-\tau(\mathbf{0})|:\ \|\psi\|\le s,\ \psi\in U_4\}
\end{aligned}$$

The assumed continuity of the respective functions yields $\omega_i(s)\to 0$ as $s\to 0+$ for $i=1,2,3,4,5$. To simplify formulas later we define

$$\omega(s)=\max\{\omega_1(s),\omega_2(s),\omega_3(s),\omega_4(s),\omega_5(s)\}.$$

Then ω is a monotone nondecreasing function and $\omega(s)\to 0$ as $s\to 0+$.

Proposition 2.1 *Suppose x is a solution of the IVP* (1.1)-(1.2) *satisfying*

$$|x(t)|\le\varrho, \qquad \text{for}\quad t\in[-r,T] \tag{2.3}$$

for some $T>0$. Then there exist constants $N_1>1$ and $\gamma>0$ such that

$$|x(t)|\le N_1e^{\gamma t}\|\varphi\|, \qquad t\in[0,T].$$

Proof Integration of (1.1) from 0 to t yields

$$x(t)=g(x(t-\sigma(x_t)))+\varphi(0)-g(\varphi(-\sigma(\varphi)))+\int_0^t f(x_s,x(s-\tau(x_s)))\,ds,$$

hence, using (2.1) and (2.2) we get

$$\begin{aligned}
|x(t)| &\le |g(x(t-\sigma(x_t)))-g(0)|+|\varphi(0)|+|g(\varphi(-\sigma(\varphi)))-g(0)|\\
&\quad+\int_0^t |f(x_s,x(s-\tau(x_s)))-f(\mathbf{0},0)|\,ds\\
&\le c|x(t-\sigma(x_t))|+\|\varphi\|+c|\varphi(-\sigma(\varphi))|+M\int_0^t(\|x_s\|+|x(s-\tau(x_s))|)\,ds\\
&\le c\max_{-r\le u\le t}|x(u)|+(1+c)\|\varphi\|+2M\int_0^t\max_{-r\le u\le s}|x(u)|\,ds, \qquad t\in[0,T].
\end{aligned}$$

Since the right-hand-side is monotone increasing in t, it implies (see, e.g., Lemma 2.1 in [17])

$$\max_{-r\le u\le t}|x(u)| \le c\max_{-r\le u\le t}|x(u)| + (1+c)\|\varphi\| + 2M\int_0^t \max_{-r\le u\le s}|x(u)|\,ds,$$

and so

$$|x(t)| \le \max_{-r\le u\le t}|x(u)| \le \frac{1+c}{1-c}\|\varphi\| + \frac{2M}{1-c}\int_0^t \max_{-r\le u\le s}|x(u)|\,ds, \qquad t\in[0,T].$$

Consequently the statement of the lemma follows with $N_1 = \frac{1+c}{1-c}$ and $\gamma = \frac{2M}{1-c}$ from Gronwall's inequality. $\square$

We can rewrite equation (1.1) as

$$\frac{d}{dt}\Big(x(t) - g'(0)x(t-\sigma(\mathbf{0})) - G(t)\Big) = D_1 f(\mathbf{0},0)x_t + D_2 f(\mathbf{0},0)x(t-\tau(\mathbf{0})) + F(t), \qquad t\ge 0,$$

where

$$G(t) = g\Big(x(t-\sigma(x_t))\Big) - g'(0)x(t-\sigma(\mathbf{0}))$$

and

$$F(t) = f\Big(x_t, x(t-\tau(x_t))\Big) - D_1 f(\mathbf{0},0)x_t - D_2 f(\mathbf{0},0)x(t-\tau(\mathbf{0})).$$

We define the fundamental solution of (1.3) as the $n\times n$ matrix solution of the initial value problem

$$\frac{d}{dt}\Big(V(t) - g'(0)V(t-\sigma(\mathbf{0}))\Big) = D_1 f(\mathbf{0},0)V_t + D_2 f(\mathbf{0},0)V(t-\tau(\mathbf{0})), \qquad t\ge 0, \tag{2.4}$$

$$V(t) = \begin{cases} I, & t=0,\\ 0 & t<0. \end{cases} \tag{2.5}$$

Here I and 0 denote the $n\times n$ identity and the zero matrices, respectively. We comment that $D_1 f(\mathbf{0},0)$ is a bounded linear functional on the space C, but using Hahn–Banach-Theorem, we can extend it to the space of bounded functions defined on $[-r,0]$. In (2.4) this extension is used, which is denoted by $D_1 f(\mathbf{0},0)$, as well.

The variation-of-constants formula (see, e.g., [15]) yields

$$x(t) = y(t) + G(t) - V(t)G(0) - \int_0^t d_s[V(t-s)]\,G(s) + \int_0^t V(t-s)F(s)\,ds, \quad t\ge 0, \tag{2.6}$$

where y is the solution of (1.3) corresponding to initial condition (1.2). It is easy to check that V is continuously differentiable on the intervals $(k\sigma(\mathbf{0}),(k+1)\sigma(\mathbf{0}))$, it is right-continuous at the points $k\sigma(\mathbf{0})$, has left-sided limits at the points $k\sigma(\mathbf{0})$, and $V(k\sigma(\mathbf{0})) - V(k\sigma(\mathbf{0})-) = (g'(0))^k$ for $k=0,1,\ldots$. Consequently (2.6) can be rewritten as

$$\begin{aligned} x(t) &= y(t) - V(t)G(0) + \sum_{k=0}^{\left[\frac{t}{\sigma(\mathbf{0})}\right]} (g'(0))^k G(t-k\sigma(\mathbf{0})) + \int_0^t V'(t-s)G(s)\,ds \\ &\quad + \int_0^t V(t-s)F(s)\,ds, \qquad t\ge 0, \end{aligned} \tag{2.7}$$

where $[\cdot]$ is the greatest integer part function.

If the trivial solution of (1.3) is exponentially stable with order α, i.e., (1.4) holds, then it is known (see, e.g., [15]), that there exists $K_2 \geq 1$ such that

$$|V(t)| \leq K_2 e^{-\alpha t}, \qquad t \geq 0. \tag{2.8}$$

The next result shows that in this case the derivative of V is also exponentially bounded.

Proposition 2.2 *Suppose the trivial solution of* (1.3) *is exponentially stable with order α, i.e., V satisfies* (2.8). *Then for any $0 < \beta \leq \alpha$ satisfying*

$$ce^{\beta\sigma(\mathbf{0})} < 1 \tag{2.9}$$

there exists $K_3 \geq 1$ such that

$$|V'(t)| \leq K_3 e^{-\beta t}, \qquad a.e.\ t \geq 0. \tag{2.10}$$

Proof Rewritting (2.4) we get

$$V'(t) = g'(0)V'(t-\sigma(\mathbf{0})) + D_1 f(\mathbf{0},0)V_t + D_2 f(\mathbf{0},0)V(t-\tau(\mathbf{0})), \qquad t \neq k\sigma(\mathbf{0}), (k=0,1,\dots),$$

and hence, the definition of c and M and relation (2.8) imply

$$\begin{aligned} |V'(t)| &\leq c|V'(t-\sigma(\mathbf{0}))| + M(\|V_t\| + |V(t-\tau(\mathbf{0}))|) \\ &\leq c|V'(t-\sigma(\mathbf{0}))| + 2MK_2 e^{-\alpha(t-r)}, \qquad t \neq k\sigma(\mathbf{0}), (k=0,1,\dots). \end{aligned}$$

Consider a $0 < \beta < \alpha$ satisfying (2.9). Multiplying both sides of the inequality by $e^{\beta t}$ we get

$$|V'(t)|e^{\beta t} \leq c|V'(t-\sigma(\mathbf{0}))|e^{\beta t} + 2MK_2 e^{-\alpha(t-r)}e^{\beta t},$$

and therefore the function $w(t) = |V'(t)|e^{\beta t}$ satisfies

$$w(t) \leq \tilde{c}w(t-\sigma(\mathbf{0})) + \tilde{K}, \qquad t \geq 0,\ t \neq k\sigma(\mathbf{0}),\ (k=0,1,\dots) \tag{2.11}$$

with $\tilde{K} = 2MK_2 e^{\alpha r}$ and $\tilde{c} = ce^{\beta\sigma(\mathbf{0})}$. It follows from (2.9) that $0 < \tilde{c} < 1$. Since $w(t) = 0$ for $t \in (0, \sigma(\mathbf{0}))$, (2.11) implies

$$w(t) \leq \tilde{K}, \qquad t \in (0, \sigma(\mathbf{0})).$$

But then (2.11) yields

$$w(t) \leq (\tilde{c}+1)\tilde{K}, \qquad t \in (\sigma(\mathbf{0}), 2\sigma(\mathbf{0})),$$

and by induction

$$w(t) \leq (\tilde{c}^k + \dots + \tilde{c} + 1)\tilde{K}, \qquad t \in (k\sigma(\mathbf{0}), (k+1)\sigma(\mathbf{0})).$$

Consequently

$$w(t) \leq \frac{\tilde{K}}{1-\tilde{c}}, \qquad t \geq 0,\ t \neq k\sigma(\mathbf{0}),\ k = 0, 1, \ldots,$$

which implies the statement with $K_3 = \frac{\tilde{K}}{1-\tilde{c}}$. □

For simplicity of the presentation we extend $\varphi(t)$ to $(-\infty, -r)$ by $\varphi(t) = \varphi(-r)$. For a fixed solution x of the IVP (1.1)-(1.2) we introduce the following sequence of functions

$$\alpha_0(t) \equiv t, \quad \alpha_1(t) \equiv t - \sigma(x_t), \quad \alpha_{j+1}(t) \equiv \alpha_1(\alpha_j(t)) \quad \text{for} \quad j = 1, 2, \ldots. \tag{2.12}$$

For the sake of simplicity the dependence of α_j on x is omitted in the notation, but it should always be kept in mind. It is easy to see that

$$\alpha_j(t) = t - \sum_{k=0}^{j-1} \sigma(x_{\alpha_k(t)}), \qquad j = 1, 2, \ldots. \tag{2.13}$$

Assumption (H3) (i) yields that $0 \leq \sigma(x_t) \leq r$ for all t, therefore

$$t - jr \leq \alpha_j(t) \leq t \qquad \text{for} \quad t \geq 0 \quad \text{and} \quad j = 0, 1, \ldots. \tag{2.14}$$

It follows from (2.13) that

$$\alpha_j(t) = t - j\sigma(\mathbf{0}) - \sum_{k=0}^{j-1} \Big(\sigma(x_{\alpha_k(t)}) - \sigma(\mathbf{0}) \Big), \qquad j = 1, 2, \ldots.$$

Suppose $0 \leq t_1 \leq t_2$. Then the definition of ω_4 implies

$$\begin{aligned} |\alpha_j(t_2) - \alpha_j(t_1)| &\leq t_2 - t_1 + \sum_{k=0}^{j-1} \Big| \sigma(x_{\alpha_k(t_2)}) - \sigma(\mathbf{0}) \Big| + \sum_{k=0}^{j-1} \Big| \sigma(x_{\alpha_k(t_1)}) - \sigma(\mathbf{0}) \Big| \\ &\leq t_2 - t_1 + \sum_{k=0}^{j-1} \Big(\omega_4(\|x_{\alpha_k(t_2)}\|) + \omega_4(\|x_{\alpha_k(t_1)}\|) \Big) \\ &\leq t_2 - t_1 + 2j\omega_4 \Big(\max_{-r \leq u \leq t_2} |x(u)| \Big), \quad j = 0, 1, \ldots. \end{aligned} \tag{2.15}$$

For a fixed x we introduce the simplifying notation

$$\eta(t) = \max_{-r \leq u \leq t} |x(u)|.$$

The proof of our main result will be based on the following proposition, which follows the idea of Proposition 2 in [13].

Proposition 2.3 *Assume (H1)–(H4). Let x be a solution of the IVP* (1.1)-(1.2) *satisfying* (2.3) *for some $T > 0$, and $0 \le t_1 \le t_2 \le T$. Then*

$$|x(t_2) - x(t_1)| \le \left(2c^{\frac{t_1}{r}} + \frac{2M}{1-c}(t_2 - t_1) + \frac{4cM}{(1-c)^2}\omega_4(\eta(t_2))\right)\eta(t_2). \tag{2.16}$$

Proof Integrating (1.1) from t_1 to t_2 we get

$$x(t_2) - x(t_1) = g(x(\alpha_1(t_2))) - g(x(\alpha_1(t_1))) + \int_{t_1}^{t_2} (f(x_s, x(s - \tau(\mathbf{0}))) - f(\mathbf{0}, 0))\, ds,$$

consequently (2.1) and (2.2) yield

$$|x(t_2) - x(t_1)| \le c|x(\alpha_1(t_2)) - x(\alpha_1(t_1))| + M \int_{t_1}^{t_2} (\|x_s\| + |x(s - \tau(\mathbf{0}))|)\, ds. \tag{2.17}$$

Let n be defined by

$$n = n(t_1) = \left[\frac{t_1}{r}\right], \tag{2.18}$$

where $[\cdot]$ is the greatest integer part function, then (2.14) yields

$$0 \le t_1 - nr \le t_1 - jr \le t_2 - jr \le \alpha_j(t_2) \le t_2, \qquad j = 0, 1, \dots, n,$$

and so

$$0 \le \alpha_j(t_i) \le t_2, \quad (j = 0, 1, \dots, n,\ i = 1, 2), \qquad -r \le \alpha_{n+1}(t_i) \le t_2, \qquad i = 1, 2. \tag{2.19}$$

Applying inequality (2.17) n times gives relation

$$\begin{aligned} |x(t_2) - x(t_1)| \ &\le\ c^{n+1}\Big|x(\alpha_{n+1}(t_2)) - x(\alpha_{n+1}(t_1))\Big| \\ &\quad + M\sum_{j=0}^{n} c^j \left|\int_{\alpha_j(t_1)}^{\alpha_j(t_2)} \Big(\|x_s\| + |x(s - \tau(\mathbf{0}))|\Big) ds\right|. \end{aligned} \tag{2.20}$$

Therefore relations (2.19) and the definition of η imply

$$|x(t_2) - x(t_1)| \le \Big(2c^{n+1} + 2M\sum_{j=0}^{n} c^j|\alpha_j(t_2) - \alpha_j(t_1)|\Big)\eta(t_2).$$

Then (2.15), $\sum_{j=0}^{\infty} c^j = \frac{1}{1-c}$ and $\sum_{j=1}^{\infty} jc^j = \frac{c}{(1-c)^2}$ yield

$$\begin{aligned} |x(t_2) - x(t_1)| \ &\le\ \Big(2c^{n+1} + 2M\sum_{j=0}^{n} c^j\Big((t_2 - t_1) + j2\omega_4(\eta(t_2))\Big)\Big)\eta(t_2) \\ &\le\ \left(2c^{n+1} + \frac{2M}{1-c}(t_2 - t_1) + \frac{4cM\omega_4(\eta(t_2))}{(1-c)^2}\right)\eta(t_2). \end{aligned}$$

Hence the statement of the proposition follows from the inequality $\frac{t_1}{r} - 1 < n \le \frac{t_1}{r}$. □

Introduce the positive constant

$$\nu_0 = -\frac{\log c}{r}.$$

Proposition 2.4 *Assume (H1)–(H4). Let x be a solution of the IVP* (1.1)-(1.2) *satisfying* (2.3) *for some $T>0$. Then there exists a constant $N_2>0$ independent of T such that*

$$|G(t)| \le \Big(2e^{-\nu_0 t} + N_2\omega(\eta(t))\Big)\eta(t), \qquad t\in[0,T], \tag{2.21}$$

and

$$|G(t)| \le 2c \max_{t-r\le u\le t} |x(u)|, \qquad t\ge 0. \tag{2.22}$$

Proof Assumption (H1) (i) and the definition of ω_1 yield

$$\begin{aligned} |g(u)-g(\tilde u)-g'(\tilde u)(u-\tilde u)| &\le \sup_{s\in(0,1)} |g'(\tilde u + s(u-\tilde u)) - g'(\tilde u)||u-\tilde u| \\ &\le \omega_1(|u-\tilde u|)|u-\tilde u|, \qquad u,\tilde u\in U_1. \end{aligned} \tag{2.23}$$

Applying Proposition 2.3 with

$$t_1 = \min\{t-\sigma(x_t), t-\sigma(\mathbf{0})\}, \qquad t_2=\max\{t-\sigma(x_t), t-\sigma(\mathbf{0})\}$$

and (2.23) we get

$$\begin{aligned} &|G(t)| \\ &\le \Big|g(x(t-\sigma(x_t))) - g(0) - g'(0)x(t-\sigma(x_t))\Big| + \Big|g'(0)\Big(x(t-\sigma(x_t)) - x(t-\sigma(\mathbf{0}))\Big)\Big| \qquad (2.24)\\ &\le \omega_1(|x(t-\sigma(x_t))|)|x(t-\sigma(x_t))| + c\left(2c^{\frac{t_1}{r}} + \frac{2M}{1-c}|\sigma(x_t)-\sigma(\mathbf{0})| + \frac{4cM\omega_4(\eta(t))}{(1-c)^2}\right)\eta(t). \end{aligned}$$

Since $t_1\ge t-r$, we get

$$\begin{aligned} |G(t)| &\le \omega_1(\eta(t))\eta(t) + c\left(2c^{\frac{t-r}{r}} + \frac{2M}{1-c}\omega_4(\|x_t\|) + \frac{4cM\omega_4(\eta(t))}{(1-c)^2}\right)\eta(t) \\ &\le \left(\omega_1(\eta(t)) + 2e^{-\nu_0 t} + \frac{c+c^2}{(1-c)^2}2M\omega_4(\eta(t))\right)\eta(t), \end{aligned}$$

which yields (2.21).

To prove (2.22) consider the obvious estimates

$$|G(t)| \le |g(x(t-\sigma(x_t))) - g(0)| + c|x(t-\sigma(\mathbf{0}))| \le c|x(t-\sigma(x_t))| + c|x(t-\sigma(\mathbf{0}))|,$$

which yields (2.22). □

We can estimate F analogously to (2.21) and (2.22).

Proposition 2.5 *Assume (H1)–(H4). Let x be a solution of the IVP* (1.1)-(1.2) *satisfying* (2.3) *for some $T>0$. Then there exist constants $N_3>0$ and $N_4>0$ independent of T such that*

$$|F(t)| \le \Big(N_3 e^{-\nu_0 t} + N_4\omega(\eta(t))\Big)\eta(t), \qquad t\in[0,T], \tag{2.25}$$

and

$$|F(t)| \le 4M \max_{t-r\le u\le t} |x(u)|, \qquad t\ge 0. \tag{2.26}$$

Proof Assumption (H2) (i) and the definition of ω_2 and ω_3 yield

$$\begin{aligned}
&|f(\psi,u) - f(\tilde\psi,\tilde u) - D_1 f(\tilde\psi,\tilde u)(\psi-\tilde\psi) - D_2 f(\tilde\psi,\tilde u)(u-\tilde u)| \\
&\quad\leq \sup_{s\in(0,1)} |D_1 f(\tilde\psi + s(\psi-\tilde\psi), \tilde u + s(u-\tilde u)) - D_1 f(\tilde\psi,\tilde u)|\|\psi-\tilde\psi\| \\
&\qquad + \sup_{s\in(0,1)} |D_2 f(\tilde\psi + s(\psi-\tilde\psi), \tilde u + s(u-\tilde u)) - D_2 f(\tilde\psi,\tilde u)||u-\tilde u| \\
&\quad\leq \omega_2\Big(\max\{\|\psi-\tilde\psi\|, |u-\tilde u|\}\Big)\|\psi-\tilde\psi\| + \omega_3\Big(\max\{\|\psi-\tilde\psi\|, |u-\tilde u|\}\Big)|u-\tilde u|, \\
&\qquad\quad \psi,\tilde\psi\in C,\ u,\tilde u\in\mathbb{R}^n.
\end{aligned}$$

Hence, by Proposition 2.3,

$$\begin{aligned}
|F(t)| &\leq \Big|f\Big(x_t, x(t-\tau(x_t))\Big) - f(\mathbf{0},0) - D_1 f(\mathbf{0},0)x_t - D_2 f(\mathbf{0},0)x(t-\tau(x_t))\Big| \\
&\quad + \Big|D_2 f(\mathbf{0},0)\Big(x(t-\tau(x_t)) - x(t-\tau(\mathbf{0}))\Big)\Big| \\
&\leq \omega_2\Big(\max\{\|x_t\|, |x(t-\tau(x_t))|\}\Big)\|x_t\| + \omega_3\Big(\max\{\|x_t\|, |x(t-\tau(x_t))|\}\Big)|x(t-\tau(x_t))| \\
&\quad + M\left(2c^{\frac{t-r}{r}} + \frac{2M}{1-c}\omega_5(\|x_t\|) + \frac{4cM\omega_4(\eta(t))}{(1-c)^2}\right)\eta(t) \\
&\leq \left(2\omega(\eta(t)) + 2Mc^{\frac{t-r}{r}} + \frac{1+c}{(1-c)^2}2M^2\omega(\eta(t))\right)\eta(t), \qquad t\in[0,T],
\end{aligned}$$

which implies (2.25).

Relation (2.26) follows directly from the definition of F and (2.2). $\square$

3 Proof of stability

In this section we give the first part of the proof of Theorem 1.1, we show the stability of the trivial solution of (1.1), under the assumption of Theorem 1.1.

Suppose φ is such that $\|\varphi\| < \varrho$, and let x be any corresponding solution. Let $T > 0$ be such that $|x(t)| < \varrho$ for $t\in[-r,T)$.

Let $m(t) := \left[\frac{t}{\sigma(\mathbf{0})}\right]$. Then the variation-of-constant formula (2.7) yields

$$\begin{aligned}
|x(t)| &\leq |y(t)| + |V(t)||G(0)| + \sum_{k=0}^{m(t)} c^k |G(t-k\sigma(\mathbf{0}))| + \int_0^t |V'(t-s)||G(s)|\,ds \\
&\quad + \int_0^t |V(t-s)||F(s)|\,ds, \qquad t\geq 0. \qquad (3.1)
\end{aligned}$$

Combining (2.8) and (2.22) we get

$$|V(t)||G(0)| \leq K_2 e^{-\alpha t} 2c\|\varphi\|, \qquad t\geq 0. \qquad (3.2)$$

Fix a constant ν_0^* such that

$$0 < \nu_0^* < \nu_0.$$

We need the following estimate.

Proposition 3.1 *There exist positive constants N_5 and N_6 such that*

$$\sum_{k=0}^{m(t)} c^k |G(t - k\sigma(\mathbf{0}))| \leq \Big(N_5 e^{-\nu_0^* t} + N_6 \omega(\eta(t))\Big)\eta(t), \qquad t \in [0, T]. \tag{3.3}$$

Proof Using (2.21) and the definition of ν_0 we get

$$\begin{aligned}
\sum_{k=0}^{m(t)} c^k |G(t - k\sigma(\mathbf{0}))| &\leq \sum_{k=0}^{m(t)} c^k \Big(2e^{-\nu_0(t - k\sigma(\mathbf{0}))} + N_2\omega(\eta(t - k\sigma(\mathbf{0})))\Big)\eta(t - k\sigma(\mathbf{0})) \\
&\leq \Big(2e^{-\nu_0 t} \sum_{k=0}^{m(t)} e^{-k\nu_0(r - \sigma(\mathbf{0}))} + N_6\omega(\eta(t))\Big)\eta(t), \qquad t \in [0, T],
\end{aligned}$$

where $N_6 = \frac{N_2}{1-c}$. We distinguish two cases. First suppose $\sigma(\mathbf{0}) < r$. Then

$$\sum_{k=0}^{m(t)} e^{-k\nu_0(r - \sigma(\mathbf{0}))} < \frac{1}{1 - e^{-\nu_0(r - \sigma(\mathbf{0}))}}.$$

Now consider the case when $\sigma(\mathbf{0}) = r$. Then

$$\sum_{k=0}^{m(t)} e^{-k\nu_0(r - \sigma(\mathbf{0}))} = m(t) + 1 = \left[\frac{t}{r}\right] + 1 \leq \frac{t}{r} + 1.$$

Now select $N_5 \geq \frac{2}{1 - e^{-\nu_0(r - \sigma(\mathbf{0}))}}$ such that

$$2\left(\frac{t}{r} + 1\right) \leq N_5 e^{(\nu_0 - \nu_0^*)t}, \qquad t \geq 0.$$

Then in both cases

$$2e^{-\nu_0 t} \sum_{k=0}^{m(t)} e^{-k\nu_0(r - \sigma(\mathbf{0}))} \leq N_5 e^{-\nu_0^* t}, \qquad t \geq 0,$$

hence (3.3) holds. □

Fix a constant β such that

$$0 < \beta \leq \alpha \qquad \text{and} \qquad \beta < \nu_0. \tag{3.4}$$

Note that the second inequality of (3.4) implies (2.9), so for such β estimate (2.10) holds.

Relations (2.10) and (2.21) yield

$$\int_0^t |V'(t-s)||G(s)|\,ds \le \int_0^t K_3 e^{-\beta(t-s)}\Big(2e^{-\nu_0 s} + N_2\omega(\eta(s))\Big)\eta(s)\,ds, \qquad t \in [0,T], \quad (3.5)$$

and similarly, (2.8) and (2.25) imply

$$\int_0^t |V(t-s)||F(s)|\,ds \le \int_0^t K_2 e^{-\alpha(t-s)}\Big(N_3 e^{-\nu_0 s} + N_4\omega(\eta(s))\Big)\eta(s)\,ds, \qquad t \in [0,T]. \quad (3.6)$$

Combining (3.1) with the above etimates (3.2), (3.3), (3.5) and (3.6), and using that $\beta \le \alpha$, we get

$$\begin{aligned} |x(t)| \ \le\ & N_7 e^{-\alpha t}\|\varphi\| + \Big(N_5 e^{-\nu_0^* t} + N_6\omega(\eta(t))\Big)\eta(t) \\ & + \int_0^t e^{-\beta(t-s)}\Big(N_8 e^{-\nu_0 s} + N_9\omega(\eta(s))\Big)\eta(s)\,ds, \qquad t \in [0,T], \end{aligned} \quad (3.7)$$

where $N_7 = K_1 + 2K_2 c$, $N_8 = 2K_3 + K_2 N_3$ and $N_9 = K_3 N_2 + K_2 N_4$.

Define the constant

$$\nu_1 = \nu_0 - \beta.$$

Then it follows from (3.4) that $0 < \nu_1 < \nu_0$.

Now we are ready to show the stability of the trivial solution of (1.1). Let $\varepsilon_0 > 0$ be such that

$$\varepsilon_0 \le \varrho, \qquad N_6\omega(\varepsilon_0) < \frac{1}{6} \qquad \text{and} \qquad \frac{N_9\omega(\varepsilon_0)}{\beta} < \frac{1}{6}.$$

Similarly, fix $T_0 > 0$ such that

$$N_5 e^{-\nu_0^* T_0} < \frac{1}{6} \qquad \text{and} \qquad \frac{N_8}{\nu_1} e^{-\beta T_0} < \frac{1}{6}.$$

Let $0 < \varepsilon \le \varepsilon_0$ be arbitrarily fixed, and let $\delta = \delta(\varepsilon)$ be such that

$$0 < \delta < \min\Big(\varrho, \frac{\varepsilon}{N_1} e^{-\gamma T_0}, \frac{\varepsilon}{6N_7}\Big).$$

Fix an initial function satisfying $\|\varphi\| < \delta$, and let x be a corresponding solution. Then Proposition 2.1 yields

$$|x(t)| \le N_1 e^{\gamma T_0}\|\varphi\|, \qquad t \in [0, T_0],$$

so $|x(t)| < \varepsilon$ holds for $t \in [0, T_0]$.

Suppose there exists $S > 0$ such that $|x(t)| < \varepsilon$ for $t \in [0, S)$, and $|x(S)| = \varepsilon$. Then $T_0 < S \le T$, and (3.7) implies for $t = S$

$$\begin{aligned} \varepsilon \ \le\ & N_7 e^{-\alpha S}\delta + \Big(N_5 e^{-\nu_0^* S} + N_6\omega(\varepsilon)\Big)\varepsilon + \int_0^S e^{-\beta(S-s)}\Big(N_8 e^{-\nu_0 s} + N_9\omega(\varepsilon)\Big)\varepsilon\,ds \\ \le\ & N_7\delta + \Big(N_5 e^{-\nu_0^* T_0} + N_6\omega(\varepsilon_0)\Big)\varepsilon \end{aligned}$$

$$
\begin{aligned}
&+N_8\varepsilon e^{-\beta S}\int_0^S e^{-\nu_1 s}\,ds + N_9\omega(\varepsilon_0)\varepsilon e^{-\beta S}\int_0^S e^{\beta s}\,ds \\
\le\ & \frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+N_8\varepsilon e^{-\beta S}\frac{e^{-\nu_1 S}-1}{-\nu_1}+N_9\omega(\varepsilon_0)\varepsilon e^{-\beta S}\frac{e^{\beta S}-1}{\beta} \\
\le\ & \frac{\varepsilon}{2}+\frac{N_8}{\nu_1}e^{-\beta S}\varepsilon+\frac{N_9\omega(\varepsilon_0)}{\beta}\varepsilon \\
\le\ & \frac{5\varepsilon}{6}.
\end{aligned}
$$

This contradiction showes that $|x(t)| < \varepsilon$ holds for all $t > 0$, i.e., the tivial solution of (1.1) is stable.

4 Proof of exponential stability

Now we give the second part of the proof of Theorem 1.1, we show the exponential stability of the trivial solution of (1.1), under the assumptions of Theorem 1.1.

From the stability of the trivial solution there exists $\delta_1 > 0$ so that if $\|\varphi\| < \delta_1$, then any corresponding solution x satisfies $|x(t)| < \varrho$ for $t \ge 0$.

Let $0 < \theta < \beta$ be fixed, and define

$$
\nu_2 = \nu_0 - \theta = -\frac{\log c}{r} - \theta. \tag{4.1}
$$

Then $0 < \nu_1 < \nu_2 < \nu_0$. Let ξ be defined by

$$
\xi(t) = \max_{-r\le u\le t}\Big(e^{\theta u}|x(u)|\Big). \tag{4.2}
$$

We will need the following variant of Proposition 2.3.

Proposition 4.1 *Assume (H1)–(H4). Let x be a solution of the IVP* (1.1)-(1.2) *corresponding to an initial function φ satisfying $\|\varphi\| < \delta_1$, $0 \le t_1$, let $t, t_2 \in [t_1, t_1 + r]$, and let ξ be defined by* (4.2). *Then*

$$
e^{\theta t}|x(t_2)-x(t_1)| \le e^{2\theta r}\Big(2e^{-\nu_2 t_1}+\frac{2M}{1-e^{-\nu_2 r}}(t_2-t_1)+\frac{4Me^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega_4(\eta(t_2))\Big)\xi(t_2). \tag{4.3}
$$

Proof We use the notation of the proof of Proposition 2.3. Relation (2.20) implies

$$
\begin{aligned}
&e^{\theta t}|x(t_2)-x(t_1)| \\
&\quad\le\ e^{\theta t}c^{n+1}\Big(e^{-\theta\alpha_{n+1}(t_2)}e^{\theta\alpha_{n+1}(t_2)}|x(\alpha_{n+1}(t_2))|+e^{-\theta\alpha_{n+1}(t_1)}e^{\theta\alpha_{n+1}(t_1)}|x(\alpha_{n+1}(t_1))|\Big) \\
&\qquad +M\sum_{j=0}^{n}c^j e^{\theta t}\bigg|\int_{\alpha_j(t_1)}^{\alpha_j(t_2)}\Big(\|e^{-\theta(s+\cdot)}e^{\theta(s+\cdot)}x_s\|+e^{-\theta(s-\tau(\mathbf{0}))}e^{\theta(s-\tau(\mathbf{0}))}|x(s-\tau(\mathbf{0}))|\Big)ds\bigg|.
\end{aligned}
$$

Since $-r \le \alpha_{n+1}(t_1) \le t_1 \le t_2$ and $-r \le t_1 - (n+1)r \le t_2 - (n+1)r \le \alpha_{n+1}(t_2) \le t_2$, it follows

$$e^{\theta t}|x(t_2) - x(t_1)| \le e^{\theta(t_1+r)} 2c^{\frac{t_1}{r}} e^{\theta r}\xi(t_2) + 2Me^{\theta r}\xi(t_2)\sum_{j=0}^{n} c^j \left| \int_{\alpha_j(t_1)}^{\alpha_j(t_2)} e^{\theta(t-s)}\, ds \right|.$$

Relation (2.13) yields

$$t - \alpha_j(t_i) = t_i - \alpha_j(t_i) + t - t_i \le \sum_{k=0}^{j-1} \sigma(x_{\alpha_k(t_i)}) + |t - t_i| \le jr + r, \qquad i = 1,2, \quad j = 0,\dots,n,$$

and therefore, using (2.15) and (4.1), we get

$$\begin{aligned}
e^{\theta t}|x(t_2) - x(t_1)| &\le 2e^{2\theta r}e^{-\nu_2 t_1}\xi(t_2) + 2Me^{\theta r}\Big(\sum_{j=0}^{n} c^j e^{\theta(j+1)r}\Big)|\alpha_j(t_2) - \alpha_j(t_1)|\xi(t_2) \\
&\le 2e^{2\theta r}e^{-\nu_2 t_1}\xi(t_2) + 2Me^{2\theta r}\Big(\sum_{j=0}^{n} e^{-j\nu_2 r}\Big(t_2 - t_1 + j2\omega_4(\eta(t_2))\Big)\Big)\xi(t_2) \\
&\le e^{2\theta r}\Big(2e^{-\nu_2 t_1} + \frac{2M}{1-e^{-\nu_2 r}}(t_2 - t_1) + \frac{4Me^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega_4(\eta(t_2))\Big)\xi(t_2).
\end{aligned}$$

□

With an application of (4.3) we generalize Propositions 2.4 and 2.5.

Proposition 4.2 *Assume (H1)–(H4). Let x be a solution of the IVP* (1.1)-(1.2) *corresponding to an initial function φ satisfying $\|\varphi\| < \delta_1$. Then there exist positive constants N_{10}, N_{11}, N_{12} and N_{13} such that*

$$e^{\theta t}|G(t)| \le \Big(N_{10}e^{-\nu_2 t} + N_{11}\omega(\eta(t))\Big)\xi(t), \qquad t \ge 0, \tag{4.4}$$

and

$$e^{\theta t}|F(t)| \le \Big(N_{12}e^{-\nu_2 t} + N_{13}\omega(\eta(t))\Big)\xi(t), \qquad t \ge 0. \tag{4.5}$$

Proof We proceed as in the proof of Proposition 2.4, but we use Proposition 4.1 instead of Proposition 2.3. Consequently we get from (2.24)

$$\begin{aligned}
e^{\theta t}|G(t)| &\le e^{\theta t}\omega_1(|x(t-\sigma(x_t))|)|x(t-\sigma(x_t))| \\
&\quad + 2ce^{2\theta r}\Big(e^{-\nu_2(t-r)} + \frac{M}{1-e^{-\nu_2 r}}|\sigma(x_t) - \sigma(\mathbf{0})| + \frac{2Me^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega_4(\eta(t))\Big)\xi(t) \\
&\le \omega_1(\eta(t))e^{\theta\sigma(x_t)}e^{\theta(t-\sigma(x_t))}|x(t-\sigma(x_t))| \\
&\quad + 2ce^{2\theta r}\Big(e^{-\nu_2(t-r)} + \frac{M + Me^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega_4(\eta(t))\Big)\xi(t) \\
&\le \left(\omega_1(\eta(t))e^{\theta r} + 2ce^{2\theta r}\Big(e^{-\nu_2(t-r)} + \frac{M + Me^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega_4(\eta(t))\Big)\right)\xi(t),
\end{aligned}$$

which yields (4.4).

Similar to the proof of Propositions 2.5 we get

$$\begin{aligned}
e^{\theta t}|F(t)| &\leq \omega_2(\max\{\|x_t\|,|x(t-\tau(x_t))|\})e^{\theta t}\|x_t\| \\
&\quad +\omega_3(\max\{\|x_t\|,|x(t-\tau(x_t))|\})e^{\theta t}|x(t-\tau(x_t))| \\
&\quad +2Me^{2\theta r}\Big(e^{-\nu_2(t-r)}+\frac{M}{1-e^{-\nu_2 r}}|\tau(x_t)-\tau(\mathbf{0})|+\frac{2Me^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega_4(\eta(t))\Big)\xi(t) \\
&\leq \omega(\eta(t))e^{\theta t}\Big(\|e^{-\theta(t+\cdot)}e^{\theta(t+\cdot)}x_t\|+e^{-\theta(t-\tau(x_t))}e^{\theta(t-\tau(x_t))}|x(t-\tau(x_t))|\Big) \\
&\quad +2Me^{2\theta r}\Big(e^{-\nu_2(t-r)}+\frac{M}{1-e^{-\nu_2 r}}\omega_5(\|x_t\|)+\frac{2Me^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega_4(\eta(t))\Big)\xi(t) \\
&\leq \Big(2e^{\theta r}\omega(\eta(t))+2Me^{2\theta r}e^{-\nu_2(t-r)}+2\frac{M^2+M^2e^{-\nu_2 r}}{(1-e^{-\nu_2 r})^2}\omega(\eta(t))\Big)\xi(t),
\end{aligned}$$

therefore (4.5) holds. □

Now let ν_2^* be such that

$$0<\nu_2^*<\nu_2,$$

and introduce the constant

$$\nu_3=-\frac{\log c}{\sigma(\mathbf{0})}-\theta.$$

Then it is easy to see that $0<\nu_1<\nu_2\leq\nu_3$ holds.

Proposition 4.3 *Assume (H1)–(H4). Let x be a solution of the IVP* (1.1)-(1.2) *corresponding to an initial function φ satisfying $\|\varphi\|<\delta_1$. Then there exist positive constants N_{14} and N_{15} such that*

$$e^{\theta t}\sum_{k=0}^{m(t)}c^k|G(t-k\sigma(\mathbf{0}))|\leq\Big(N_{14}e^{-\nu_2^* t}+N_{15}\omega(\eta(t))\Big)\eta(t),\qquad t\geq 0. \tag{4.6}$$

Proof Using (4.4) we get

$$\begin{aligned}
&\sum_{k=0}^{m(t)}c^k e^{\theta k\sigma(\mathbf{0})}e^{\theta(t-k\sigma(\mathbf{0}))}|G(t-k\sigma(\mathbf{0}))| \\
&\leq \sum_{k=0}^{m(t)}c^k e^{\theta k\sigma(\mathbf{0})}\Big(N_{10}e^{-\nu_2(t-k\sigma(\mathbf{0}))}+N_{11}\omega(\eta(t-k\sigma(\mathbf{0})))\Big)\xi(t-k\sigma(\mathbf{0})) \\
&\leq \sum_{k=0}^{m(t)}e^{-k\nu_3\sigma(\mathbf{0})}\Big(N_{10}e^{-\nu_2(t-k\sigma(\mathbf{0}))}+N_{11}\omega(\eta(t))\Big)\xi(t) \\
&\leq \Big(N_{10}e^{-\nu_2 t}\sum_{k=0}^{m(t)}e^{-k(\nu_3-\nu_2)\sigma(\mathbf{0})}+N_{15}\omega(\eta(t))\Big)\eta(t),\qquad t\geq 0,
\end{aligned}$$

where $N_{15} = \frac{N_{11}}{1-e^{-\nu_3\sigma(\mathbf{0})}}$. The existence of N_{14} can be argued as in the proof of Proposition 3.1. $\square$

We will specify $T_1 > 0$ later. We multiply both sides of (3.1) by $e^{\theta t}$ and using relations (2.8), (2.10) and (3.2) we get for $t \geq T_1$

$$\begin{aligned}
e^{\theta t}|x(t)| &\leq N_7 e^{-(\alpha-\theta)t}\|\varphi\| + \sum_{k=0}^{m(t)} c^k e^{\theta t}|G(t-k\sigma(\mathbf{0}))| \\
&\quad + e^{\theta t}\int_0^{T_1} K_3 e^{-\beta(t-s)}|G(s)|\,ds + e^{\theta t}\int_0^{T_1} K_2 e^{-\alpha(t-s)}|F(s)|\,ds \\
&\quad + e^{\theta t}\int_{T_1}^{t} K_3 e^{-\beta(t-s)}e^{-\theta s}e^{\theta s}|G(s)|\,ds + e^{\theta t}\int_{T_1}^{t} K_2 e^{-\alpha(t-s)}e^{-\theta s}e^{\theta s}|F(s)|\,ds.
\end{aligned}$$

For $s \in [0, T_1]$ we use estimates (2.22) and (2.26) combined with Proposition 2.1, so we get

$$|G(s)| \leq 2cN_1 e^{\gamma T_1}\|\varphi\| \quad \text{and} \quad |F(s)| \leq 4MN_1 e^{\gamma T_1}\|\varphi\|, \qquad s \in [0, T_1].$$

Then, using the fact that $\beta \leq \alpha$ and applying Proposition 4.2, we can find constants $N_{16} = N_{16}(T_1)$, N_{17} and N_{18} such that

$$\begin{aligned}
e^{\theta t}|x(t)| &\leq N_7\|\varphi\| + \Big(N_{14}e^{-\nu_2^* t} + N_{15}\omega(\eta(t))\Big)\xi(t) + e^{-(\beta-\theta)t}\int_0^{T_1} e^{\beta s}N_{16}\|\varphi\|\,ds \\
&\quad + e^{-(\beta-\theta)t}\int_{T_1}^{t} e^{(\beta-\theta)s}\Big(N_{17}e^{-\nu_2 s} + N_{18}\omega(\eta(s))\Big)\xi(s)\,ds, \qquad t \geq T_1. \qquad (4.7)
\end{aligned}$$

Let $T_1 > 0$ be such that

$$N_{14}e^{-\nu_2^* T_1} < \frac{1}{6} \qquad \text{and} \qquad N_{17}e^{-\nu_2 T_1} < \frac{1}{6}(\beta-\theta).$$

Since the trivial solution of (1.1) is stable, there exists δ_2 such that

$$N_{15}\omega(\eta(t)) < \frac{1}{6}, \qquad \text{and} \qquad N_{18}\omega(\eta(t)) < \frac{1}{6}(\beta-\theta), \qquad t \geq 0$$

for $\|\varphi\| < \delta_2$.

Let $\delta = \min(\delta_1, \delta_2)$. Then, for $\|\varphi\| < \delta$ and $t \geq T_1$, (4.7) yields

$$\begin{aligned}
e^{\theta t}|x(t)| &\leq N_7\|\varphi\| + \frac{1}{3}\xi(t) + \frac{e^{\beta T_1}N_{16}}{\beta}\|\varphi\| + \frac{\beta-\theta}{3}\xi(t)e^{-(\beta-\theta)t}\int_{T_1}^{t} e^{(\beta-\theta)s}\,ds \\
&\leq \Big(N_7 + \frac{e^{\beta T_1}N_{16}}{\beta}\Big)\|\varphi\| + \frac{2}{3}\xi(t), \qquad t \geq T_1.
\end{aligned}$$

For $t \in [0, T_1]$ we use Proposition 2.1 to get

$$e^{\theta t}|x(t)| \leq e^{\theta T_1}N_1 e^{\gamma T_1}\|\varphi\|, \qquad \|\varphi\| < \delta.$$

Therefore, if we define

$$K = \max\left(3\Big(N_7 + \frac{e^{\beta T_1} N_{16}}{\beta}\Big), 3e^{\theta T_1} N_1 e^{\gamma T_1}\right),$$

then

$$e^{\theta t}|x(t)| \le \frac{1}{3}K\|\varphi\| + \frac{2}{3}\xi(t), \qquad t \ge 0$$

is satisfied. Since $N_7 > 1$, and therefore $K > 3$, and the right-hand-side is monotone increasing in t, it follows (see, e.g., Lemma 2.1 in [17])

$$\xi(t) \quad \le \quad \frac{1}{3}K\|\varphi\| + \frac{2}{3}\xi(t), \qquad t \ge 0,$$

and so

$$\xi(t) \le K\|\varphi\|, \qquad t \ge 0.$$

Consequently

$$|x(t)| \le Ke^{-\theta t}\|\varphi\|, \qquad t \ge 0, \quad \|\varphi\| < \delta.$$

This concludes the proof of Theorem 1.1.

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