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# Differentiability of Solutions with respect to Parameters in Neutral Differential Equations with State-Dependent Delays*

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**Abstract**

In this paper we consider a class of nonlinear neutral differential equations with state-dependent delays. We study well-posedness and continuous dependence issues and differentiability of the parameter map with respect to the initial function and other possibly infinite dimensional parameters in a pointwise sense and also in the $C$ and $W^{1,\infty}$-norms.

## 1 Introduction

In this paper we consider state-dependent neutral functional differential equations (SD-NFDEs) of the form

$$\frac{d}{dt}\Big(x(t) - g(t, x(t-\eta(t)))\Big) = f\Big(t, x_t, x(t-\tau(t, x_t, \sigma)), \theta\Big) \qquad t \in [0, T], \tag{1.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{1.2}$$

Here $\theta \in \Theta$ and $\sigma \in \Sigma$ represent parameters in the function $f$ and in the delay function $\tau$, where $\Theta$ and $\Sigma$ are normed linear spaces with norms $|\cdot|_\Theta$ and $|\cdot|_\Sigma$, respectively. The solution segment function $x_t$ is defined by $x_t(s) = x(t+s)$, $s \in [-r, 0]$. (See Section 2 below for the detailed assumptions on the initial value problem (IVP) (1.1)-(1.2).)

The study of state-dependent delay differential equations (SD-DDEs), i.e., the case when $g \equiv 0$ in (1.1) is an active research area. We refer to [22] for a recent survey on this topic with more than 220 references. In spite of that one of the first model appeared in the literature with state-dependent delays, the mathematical model for a two-body problem of classical electrodynamics introduced by Driver [8, 9, 10] involves NFDEs with state-dependent delays, much less work is devoted to SD-NFDEs [1, 3, 4, 6, 12, 21, 26, 34, 35]. Most of the above papers deal with SD-NFDEs of the form

$$x'(t) = h\Big(t, x(t), x(t-\tau(t, x(t))), x'(t-\sigma(t, x(t)))\Big). \tag{1.3}$$

This equation is called in [29] as "explicit" SD-NFDE contrary to the "implicit" SD-NFDE (1.1). Well-posedness of such "explicit" SD-NFDEs was investigated in [11, 25].

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Our equation (1.1) can be considered as a natural "generalization" of the "usual" NFDEs with time-dependent delays of the form

$$\frac{d}{dt}D(t, x_t) = f(t, x_t), \tag{1.4}$$

but (1.4) may also contain (1.1) depending on appropriate conditions on $D$ and $f$, like in [22]. The basic well-posedness theory for several classes of (1.4), especially for the linear case, is well-developed [2, 5, 14, 16, 31]. Existence, uniqueness and numerical approximation of solutions was studied in [20] for SD-NFDEs of the form

$$\frac{d}{dt}\Big(x(t) - q(t)x(t - \sigma(t, x(t)))\Big) = f\Big(t, x(t), x(t - \tau(t, x(t)))\Big),$$

and numerical approximation issues were discussed in [29] for such equations.

Differentiability of solutions with respect to (wrt) parameters is an important qualitative question, but it also has natural application in the problem of identification of parameters (see [19]). But even for simple constant delay equations this problem leads to technical difficulties if the parameter is the delay. Namely, at some point in the proof it is needed to compute the derivative of a composite function when the outer function is not differentiable, it is only Lipschitz continuous. To overcome these technicalities, Hale and Ladeira [15] proved differentiability of the solution wrt the delay using the $W^{1,1}$ norm on the state space of solutions. The $W^{1,p}$-norm of a function $\psi\colon [-r, 0] \to \mathbb{R}^n$ is defined by $|\psi|_{W^{1,p}} = \Big(\int_{-r}^{0} |\psi(s)|^p + |\dot\psi(s)|^p\, ds\Big)^{1/p}$, $1 \le p < \infty$.

The same difficulty arises in SD-DDEs. The case when the solution corresponding to a parameter is continuously differentiable can be treated relatively easily, and it was investigated in [18]. Related is the work of Walther [32, 33], where the well-posedness of autonomous SD-DDEs is obtained restricting the state-space of solutions to the space of continuously differentiable functions. Walther also obtained differentiability of the solution with respect to the initial function in this space. Differentiability of solutions of SD-DDEs wrt parameters under less restrictive conditions was investigated in [23] in the case when the solutions are not continuously differentiable, only Lipschitz continuous functions. In this case differentiability wrt the parameters was obtained in the $W^{1,p}$ norm.

The organization of the paper is the following. In Section 2 we introduce some notations, assumptions and formulate some basic results will be used in the rest of the paper. In Section 3 we discuss well-posedness of the IVP (1.1)-(1.2), and then in Section 4, using and improving the method of [18] applied for SD-DDEs, we study differentiability of solutions wrt parameters for the IVP (1.1)-(1.2) in a pointwise-sense and using the $C$ and $W^{1,\infty}$-norms on the state-space.

The parameters we consider in this paper are restricted to the initial function and other parameters in the function $f$ and in the delay function $\tau$. The dependence of the solution on $\eta$ (or some parts of $\eta$) is somewhat more complicated. A simple example was given in [28] to show that a solution may not be differentiable wrt $\eta$ even when both delays are constants.

Note that for simplicity we present our results for the single delay case, but all our results can be easily extended to the multiple delay case.

## 2 Notations, assumptions and preliminaries

Throughout this paper a fixed norm on $\mathbb{R}^n$ and the corresponding matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$.

In a normed linear space $(X, |\cdot|_X)$ the open ball around a point $x_0$ with radius $R$ is denoted by $\mathcal{B}_X(x_0; R)$, i.e., $\mathcal{B}_X(x_0; R) = \{x \in X : |x - x_0|_X < R\}$, and the corresponding closed ball by $\overline{\mathcal{B}}_X(x_0; R)$. Similarly, an open neighborhood of a set $M \subset X$ with radius $R$ is denoted by $\mathcal{B}_X(M; R)$, i.e., $\mathcal{B}_X(M; R) = \{x \in X : \text{there exists } y \in M \text{ such that } |x - y|_X < R\}$. The closure of this neighborhood is denoted by $\overline{\mathcal{B}}_X(M; R)$.

The space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ and the usual supremum norm on it are denoted by $C$ and $|\cdot|_C$, respectively. The $L^\infty$-norm of an absolutely continuous function $\psi: [-r, 0] \to \mathbb{R}^n$ is defined by $|\psi|_{L^\infty} = \operatorname{ess\,sup}\{|\dot{\psi}(s)| : s \in [-r, 0]\}$. The space of absolutely continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ with essentially bounded derivatives is denoted by $W^{1,\infty}$. The corresponding norm on $W^{1,\infty}$ is $|\psi|_{W^{1,\infty}} = \max\{|\psi|_C, |\dot{\psi}|_{L^\infty}\}$.

The space of bounded linear operators between normed linear spaces $X$ and $Y$ is denoted by $\mathcal{L}(X, Y)$, and the norm on it is $|\cdot|_{\mathcal{L}(X,Y)}$.

The partial derivatives of a function $F(x_1, x_2, \ldots, x_n)$ wrt its first, second, etc. arguments are denoted by $D_1F$, $D_2F$, etc., and the derivative of a single variable function $v(t)$ wrt $t$ is denoted by $\dot{v}$. Note that all derivatives we use in this paper are Fréchet derivatives.

Next we list our assumptions on the SD-NFDE (1.1) and the associated initial function we will use throughout this paper.

Let $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset \Sigma$, and $\Omega_5 \subset \mathbb{R}^n$ be open subsets of the respective spaces. $T > 0$ is finite or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$. We assume:

(A1) (i) $f\colon \Big([0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \subset \mathbb{R} \times C \times \mathbb{R}^n \times \Theta\Big) \to \mathbb{R}^n$ is continuous,

(ii) $f(t, \psi, u, \theta)$ is locally Lipschitz continuous in $\psi$, $u$ and $\theta$ in the following sense: for every finite $\alpha \in (0, T]$, for every compact subsets $M_1 \subset \Omega_1$ and $M_2 \subset \Omega_2$ of $C$ and $\mathbb{R}^n$, respectively, and for every closed and bounded subset $M_3 \subset \Omega_3$ of $\Theta$ there exists a constant $L_1 = L_1(\alpha, M_1, M_2, M_3)$ such that

$$|f(t, \psi, u, \theta) - f(t, \bar{\psi}, \bar{u}, \bar{\theta})| \leq L_1\Big(|\psi - \bar{\psi}|_C + |u - \bar{u}| + |\theta - \bar{\theta}|_\Theta\Big),$$

for $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_1$, $u, \bar{u} \in M_2$ and $\theta, \bar{\theta} \in M_3$,

(iii) $f$ is continuously differentiable wrt its second, third and fourth variables;

(A2) (i) $\tau\colon \Big([0, T] \times \Omega_1 \times \Omega_4 \subset \mathbb{R} \times C \times \Sigma\Big) \to \mathbb{R}$ is continuous, and

$$0 \leq \tau(t, \psi, \sigma) \leq r, \qquad \text{for } t \in [0, T],\ \psi \in \Omega_1 \text{ and } \sigma \in \Omega_4,$$

(ii) $\tau(t, \psi, \sigma)$ is locally Lipschitz continuous in $\psi$ and $\sigma$ in the following sense: for every finite $\alpha \in (0, T]$, for every compact subset $M_1 \subset \Omega_1$ of $C$, and for every closed and bounded subset $M_4 \subset \Omega_4$ of $\Sigma$ there exists a constant $L_2 = L_2(\alpha, M_1, M_4)$ such that

$$|\tau(t, \psi, \sigma) - \tau(t, \bar{\psi}, \bar{\sigma})| \leq L_2\Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma\Big)$$

for $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_1$ and $\sigma, \bar{\sigma} \in M_4$,

(iii) $\tau$ is continuously differentiable wrt its second and third variables,

(iv) $D_2\tau(t,\psi,\sigma)$ and $D_3\tau(t,\psi,\sigma)$ are locally Lipschitz continuous in $\psi$ and $\sigma$, i.e., for every finite $\alpha \in (0,T]$, for every compact subset $M_1 \subset \Omega_1$ of $C$, and for every closed and bounded subset $M_4 \subset \Omega_4$ of $\Sigma$ there exists $L_3 = L_3(\alpha, M_1, M_4)$ such that

$$\left|D_2\tau(t,\psi,\sigma) - D_2\tau(t,\bar\psi,\bar\sigma)\right|_{\mathcal{L}(C,\mathbb{R})} \le L_3\Big(|\psi-\bar\psi|_C + |\sigma-\bar\sigma|_\Sigma\Big),$$

and

$$\left|D_3\tau(t,\psi,\sigma) - D_3\tau(t,\bar\psi,\bar\sigma)\right|_{\mathcal{L}(\Sigma,\mathbb{R})} \le L_3\Big(|\psi-\bar\psi|_C + |\sigma-\bar\sigma|_\Sigma\Big)$$

hold for all $t \in [0,\alpha]$, $\psi, \bar\psi \in M_1$ and $\sigma, \bar\sigma \in M_4$;

(A3) $\varphi \in W^{1,\infty}$;

(A4) (i) $g\colon \Big([0,T]\times\Omega_5 \subset \mathbb{R}\times\mathbb{R}^n\Big) \to \mathbb{R}^n$ is continuously differentiable wrt its both variables,

(ii) $D_1 g$ and $D_2 g$ are locally Lipschitz continuous, i.e., for every $\alpha \in (0,T]$ and compact subset $M_5 \subset \Omega_5$ of $\mathbb{R}^n$ there exists $L_4 = L_4(\alpha, M_5)$ such that $|D_i g(t,u) - D_i g(t,\bar u)| \le L_4|u-\bar u|$ for $i = 1,2$, $t \in [0,\alpha]$ and $u, \bar u \in M_5$,

(iii) $D_1 g$ and $D_2 g$ are continuously differentiable wrt their second variables;

(A5) (i) $\eta\colon [0,T] \to \mathbb{R}$ is continuous,

(ii) there exists a positive constant $\eta_0$ such that $0 < \eta_0 \le \eta(t) \le r$ for $t \in [0,T]$, and

(iii) $\eta$ is locally Lipschitz continuous on $[0,T]$, i.e., for every finite $\alpha \in (0,T]$ there exists $L_5 = L_5(\alpha)$ such that $|\eta(t) - \eta(\bar t)| \le L_5|t-\bar t|$ for $t, \bar t \in [0,\alpha]$,

(iv) $\dot\eta(0+)$ exists.

Note that assumptions (A1)–(A3) are identical to those used in [23] for SD-DDEs, i.e., for the case when $g \equiv 0$. (See also [7] or [23] for well-posedness of SD-DDEs.) We refer to [23] for further comments on the particular definition of local Lipschitz continuity we use in (A1) (ii) and (A2) (ii).

We introduce the following function:

$$\Lambda : \Big([0,T]\times\Omega_1\times\Omega_4 \subset \mathbb{R}\times C\times\Sigma\Big) \to \mathbb{R}^n, \quad \Lambda(t,\psi,\sigma) = \psi(-\tau(t,\psi,\sigma)). \tag{2.1}$$

With this notation we can rewrite (1.1) simply as:

$$\frac{d}{dt}\Big(x(t) - g(t, x(t-\eta(t)))\Big) = f(t, x_t, \Lambda(t, x_t, \sigma), \theta), \qquad t \in [0,T].$$

Let $\alpha > 0$, $M_1 \subset \Omega_1$ be a compact subset of $C$, $M_4 \subset \Omega_4$ be a closed and bounded subset of $\Sigma$, and $L_2 = L_2(\alpha, M_1, M_4)$ be the constant from (A2) (ii). It follows from the definition of $\Lambda$, (A2) (ii) and the Mean Value Theorem that

$$\begin{aligned}
|\Lambda(t,\psi,\sigma) - \Lambda(t,\bar\psi,\bar\sigma)| &\le |\bar\psi(-\tau(t,\psi,\sigma)) - \bar\psi(-\tau(t,\bar\psi,\bar\sigma))| + |\psi(-\tau(t,\psi,\sigma)) - \bar\psi(-\tau(t,\psi,\sigma))| \\
&\le L_2|\bar\psi|_{W^{1,\infty}}(|\psi-\bar\psi|_C + |\sigma-\bar\sigma|_\Sigma) + |\psi-\bar\psi|_C
\end{aligned} \tag{2.2}$$

for $t \in [0,\alpha]$, $\psi, \bar\psi \in M_1$, $\bar\psi \in W^{1,\infty}$ and $\sigma, \bar\sigma \in M_4$.

We define the parameter space $\Gamma = W^{1,\infty}\times\Sigma\times\Theta$, and use the notation $\gamma = (\varphi,\sigma,\theta)$ or $\gamma = (\gamma^\varphi, \gamma^\sigma, \gamma^\theta)$ for the components of $\gamma \in \Gamma$, and $|\gamma|_\Gamma = |\varphi|_{W^{1,\infty}} + |\sigma|_\Sigma + |\theta|_\Theta$ for the norm on $\Gamma$.

The solution of the IVP (1.1)-(1.2) corresponding to a parameter $\gamma$ and its segment function at $t$ are denoted by $x(t;\gamma)$ and $x(\cdot;\gamma)_t$, respectively.

Introduce the set of feasible parameters

$$\Pi = \Big\{(\varphi,\sigma,\theta)\in\Gamma\colon\ \varphi\in\Omega_1,\quad \varphi(-\tau(0,\varphi,\sigma))\in\Omega_2,\quad \theta\in\Omega_3,\quad \sigma\in\Omega_4,\quad \varphi(-\eta(0))\in\Omega_5\Big\},$$

and define the parameter set

$$\mathcal{M} = \Big\{(\varphi,\sigma,\theta)\in\Pi:\quad \varphi\in C^1,\quad \dot\varphi(0-) = D_1 g(0,\varphi(-\eta(0))) + D_2 g(0,\varphi(-\eta(0)))\dot\varphi(-\eta(0))(1-\dot\eta(0+)) + f(0,\varphi,\Lambda(0,\varphi,\sigma),\theta)\Big\}.$$

Note that analogous conditions were used for neutral FDEs in order to guarantee the existence of a continuous semiflow on a subset of $C^1$ in [27].

In the next lemma we formalize a method used frequently in functional inequalities (see, e.g., in [13]) and which will be used in the sequel, as well.

**Lemma 2.1** *Suppose $g\colon [0,\alpha]\times[0,\infty)^3\to[0,\infty)$ is monotone increasing in all variables, i.e., if $0\le t_i\le s_i$ for $i=1,2,3,4$, then $g(t_1,t_2,t_3,t_4)\le g(s_1,s_2,s_3,s_4)$; $\lambda\colon [0,\alpha]\to[0,\infty)$ is such that $\lambda_0\le\lambda(t)$ for $t\in[0,\alpha]$ for some $\lambda_0>0$; $u\colon [-r,\alpha]\to[0,\infty)$ is such that*

$$u(t)\le g(t,u(t),u(t-\lambda(t)),|u_t|_C),\qquad t\in[0,\alpha], \tag{2.3}$$

*and*

$$|u_0|_C\le g(0,u(0),u(-\lambda(0)),|u_0|_C). \tag{2.4}$$

*Then*

$$v(t)\le g(t,v(t),v(t-\lambda_0),v(t)),\qquad t\in[0,\alpha], \tag{2.5}$$

*where $v(t)=\sup\{u(s)\colon\ s\in[-r,t]\}$.*

**Proof** It follows from (2.3), $\lambda_0\le\lambda(t)$, the definition of $v(t)$ and the monotonicity of $g$ that if $0\le s\le t\le\alpha$, then

$$\begin{aligned} u(s) &\le g(s,u(s),u(s-\lambda(s)),|u_s|_C)\\ &\le g(s,v(s),v(s-\lambda_0),v(s))\\ &\le g(t,v(t),v(t-\lambda_0),v(t)). \end{aligned}$$

Then taking the supremum of the left hand side for $s\in[0,t]$ we get

$$\sup\{u(s)\colon\ s\in[0,t]\}\le g(t,v(t),v(t-\lambda_0),v(t)).$$

This, combined with (2.4), implies (2.5). □

Finally, we recall the following two results which will be used later.

**Lemma 2.2 (see [13])** *Let $a > 0$, $b \ge 0$, $r_1 > 0$, $r_2 \ge 0$, $r = \max\{r_1, r_2\}$, and $v\colon [0,\alpha] \to [0,\infty)$ be continuous and nondecreasing. Let $u\colon [-r,\alpha] \to [0,\infty)$ be continuous and satisfy the inequality*

$$u(t) \le v(t) + bu(t-r_1) + a\int_0^t u(s-r_2)\,ds, \qquad t \in [0,\alpha].$$

*Then $u(t) \le d(t)e^{ct}$ for $t \in [0,\alpha]$, where $c$ is the unique positive solution of $cbe^{-cr_1} + ae^{-cr_2} = c$, and*

$$d(t) = \max\left\{\frac{v(t)}{1-be^{-cr_1}}, \max_{-r\le s\le 0} e^{-cs}u(s)\right\}, \qquad t \in [0,\alpha].$$

**Lemma 2.3 (see, e.g., [30])** *Suppose that $X$ and $Y$ are normed linear spaces, and $U$ is an open subset of $X$, and $F : U \to Y$ is differentiable. Let $x, y \in U$ such that $y + \nu(x-y) \in U$ for $\nu \in [0,1]$. Then*

$$|F(y) - F(x) - F'(x)(y-x)|_Y \le |x-y|_X \sup_{0<\nu<1} |F'(y+\nu(x-y)) - F'(x)|_{\mathcal{L}(X,Y)}.$$

## 3 Well-posedness and continuous dependence on parameters

In this section we show that under the assumptions listed in the previous section the IVP (1.1)-(1.2) has a unique solution which depends continuously on the parameters $\varphi$, $\sigma$ and $\theta$ in the $W^{1,\infty}$-norm.

By a solution of the IVP (1.1)-(1.2) we mean a continuous function defined on an interval $[-r,\alpha]$, such that (i) $t \mapsto x(t) - g(t, x(t-\eta(t)))$ is differentiable for $t \in [0,\alpha]$, (at the ends of the interval one sided derivatives exist); (ii) $x$ satisfies (1.1) for $t \in [0,\alpha]$, and (iii) $x$ satisfies the initial condition (1.2).

**Theorem 3.1** *Assume (A1) (i), (ii), (A2) (i), (ii), (A3), (A4) (i), (ii) and (A5) (i)–(iii), and let $\bar\gamma \in \Pi$. Then there exist $\delta > 0$ and $0 < \alpha \le T$ finite numbers such that*

*(i)* $\overline{\mathcal{B}}_\Gamma(\bar\gamma;\,\delta) \subset \Pi$;

*(ii) the IVP* (1.1)-(1.2) *has a unique solution $x(t;\gamma)$ on $[0,\alpha]$ for all $\gamma \in \mathcal{B}_\Gamma(\bar\gamma;\,\delta)$;*

*(iii) there exist $M_1 \subset \Omega_1$, $M_2 \subset \Omega_2$ and $M_5 \subset \Omega_5$, compact subsets of $C$ and $\mathbb{R}^n$, respectively, and $M_3 \subset \Omega_3$ and $M_4 \subset \Omega_4$ closed and bounded subsets of $\Theta$ and $\Sigma$, respectively, such that*

$$x(\cdot;\gamma)_t \in M_1, \quad \Lambda(t, x(\cdot;\gamma)_t, \sigma) \in M_2, \quad \theta \in M_3, \quad \sigma \in M_4, \quad \text{and} \quad x(t-\eta(t);\gamma) \in M_5 \tag{3.1}$$

*for $t \in [0,\alpha]$, $\gamma = (\varphi,\sigma,\theta) \in \mathcal{B}_\Gamma(\bar\gamma;\,\delta)$;*

*(iv) $x(\cdot;\gamma)_t \in W^{1,\infty}$ for $t \in [0,\alpha]$, $\gamma \in \mathcal{B}_\Gamma(\bar\gamma;\,\delta)$, and there exists $L = L(\alpha,\delta)$, such that*

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_{W^{1,\infty}} \le L|\gamma - \bar\gamma|_\Gamma \qquad \text{for } t \in [0,\alpha],\ \gamma \in \mathcal{B}_\Gamma(\bar\gamma;\,\delta). \tag{3.2}$$

*(v) Moreover assume (A5) (iv). Then the function $x(\cdot;\gamma)\colon [-r,\alpha] \to \mathbb{R}^n$ is continuously differentiable for $\gamma \in \mathcal{M} \cap \mathcal{B}_\Gamma(\bar\gamma;\,\delta)$.*

**Proof** Since $\bar\varphi \in \Omega_1$ and $\Omega_1$ is open in $C$, there exists $\varepsilon_1^* > 0$ such that $\overline{\mathcal{B}}_C(\bar\varphi;\, \varepsilon_1^*) \subset \Omega_1$, therefore $\overline{\mathcal{B}}_{W^{1,\infty}}(\bar\varphi;\, \varepsilon_1^*) \subset \Omega_1$, as well. $\Omega_3$ and $\Omega_4$ are open sets of their respective spaces, hence there exist $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ such that $M_3 := \overline{\mathcal{B}}_\Theta\big(\bar\theta;\, \varepsilon_3\big) \subset \Omega_3$ and $M_4 := \overline{\mathcal{B}}_\Sigma(\bar\sigma;\, \varepsilon_4) \subset \Omega_4$. It follows from the definition of $\Pi$ that $\bar\varphi(-\tau(0,\bar\varphi,\bar\sigma)) \in \Omega_2$, moreover, $\Omega_2$ is open in $\mathbb{R}^n$, so there exists $\varepsilon_2^* > 0$ such that $\overline{\mathcal{B}}_{\mathbb{R}^n}(\bar\varphi(-\tau(0,\bar\varphi,\bar\sigma));\, \varepsilon_2^*) \subset \Omega_2$. We have that $\overline{\mathcal{B}}_{W^{1,\infty}}(\bar\varphi;\, \varepsilon_1^*)$ is compact in $C$ by Arsela-Ascoli's Theorem since it is a bounded subset of $W^{1,\infty}$. Let $L_2^*$ be the Lipschitz constant from (A2) (ii) corresponding to any $\alpha > 0$ and to the sets $\overline{\mathcal{B}}_{W^{1,\infty}}(\bar\varphi;\, \varepsilon_1^*)$ and $M_4$. Then applying (2.2) it follows

$$\begin{aligned}|\varphi(-\tau(0,\varphi,\sigma)) - \bar\varphi(-\tau(0,\bar\varphi,\bar\sigma))| &\le |\varphi-\bar\varphi|_C + L_2^*|\bar\varphi|_{W^{1,\infty}}(|\varphi-\bar\varphi|_C + |\sigma-\bar\sigma|_\Sigma)\\ &\le (L_2^*|\bar\varphi|_{W^{1,\infty}}+1)(|\varphi-\bar\varphi|_{W^{1,\infty}} + |\sigma-\bar\sigma|_\Sigma)\end{aligned}$$

for $\varphi, \bar\varphi \in \overline{\mathcal{B}}_{W^{1,\infty}}(\bar\varphi;\, \varepsilon_1^*)$ and $\sigma, \bar\sigma \in M_4$. Let $\varepsilon_5^* > 0$ be such that $\overline{\mathcal{B}}_{\mathbb{R}^n}(\bar\varphi(-\eta(0));\, \varepsilon_5^*) \subset \Omega_5$. From the assumed continuity of $\eta$ it follows that there exists $0 < \alpha_0 \le \eta_0$ such that $|\bar\varphi(t-\eta(t)) - \bar\varphi(-\eta(0))| < \varepsilon_5^*/2$ for $t \in [0,\alpha_0]$. Then $t - \eta(t) \le 0$ and $\varphi(t-\eta(t)) \in \Omega_5$ for $t \in [0,\alpha_0]$, and $\varphi \in \mathcal{B}_{W^{1,\infty}}(\bar\varphi;\, \varepsilon_5^*/2)$. Let

$$\delta_1 = \min(\varepsilon_1^*, \varepsilon_2^*/(L_2^*|\bar\varphi|_{W^{1,\infty}}+1), \varepsilon_3, \varepsilon_4, \varepsilon_5^*/2).$$

Then (i) holds with $\delta = \delta_1$.

Fix $\gamma = (\varphi,\sigma,\theta) \in \overline{\mathcal{B}}_\Gamma(\bar\gamma;\, \delta_1)$. We can use the method of steps to show that the IVP (1.1)-(1.2) corresponding to $\gamma$ has a unique solution. By assumption (A5) (ii) Equation (1.1) is equivalent to

$$\frac{d}{dt}\Big(x(t) - \lambda(t)\Big) = f(t, x_t, \Lambda(t,x_t,\sigma), \theta) \qquad \text{for } t \in [0,\eta_0],$$

where $\lambda(t) = g(t, \varphi(t-\eta(t)))$. Assumptions (A3), (A4) (i) and (A5) (iii) yield that $\lambda$ is Lipschitz continuous, therefore it is also a.e. differentiable. Then $x$ is also a.e. differentiable, since $x - \lambda$ is differentiable. Hence it is easy to see that the equation is equivalent to the SD-DDE

$$\dot x(t) = \dot\lambda(t) + f(t, x_t, \Lambda(t,x_t,\sigma), \theta), \qquad \text{a.e. } t \in [0,\eta_0]. \tag{3.3}$$

It follows from an obvious generalization of a result in [24] (see also [17]) that the IVP (3.3)-(1.2) has a unique solution on an interval $[-r,\alpha_1]$, $\alpha_1 \le \eta_0$. If $\alpha_1 = \eta_0$, repeating the previous step we can extend the solution to $[\alpha_1,\alpha_2]$ with $\alpha_2 \in [\eta_0, 2\eta_0]$, and so on. We get that the IVP (1.1)-(1.2) corresponding to parameter $\gamma$ has a unique solution $x(t;\gamma)$ for $t \in [-r,\tilde\alpha]$ for some $\tilde\alpha = \tilde\alpha(\gamma) > 0$. Moreover, the above method of steps argument yields easily that $x_t \in W^{1,\infty}$ for $t \in [0,\tilde\alpha]$. We will show that $\tilde\alpha(\gamma)$ can be selected independently of $\gamma$ if $0 < \delta \le \delta_1$ is small enough.

Let $\bar\gamma = (\bar\varphi,\bar\sigma,\bar\theta) \in \Pi$, and $x(t;\bar\gamma)$ be the corresponding solution of the IVP (1.1)-(1.2) on an interval $[-r,\alpha]$ for some $\alpha > 0$. Define $M_1^* = \{x(\cdot;\bar\gamma)_t : t \in [0,\alpha]\}$, $M_2^* = \{\Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma) : t \in [0,\alpha]\}$, and $M_5^* = \{x(t-\eta(t);\bar\gamma) : t \in [0,\alpha]\}$. Clearly $M_i^* \subset \Omega_i$ $(i = 1,2,5)$. Moreover, $M_1^*$, $M_2^*$ and $M_5^*$ are compact subsets of $C$ and $\mathbb{R}^n$, respectively, since $t \mapsto x(\cdot;\bar\gamma)_t$, $t \mapsto \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma)$ and $t \mapsto x(t-\eta(t);\bar\gamma)$ are continuous functions on $[0,\alpha]$. Therefore there exist $\varepsilon_i > 0$ $(i = 1,2,5)$ such that $\overline{\mathcal{B}}_C(M_1^*;\, \varepsilon_1) \subset \Omega_1$, $M_2 := \overline{\mathcal{B}}_{\mathbb{R}^n}(M_2^*;\, \varepsilon_2) \subset \Omega_2$, and $M_5 := \overline{\mathcal{B}}_{\mathbb{R}^n}(M_5^*;\, \varepsilon_5) \subset \Omega_5$, since $\Omega_i$ $(i = 1,2,5)$ are open sets in $C$ and $\mathbb{R}^n$, respectively. Clearly, $M_2$ and $M_5$ are compact subsets of $\mathbb{R}^n$. Let $M_1 = \overline{\mathcal{B}}_{W^{1,\infty}}(M_1^*;\, \varepsilon_1)$. We have $M_1 \subset \Omega_1$, and it is compact in $C$ by Arsela-Ascoli's Theorem.

Let $L_2 = L_2(\alpha, M_1, M_4)$ be the constant from (A2) (ii), and define

$$\delta_2 = \min\{\delta_1, \varepsilon_1, \varepsilon_2/(L_2|\bar\varphi|_{W^{1,\infty}}+1), \varepsilon_5\}.$$

Let $\gamma = (\varphi, \sigma, \theta) \in \mathcal{B}_\Gamma(\bar{\gamma}; \delta_2)$. Then, clearly, $\theta \in M_3$ and $\sigma \in M_4$. We have from (2.2) and the definition of $|\cdot|_\Gamma$ that $|\varphi - \bar{\varphi}|_C \le |\varphi - \bar{\varphi}|_{W^{1,\infty}} < \varepsilon_1$, $|\Lambda(0, \varphi, \sigma) - \Lambda(0, \bar{\varphi}, \bar{\sigma})| \le L_2 |\bar{\varphi}|_{W^{1,\infty}}(|\varphi - \bar{\varphi}|_C + |\sigma - \bar{\sigma}|_\Sigma) + |\varphi - \bar{\varphi}|_C < \varepsilon_2$, and $|\varphi(-\eta(0)) - \bar{\varphi}(-\eta(0))| < \varepsilon_5$. Therefore there exists $0 < \alpha^\gamma \le \alpha$ such that

$$|x(\cdot;\gamma)_t - x(\cdot;\bar{\gamma})_t|_C < \varepsilon_1, \qquad |\Lambda(t, x(\cdot;\gamma)_t, \sigma) - \Lambda(t, x(\cdot;\bar{\gamma})_t, \bar{\sigma})| < \varepsilon_2 \tag{3.4}$$

and

$$|x(t - \eta(t); \gamma) - x(t - \eta(t); \bar{\gamma})| < \varepsilon_5 \tag{3.5}$$

for $t \in [0, \alpha^\gamma]$.

Let $L_1 = L_1(\alpha, M_1, M_2, M_3)$ and $L_2 = L_2(\alpha, M_1, M_4)$ be the constants from (A1) (ii) and (A2) (ii), respectively, and

$$N_1 = \max\{\max\{|D_1 g(t,u)| \,:\, t \in [0,\alpha],\ u \in M_5\}, \max\{|D_2 g(t,u)| \,:\, t \in [0,\alpha],\ u \in M_5\}\}. \tag{3.6}$$

We have for $t \in [0, \alpha^\gamma]$:

$$\begin{aligned}
&|x(t;\gamma) - x(t;\bar{\gamma})| \\
&\quad \le |g(t, x(t-\eta(t);\gamma)) - g(t, x(t-\eta(t);\bar{\gamma}))| + |\varphi(0) - \bar{\varphi}(0)| + \ |g(0, \varphi(-\eta(0))) - g(0, \bar{\varphi}(-\eta(0)))| \\
&\qquad + \int_0^t \Big| f(s, x(\cdot;\gamma)_s, \Lambda(s, x(\cdot;\gamma)_s, \sigma), \theta) - f(s, x(\cdot;\bar{\gamma})_s, \Lambda(s, x(\cdot;\bar{\gamma})_s, \bar{\sigma}), \bar{\theta}) \Big|\, ds \\
&\quad \le \ N_1 |x(t-\eta(t);\gamma) - x(t-\eta(t);\bar{\gamma})| + (1 + N_1)|\varphi - \bar{\varphi}|_C \\
&\qquad + L_1 \int_0^t \Big( |x(\cdot;\gamma)_s - x(\cdot;\bar{\gamma})_s|_C + \ |\Lambda(s, x(\cdot;\gamma)_s, \sigma) - \Lambda(s, x(\cdot;\bar{\gamma})_s, \bar{\sigma})| + |\theta - \bar{\theta}|_\Theta \Big)\, ds.
\end{aligned}$$

Let

$$N_2 = \max\{\max\{|x(t;\bar{\gamma})| \,:\, t \in [-r, \alpha]\}, \operatorname{ess\,sup}\{|\dot{x}(t;\bar{\gamma})| \,:\, t \in [-r, \alpha]\}\}. \tag{3.7}$$

Then (2.2) yields

$$|\Lambda(s, x(\cdot;\gamma)_s, \sigma) - \Lambda(s, x(\cdot;\bar{\gamma})_s, \bar{\sigma})| \le L_2 N_2 (|x(\cdot;\gamma)_s - x(\cdot;\bar{\gamma})_s|_C + \ |\sigma - \bar{\sigma}|_\Sigma) + |x(\cdot;\gamma)_s - x(\cdot;\bar{\gamma})_s|_C \tag{3.8}$$

for $s \in [0, \alpha^\gamma]$, therefore

$$\begin{aligned}
|x(t;\gamma) - x(t;\bar{\gamma})| \ \le\ & N_1 |x(t-\eta(t);\gamma) - x(t-\eta(t);\bar{\gamma})| + (1 + N_1)|\gamma - \bar{\gamma}|_\Gamma \\
& + L_1 \int_0^t \Big( |x(\cdot;\gamma)_s - x(\cdot;\bar{\gamma})_s|_C + L_2 N_2 (|x(\cdot;\gamma)_s - x(\cdot;\bar{\gamma})_s|_C \\
& + |\sigma - \bar{\sigma}|_\Sigma) + |x(\cdot;\gamma)_s - x(\cdot;\bar{\gamma})_s|_C + |\gamma - \bar{\gamma}|_\Gamma \Big)\, ds.
\end{aligned}$$

Lemma 2.1 yields

$$\xi(t; \bar{\gamma}, \gamma) \le N_1 \xi(t - \eta_0; \bar{\gamma}, \gamma) + K_1 |\gamma - \bar{\gamma}|_\Gamma + K_2 \int_0^t \xi(s; \bar{\gamma}, \gamma)\, ds, \quad t \in [0, \alpha^\gamma],$$

where $\xi(t; \bar{\gamma}, \gamma) = \sup\{|x(s;\gamma) - x(s;\bar{\gamma})| \,:\, s \in [-r, t]\}$ and $K_1 = 1 + N_1 + L_1\alpha + L_1 L_2 N_2 \alpha$ and $K_2 = L_1(2 + L_2 N_2)$. Applying Lemma 2.2 we get

$$|x(t;\gamma) - x(t;\bar{\gamma})| \le \xi(t; \bar{\gamma}, \gamma) \le d(\gamma, \bar{\gamma}) e^{ct}, \qquad t \in [-r, \alpha^\gamma], \tag{3.9}$$

where $c > 0$ is the solution of $cN_1e^{-c\eta_0} + K_2 = c$, and

$$d(\gamma, \bar\gamma) = \max\left\{\frac{K_1|\gamma - \bar\gamma|_\Gamma}{1 - N_1e^{-c\eta_0}}, e^{cr}|\varphi - \bar\varphi|_C\right\}.$$

Therefore there exists $K_3 > 0$ such that $d(\gamma, \bar\gamma) \le K_3|\gamma - \bar\gamma|_\Gamma$, so, combining this with (3.9), we get

$$|x(t;\gamma) - x(t;\bar\gamma)| \le L^*|\gamma - \bar\gamma|_\Gamma, \qquad t \in [-r, \alpha^\gamma], \quad \gamma \in \mathcal{B}_\Gamma(\bar\gamma;\, \delta_2), \tag{3.10}$$

where $L^* = K_3e^{c\alpha}$. Inequality (3.8) yields

$$|\Lambda(s, x(\cdot;\gamma)_s, \sigma) - \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma)| \le (L_2N_2(L^* + 1) + L^*)|\gamma - \bar\gamma|_\Gamma, \qquad s \in [0, \alpha^\gamma],$$

therefore if define $\delta = \min\{\delta_2, \varepsilon_1/L^*, \varepsilon_2/(L_2N_2(L^* + 1) + L^*), \varepsilon_5/L^*\}$, then $\alpha^\gamma = \alpha$ can be used in (3.4) and (3.5) for $\gamma \in \mathcal{B}_\Gamma(\bar\gamma;\, \delta)$, hence statements (ii) and (iii) of the theorem hold. Then (3.10) yields

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C \le L^*|\gamma - \bar\gamma|_\Gamma \qquad \text{for } t \in [0, \alpha], \quad \gamma \in \mathcal{B}_\Gamma(\bar\gamma;\, \delta). \tag{3.11}$$

As we have seen, $x(\cdot;\gamma)$ is a.e. differentiable, and (1.1) can be rewritten as

$$\begin{aligned} \dot x(t;\gamma) &= D_1g(t, x(t - \eta(t);\gamma)) + D_2g(t, x(t - \eta(t);\gamma))\dot x(t - \eta(t);\gamma)(1 - \dot\eta(t)) \\ &\quad + f\Big(t, x(\cdot;\gamma)_t; \Lambda(t, x(\cdot;\gamma)_t, \sigma), \theta\Big) \end{aligned} \tag{3.12}$$

for a.e. $t \in [0, \alpha]$. Let $W(\gamma) = \{t \in [0, \alpha]\colon\ t \mapsto x(t - \eta(t);\gamma)$ and $\eta$ are differentiable at $t\}$. (The Lebesgue-measure of $W(\gamma)$ is $\alpha$.) It follows from (3.12) that

$$\begin{aligned} \dot x(t;\gamma) - \dot x(t;\bar\gamma) &= D_1g(t, x(t - \eta(t);\gamma)) - D_1g(t, x(t - \eta(t);\bar\gamma)) \\ &\quad + \Big(D_2g(t, x(t - \eta(t);\gamma)) - D_2g(t, x(t - \eta(t);\bar\gamma))\Big)\dot x(t - \eta(t);\gamma)(1 - \dot\eta(t)) \\ &\quad + D_2g(t, x(t - \eta(t);\bar\gamma))\Big(\dot x(t - \eta(t);\gamma) - \dot x(t - \eta(t);\bar\gamma)\Big)(1 - \dot\eta(t)) \\ &\quad f(t, x(\cdot;\gamma)_t, \Lambda(t, x(\cdot;\gamma)_t, \sigma), \theta) - f(t, x(\cdot;\bar\gamma)_t, \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta) \end{aligned}$$

for $t \in W(\gamma) \cap W(\bar\gamma)$. Let $L_4 = L_4(\alpha, M_5)$ be the constant from (A4) (ii), and

$$N_3 = \operatorname{ess\,sup}\{|1 - \dot\eta(t)|\colon\ t \in [0, \alpha]\}. \tag{3.13}$$

Then, using (3.8), (3.11), (A1) (ii) and (A2) (ii), we get

$$\begin{aligned} &|\dot x(t;\gamma) - \dot x(t;\bar\gamma)| \\ &\quad\le L_4|x(t - \eta(t);\gamma) - x(t - \eta(t);\bar\gamma)| \\ &\qquad + L_4N_3|x(t - \eta(t);\gamma) - x(t - \eta(t);\bar\gamma)|\Big(|\dot x(t - \eta(t);\gamma) - \dot x(t - \eta(t);\bar\gamma)| + |\dot x(t - \eta(t);\bar\gamma)|\Big) \\ &\qquad + N_1N_3|\dot x(t - \eta(t);\gamma) - \dot x(t - \eta(t);\bar\gamma)| \\ &\qquad + L_1\Big(|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C + L_2N_2(|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C + |\sigma - \bar\sigma|_\Sigma) \\ &\qquad\qquad + |x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C + |\theta - \bar\theta|_\Theta\Big) \\ &\quad\le L_4L^*(1 + N_2N_3)|\gamma - \bar\gamma|_\Gamma + L_4N_3L^*|\gamma - \bar\gamma|_\Gamma|\dot x(t - \eta(t);\gamma) - \dot x(t - \eta(t);\bar\gamma)| \\ &\qquad + N_1N_3|\dot x(t - \eta(t);\gamma) - \dot x(t - \eta(t);\bar\gamma)| + L_1(2L^* + L_2N_2(L^* + 1) + 1)|\gamma - \bar\gamma|_\Gamma \\ &\quad\le K_3|\gamma - \bar\gamma|_\Gamma + K_4|\dot x(t - \eta(t);\gamma) - \dot x(t - \eta(t);\bar\gamma)|, \qquad t \in W(\gamma) \cap W(\bar\gamma), \end{aligned} \tag{3.14}$$

where $K_3 = \max\{L_4 L^*(1+N_2N_3) + L_1(2L^* + L_2N_2(L^*+1)+1), 1\}$ and $K_4 = L_4N_3L^*\delta + N_1N_3$. Define $\chi(t;\bar\gamma,\gamma) = \operatorname{ess\,sup}\{|\dot x(s;\gamma) - \dot x(s;\bar\gamma)| \colon\ s\in[-r,t]\}$. Then (3.14) yields

$$\chi(t;\bar\gamma,\gamma) \le K_3|\gamma-\bar\gamma|_\Gamma + K_4\chi(t-\eta_0;\bar\gamma,\gamma), \qquad t\in[0,\alpha]. \tag{3.15}$$

Let $m = [\alpha/\eta_0]$, where $[\cdot]$ is the greatest integer part function. For $t\in[0,\eta_0]$ (3.15) implies $\chi(t;\bar\gamma,\gamma) \le (K_3+K_4)|\gamma-\bar\gamma|_\Gamma$. Applying (3.15) inductively with the intervals $[i\eta_0,(i+1)\eta_0]$ for $i=1,\dots,m$, it is easy to check that

$$\chi(t;\bar\gamma,\gamma) \le \big(K_3(1+K_4+\cdots+K_4^m) + K_4^{m+1}\big)\,|\gamma-\bar\gamma|_\Gamma, \qquad t\in[0,\alpha].$$

Therefore (3.2) holds with $L = \max\big\{L^*, K_3(1+K_4+\cdots+K_4^m)+K_4^{m+1}\big\}$. This completes the proof of (iv).

Part (v) is obvious using the method of steps with the intervals $[i\eta_0,(i+1)\eta_0]$, $i=1,\dots,m$.

□

## 4 Differentiability wrt parameters

In this section we study differentiability of solutions of the IVP (1.1)-(1.2) wrt the initial function, $\varphi$, the parameter $\sigma$ of the delay function $\tau$ and the parameter $\theta$ of the function $f$.

First we define a few notations will be used throughout this section. Introduce

$$\begin{aligned}\omega_f(t,\bar\psi,\bar u,\bar\theta;\psi,u,\theta) &= f(t,\psi,u,\theta) - f(t,\bar\psi,\bar u,\bar\theta) - D_2f(t,\bar\psi,\bar u,\bar\theta)(\psi-\bar\psi)\\ &\quad - D_3f(t,\bar\psi,\bar u,\bar\theta)(u-\bar u) - D_4f(t,\bar\psi,\bar u,\bar\theta)(\theta-\bar\theta)\end{aligned}$$

for $t\in[0,T]$, $\bar\psi,\psi\in\Omega_1$, $\bar u,u\in\Omega_2$, and $\bar\theta,\theta\in\Omega_3$. It follows from (A1) (iii) that

$$\frac{|\omega_f(t,\bar\psi,\bar u,\bar\theta;\psi,u,\theta)|}{|\psi-\bar\psi|_C + |u-\bar u| + |\theta-\bar\theta|_\Theta} \to 0, \qquad \text{as } |\psi-\bar\psi|_C + |u-\bar u| + |\theta-\bar\theta|_\Theta \to 0. \tag{4.1}$$

Later we will need an explicit estimate of the fraction in (4.1). In order to get it we apply Lemma 2.3 and assumption (A1) (iii):

$$\begin{aligned}&|\omega_f(t,\bar\psi,\bar u,\bar\theta;\psi,u,\theta)|\\ &\quad\le \sup_{0<\nu<1}\Big(\Big|D_2f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta)) - D_2f(t,\bar\psi,\bar u,\bar\theta)\Big|_{\mathcal L(C,\mathbb R^n)}|\psi-\bar\psi|_C\\ &\qquad + \Big|D_3f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta)) - D_3f(t,\bar\psi,\bar u,\bar\theta)\Big||u-\bar u|\\ &\qquad + \Big|D_4f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta)) - D_4f(t,\bar\psi,\bar u,\bar\theta)\Big|_{\mathcal L(\Theta,\mathbb R^n)}|\theta-\bar\theta|_\Theta\Big)\end{aligned} \tag{4.2}$$

for $t\in[0,\alpha]$, $\psi\in\mathcal B_C(\bar\psi;\bar\varepsilon)$, $u\in\mathcal B_{\mathbb R^n}(\bar u;\bar\varepsilon)$ and $\theta\in\mathcal B_\Theta(\bar\theta;\bar\varepsilon)$, where $\bar\varepsilon>0$ is sufficiently small. Suppose $E := U_1\times U_2\times U_3 \subset \mathcal B_C(\bar\psi;\bar\varepsilon)\times\mathcal B_{\mathbb R^n}(\bar u;\bar\varepsilon)\times\mathcal B_\Theta(\bar\theta;\bar\varepsilon)$ is a star domain with center at $(\bar\psi,\bar u,\bar\theta)$, i.e., such that for any $(\psi,u,\theta)\in U_1\times U_2\times U_3$ it follows $(\bar\psi,\bar u,\bar\theta)+\nu(\psi-\bar\psi,u-\bar u,\theta-\bar\theta)\in U_1\times U_2\times U_3$ for any $\nu\in[0,1]$. Then

$$|\omega_f(t,\bar\psi,\bar u,\bar\theta;\psi,u,\theta)| \le \Omega_f(|\psi-\bar\psi|_C + |u-\bar u| + |\theta-\bar\theta|_\Theta;\alpha,E)\Big(|\psi-\bar\psi|_C + |u-\bar u| + |\theta-\bar\theta|_\Theta\Big) \tag{4.3}$$

for $t\in[0,\alpha]$, $(\psi,u,\theta)\in U_1\times U_2\times U_3$, where

$$\begin{aligned}\Omega_f(\varepsilon;\alpha,E) &= \sup\Big\{\max\Big(|D_2f(t,\psi,u,\theta)-D_2f(t,\tilde\psi,\tilde u,\tilde\theta)|_{\mathcal{L}(C,\mathbb{R}^n)},\\ &\qquad |D_3f(t,\psi,u,\theta)-D_3f(t,\tilde\psi,\tilde u,\tilde\theta)|,\\ &\qquad |D_4f(t,\psi,u,\theta)-D_4f(t,\tilde\psi,\tilde u,\tilde\theta)|_{\mathcal{L}(\Theta,\mathbb{R}^n)}\Big):\\ &\qquad |\psi-\tilde\psi|_C+|u-\tilde u|+|\theta-\tilde\theta|_\Theta\le\varepsilon,\quad t\in[0,\alpha],\ (\psi,u,\theta),(\tilde\psi,\tilde u,\tilde\theta)\in E\Big\}.\end{aligned}$$

We introduce the function

$$\omega_g(t,\bar u;u)=g(t,u)-g(t,\bar u)-\ D_2g(t,\bar u)(u-\bar u),\qquad t\in[0,\alpha],\quad \bar u,u\in\Omega_5.$$

Let $M_5$ be a compact subset of $\Omega_5$, and $L_4=L_4(\alpha,M_5)$ be the Lipschitz constant from (A4) (ii). Then Lemma 2.3 yields

$$|\omega_g(t,\bar u;u)|\le L_4|u-\bar u|^2,\qquad t\in[0,\alpha],\quad u,\bar u\in M_5. \tag{4.4}$$

We have seen that in the proof of Theorem 3.1 $\Lambda$ is used only with second argument of the form $x_s$, where $x_s$ is not only a $C$-function, but also a $W^{1,\infty}$-function. Therefore for the rest of the paper we restrict $\Lambda$ to this domain, but for simplicity, we use the same notation for the restriction. So we redefine $\Lambda$ as

$$\Lambda\,:\,\Big([0,T]\times(\Omega_1\cap W^{1,\infty})\times\Omega_4\subset\mathbb{R}\times W^{1,\infty}\times\Sigma\Big)\to\mathbb{R}^n,\quad \Lambda(t,\psi,\sigma)=\psi(-\tau(t,\psi,\sigma)). \tag{4.5}$$

It was shown in [18] that $\Lambda$ defined by (4.5) is differentiable wrt its second and third variables at a point $(t,\psi,\sigma)$ where $\psi\in C^1$. Since later in the paper we will need a more careful estimate than that used in [18], we include the revised proof of this result, as well.

**Lemma 4.1** *Assume (A2) (i)–(iii), and let $\Lambda$ be defined by* (4.5)*. Then the partial derivatives $D_2\Lambda(t,\psi,\sigma)$ and $D_3\Lambda(t,\psi,\sigma)$ exist for $t\in[0,T]$, $\psi\in\Omega_1\cap C^1$, $\sigma\in\Omega_4$, and*

$$D_2\Lambda(t,\psi,\sigma)h = -\dot\psi(-\tau(t,\psi,\sigma))D_2\tau(t,\psi,\sigma)h+h(-\tau(t,\psi,\sigma)),\qquad h\in W^{1,\infty}, \tag{4.6}$$

$$D_3\Lambda(t,\psi,\sigma) = -\dot\psi(-\tau(t,\psi,\sigma))D_3\tau(t,\psi,\sigma). \tag{4.7}$$

*Moreover, $D_2\Lambda(t,\cdot,\cdot)$ and $D_3\Lambda(t,\cdot,\cdot)$ are continuous on $(\Omega_1\cap C^1)\times\Omega_4$ for $t\in[0,T]$.*

**Proof** Let $\psi\in\Omega_1\cap C^1$. Introduce

$$\omega_\psi(\bar u;u)=\psi(u)-\psi(\bar u)-\dot\psi(\bar u)(u-\bar u)$$

and the modulus of continuity of $\dot\psi$

$$\Omega_{\dot\psi}(\varepsilon)=\sup\{|\dot\psi(u)-\dot\psi(\bar u)|:\ |u-\bar u|\le\varepsilon,\ u,\bar u\in[-r,0]\}.$$

The function $\dot\psi$ is continuous, therefore $\Omega_{\dot\psi}(\varepsilon)\to0$ as $\varepsilon\to0$. Lemma 2.3 yields

$$|\omega_\psi(\bar u;u)|\le\Omega_{\dot\psi}(|u-\bar u|)|u-\bar u|,\qquad u,\bar u\in[-r,0].$$

Let $\psi \in \Omega_1 \cap C^1$ and $\sigma \in \Omega_4$ be fixed, and let $h \in W^{1,\infty}$ and $k \in \Sigma$. We define

$$\begin{aligned} \omega_\Lambda(t,\psi,\sigma;\psi+h,\sigma+k) &= \Lambda(t,\psi+h,\sigma+k) - \Lambda(t,\psi,\sigma) + \dot\psi(-\tau(t,\psi,\sigma))D_2\tau(t,\psi,\sigma)h \\ &\quad -h(-\tau(t,\psi,\sigma)) + \dot\psi(-\tau(t,\psi,\sigma))D_3\tau(t,\psi,\sigma)k \end{aligned}$$

and

$$\omega_\tau(t,\psi,\sigma;\psi+h,\sigma+k) = \tau(t,\psi+h,\sigma+k) - \tau(t,\psi,\sigma) - D_2\tau(t,\psi,\sigma)h - D_3\tau(t,\psi,\sigma)k$$

for $h \in W^{1,\infty}$ and $k \in \Sigma$ such that $\psi + h \in \Omega_1$ and $\sigma + k \in \Omega_4$. Then it is easy to check that

$$\begin{aligned} &\omega_\Lambda(t,\psi,\sigma;\psi+h,\sigma+k) \\ &\quad = \omega_\psi(-\tau(t,\psi,\sigma);-\tau(t,\psi+h,\sigma+k)) - \dot\psi(-\tau(t,\psi,\sigma))\omega_\tau(t,\psi,\sigma;\psi+h,\sigma+k) \\ &\qquad +h(-\tau(t,\psi+h,\sigma+k)) - h(-\tau(t,\psi,\sigma)). \end{aligned}$$

Let $\varepsilon_1 > 0$ and $\varepsilon_4 > 0$ be such that $\overline{\mathcal{B}}_C(\psi;\varepsilon_1) \subset \Omega_1$ and $\tilde M_4 := \overline{\mathcal{B}}_\Sigma(\sigma;\varepsilon_4) \subset \Omega_4$. Then $\tilde M_1 := \overline{\mathcal{B}}_{W^{1,\infty}}(\psi;\varepsilon_1)$ is a compact subset of $C$, and $\tilde M_1 \subset \Omega_1$. Let $L_2 = L_2(t,\tilde M_1,\tilde M_4)$ be the Lipschitz constant from (A2) (ii). Then

$$\begin{aligned} &|\omega_\Lambda(t,\psi,\sigma;\psi+h,\sigma+k)| \\ &\quad \le \Omega_{\dot\psi}(|\tau(t,\psi+h,\sigma+k)-\tau(t,\psi,\sigma)|)|\tau(t,\psi+h,\sigma+k)-\tau(t,\psi,\sigma)| \\ &\qquad +|\psi|_{W^{1,\infty}}|\omega_\tau(t,\psi,\sigma;\psi+h,\sigma+k)| + |h|_{W^{1,\infty}}|\tau(t,\psi+h,\sigma+k)-\tau(t,\psi,\sigma)| \\ &\quad \le \Omega_{\dot\psi}\Big(L_2(|h|_C+|k|_\Sigma)\Big)L_2(|h|_C+|k|_\Sigma) \\ &\qquad +|\psi|_{W^{1,\infty}}|\omega_\tau(t,\psi,\sigma;\psi+h,\sigma+k)| + L_2|h|_{W^{1,\infty}}(|h|_C+|k|_\Sigma) \end{aligned} \tag{4.8}$$

for $|h|_C \le \varepsilon_1$ and $|k|_\Sigma \le \varepsilon_4$. Since $|h|_C \le |h|_{W^{1,\infty}}$, it implies

$$\begin{aligned} \frac{|\omega_\Lambda(t,\psi,\sigma;\psi+h,\sigma+k)|}{|h|_{W^{1,\infty}}+|k|_\Sigma} &\le L_2\Omega_{\dot\psi}\Big(L_2(|h|_{W^{1,\infty}}+|k|_\Sigma)\Big) \\ &\quad +|\psi|_{W^{1,\infty}}\frac{|\omega_\tau(t,\psi,\sigma;\psi+h,\sigma+k)|}{|h|_C+|k|_\Sigma} + L_2|h|_{W^{1,\infty}}. \end{aligned}$$

Consequently, $\frac{|\omega_\Lambda(t,\psi,\sigma;\psi+h,\sigma+k)|}{|h|_{W^{1,\infty}}+|k|_\Sigma} \to 0$ as $|h|_{W^{1,\infty}}+|k|_\Sigma \to 0$, since (A2) (iii) and the continuity of $\dot\psi$ yield $\Omega_{\dot\psi}\Big(L_2(|h|_{W^{1,\infty}}+|k|_\Sigma)\Big) \to 0$, and (A2) (iii) implies $\frac{|\omega_\tau(t,\psi,\sigma;\psi+h,\sigma+k)|}{|h|_C+|k|_\Sigma} \to 0$ as $|h|_{W^{1,\infty}}+|k|_\Sigma \to 0$. This concludes the proof of both (4.6) and (4.7). □

Lipschitz continuity of $D_2\tau$ and $D_3\tau$, Lemma 2.3 and (4.8) immediately yield the next estimate.

**Corollary 4.2** *Assume (A2) (i)–(iv). Let $0 < \alpha \le T$ be finite, $M_1 \subset C$ be a compact, $M_4 \subset \Sigma$ be a closed and bounded subset of the respective spaces, $L_2 = L_2(\alpha,M_1,M_4)$ and $L_3(\alpha,M_1,M_4)$ be the constants from (A2) (ii) and (iv), respectively. Then*

$$\frac{|\omega_\Lambda(t,\psi,\sigma;\psi+h,\sigma+k)|}{|h|_C+|k|_\Sigma} \le L_2\Omega_{\dot\psi}\Big(L_2(|h|_C+|k|_\Sigma)\Big) + L_3|\psi|_{W^{1,\infty}}(|h|_C+|k|_\Sigma) + L_2|h|_{W^{1,\infty}} \tag{4.9}$$

*for $t \in [0,\alpha]$, $\psi, \psi+h \in M_1$, $\sigma, \sigma+k \in M_4$.*

Let $\bar\gamma = (\bar\varphi, \bar\sigma, \bar\theta) \in \mathcal{M}$, and $x(\cdot;\bar\gamma)$ be the corresponding solution of the IVP (1.1)-(1.2) on $[0,\alpha]$. Fix $h = (h^\varphi, h^\sigma, h^\theta) \in \Gamma$, and consider the variational equation

$$\begin{aligned}
&\frac{d}{dt}\Big(z(t;\bar\gamma,h) - D_2 g(t, x(t-\eta(t);\bar\gamma))z(t-\eta(t);\bar\gamma,h)\Big) \\
&\quad = D_2 f(t, x(\cdot;\bar\gamma)_t, \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta) z(\cdot;\bar\gamma,h)_t \\
&\qquad + D_3 f(t, x(\cdot;\bar\gamma)_t, \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta)\Big(D_2\Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma) z(\cdot;\bar\gamma,h)_t \\
&\qquad + D_3\Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma) h^\sigma\Big) + D_4 f(t, x(\cdot;\bar\gamma)_t, \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta) h^\theta, \quad t \in [0,\alpha],
\end{aligned} \tag{4.10}$$

$$z(t;\bar\gamma,h) = h^\varphi(t), \qquad t \in [-r, 0]. \tag{4.11}$$

This is a linear time-dependent and state-independent NFDE for $z(\cdot;\bar\gamma,h)$, and the right-hand side of (4.10) depends continuously on $t$ and $z(\cdot;\bar\gamma,h)_t$ since $x(\cdot;\bar\gamma)_t \in C^1$ by Theorem 3.1 (v). Therefore this IVP has a unique solution, $z(\cdot;\bar\gamma,h)$, which depends linearly on $h$. The boundedness of the map $\Gamma \to \mathbb{R}^n$, $h \mapsto z(t;\bar\gamma,h)$ for each $t \in [0,\alpha]$ follows from the next theorem.

**Theorem 4.3** *Assume (A1) (i)–(iii), (A2) (i)–(iii), (A3), (A4) (i), (ii) and (A5) (i)–(iv), and let $\bar\gamma \in \mathcal{M}$. There exists $N_4 \ge 0$ such that the solution of the IVP* (4.10)-(4.11) *satisfies*

$$|z(t;\bar\gamma,h)| \le N_4 |h|_\Gamma, \qquad t \in [0,\alpha], \quad h \in \Gamma.$$

**Proof** For simplicity we use the notations

$$x(t) = x(t;\bar\gamma) \qquad \text{and} \qquad z(t) = z(t;\bar\gamma,h).$$

Integrating (4.10) from 0 to $t$ we get

$$\begin{aligned}
z(t) &= D_2 g(t, x(t-\eta(t)))z(t-\eta(t)) + h^\varphi(0) - D_2 g(0, \bar\varphi(-\eta(0)))h^\varphi(-\eta(0)) \\
&\quad + \int_0^t D_2 f(s, x_s, \Lambda(s, x_s, \bar\sigma), \bar\theta) z_s + D_3 f(s, x_s, \Lambda(s, x_s, \bar\sigma), \bar\theta)\Big(D_2\Lambda(s, x_s, \bar\sigma) z_s \\
&\quad + D_3\Lambda(s, x_s, \bar\sigma) h^\sigma\Big) + D_4 f(s, x_s, \Lambda(s, x_s, \bar\sigma), \bar\theta) h^\theta \, ds.
\end{aligned}$$

Let the constants $\delta, \alpha$ and the sets $M_1, \ldots, M_5$ be defined by Theorem 3.1, and $L_1$ be the corresponding Lipschitz constant, and $N_1, N_2$ be defined by (3.6) and (3.7), respectively. Assumptions (A1) (ii) and (iii) yield that

$$|D_2 f(t,\psi,u,\theta)|_{\mathcal{L}(C,\mathbb{R}^n)} \le L_1, \ |D_3 f(t,\psi,u,\theta)| \le L_1, \ |D_4 f(t,\psi,u,\theta)|_{\mathcal{L}(\Theta,\mathbb{R}^n)} \le L_1 \tag{4.12}$$

for $t \in [0,\alpha]$, $\psi \in M_1$, $u \in M_2$, and $\theta \in M_3$. From (4.6), (4.7) and (A2) (iii) it follows that there exists $N_5 = N_5(\alpha)$ such that

$$|D_2\Lambda(s, x_s, \bar\sigma)|_{\mathcal{L}(C,\mathbb{R}^n)} \le N_5 \quad \text{and} \quad |D_3\Lambda(s, x_s, \bar\sigma)|_{\mathcal{L}(\Sigma,\mathbb{R}^n)} \le N_5, \qquad s \in [0,\alpha]. \tag{4.13}$$

Then we get

$$|z(t)| \le N_1 |z(t-\eta(t))| + (1+N_1)|h|_\Gamma + L_1 \int_0^t |z_s|_C + N_5 |z_s|_C + N_5 |h|_\Gamma + |h|_\Gamma \, ds.$$

An application of Lemma 2.1 implies

$$\beta(t;\bar\gamma,h)\le N_1\beta(t-\eta_0;\bar\gamma,h)+K_1|h|_\Gamma+K_2\int_0^t\beta(s;\bar\gamma,h)\,ds,\qquad t\in[0,\alpha],$$

where $\beta(t;\bar\gamma,h)=\max\{|z(s)|:\ s\in[-r,t]\}$, $K_1=1+N_1+L_1(1+N_5)\alpha$ and $K_2=L_1(1+N_5)$. Then Lemma 2.2 yields

$$|z(t)|\le\beta(t;\bar\gamma,h)\le N_4|h|_\Gamma,\qquad t\in[0,\alpha],$$

where

$$N_4=\max\left\{\frac{K_1}{1-N_1e^{-c\eta_0}},e^{cr}\right\}e^{c\alpha}$$

and $c$ is the positive solution of $cN_1e^{-c\eta_0}+K_2=c$. □

Next we study differentiability of the function $x(t;\gamma)$ wrt $\gamma$. We denote this differentiation by $D_2x$.

**Theorem 4.4** *Assume (A1) (i)–(iii), (A2) (i)–(iii), (A3), (A4) (i), (ii) and (A5) (i)–(iv), and let $\bar\gamma\in\mathcal{M}$. Let $\delta>0$ and $\alpha>0$ be defined by Theorem 3.1, and $x(t;\gamma)$ be the solution of the IVP* (1.1)-(1.2) *on* $[0,\alpha]$ *for $\gamma\in\mathcal{B}_\Gamma(\bar\gamma;\,\delta)$. Then the function $x(t;\cdot):\Big(\mathcal{B}_\Gamma(\bar\gamma;\,\delta)\subset\Gamma\Big)\to\mathbb{R}^n$ is differentiable at $\bar\gamma$ for $t\in[0,\alpha]$, and*

$$D_2x(t;\bar\gamma)h=z(t;\bar\gamma,h),\qquad h\in\Gamma,$$

*where $z$ is the solution of the IVP* (4.10)-(4.11).

**Proof** Let $\bar\gamma=(\bar\varphi,\bar\sigma,\bar\theta)\in\mathcal{M}$, $\delta>0$ and $\alpha$ be as in the assumption of the theorem, and let the sets $M_1,\dots,M_5$ be defined by Theorem 3.1 (iii). Let $h=(h^\varphi,h^\sigma,h^\theta)\in\Gamma$ be such that $|h|_\Gamma<\delta$. For brevity, we use the notations

$$x(t)=x(t;\bar\gamma),\qquad y(t)=x(t;\bar\gamma+h)\qquad\text{and}\qquad z(t)=z(t;\bar\gamma,h).$$

Integrating (1.1) and (4.10), and using the definitions of $\omega_f$, $\omega_g$ and $\omega_\Lambda$ we get

$$\begin{aligned}
&y(t)-x(t)-z(t)\\
&=\ g(t,y(t-\eta(t)))-g(t,x(t-\eta(t)))-D_2g(t,x(t-\eta(t)))z(t-\eta(t))\\
&\qquad-\Big(g(0,\bar\varphi(-\eta(0))+h^\varphi(-\eta(0)))-g(0,\bar\varphi(-\eta(0)))-D_2g(0,\bar\varphi(-\eta(0)))h^\varphi(-\eta(0))\Big)\\
&\quad+\int_0^t\Big(f(s,y_s,\Lambda(s,y_s,\bar\sigma+h^\sigma),\bar\theta+h^\theta)-f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta)-D_2f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta)z_s\\
&\qquad-D_3f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta)\Big(D_2\Lambda(s,x_s,\bar\sigma)z_s+D_3\Lambda(s,x_s,\bar\sigma)h^\sigma\Big)\\
&\qquad-D_4f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta)h^\theta\Big)ds\\
&=\ \omega_g(t,x(t-\eta(t));y(t-\eta(t)))-\omega_g(0,\bar\varphi(-\eta(0));\bar\varphi(-\eta(0))+h^\varphi(-\eta(0)))\\
&\quad+D_2g(t,x(t-\eta(t)))\Big(y(t-\eta(t))-x(t-\eta(t))-z(t-\eta(t))\Big)\\
&\quad+\int_0^t\Big(\omega_f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta;y_s,\Lambda(s,y_s,\bar\sigma+h^\sigma),\bar\theta+h^\theta)+D_2f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta)(y_s-x_s-z_s)\\
&\quad+D_3f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta)\Big(\omega_\Lambda(s,x_s,\bar\sigma;y_s,\bar\sigma+h^\sigma)+D_2\Lambda(s,x_s,\bar\sigma)(y_s-x_s-z_s)\Big)\Big)ds.
\end{aligned}\tag{4.14}$$

Let $L_1 = L_1(\alpha, M_1, M_2, M_3)$, $L_2 = L_2(\alpha, M_1, M_4)$ and $L_4 = L_4(\alpha, M_5)$ be the constants from (A1) (ii), (A2) (ii) and (A4) (ii), respectively, $N_1, N_2$, $N_3$ and $N_5$ be defined by (3.6), (3.7), (3.13) and (4.13), respectively, and $L$ be the constant from Theorem 3.1. Applying (4.4) and Theorem 3.1 we have

$$|\omega_g(t, x(t-\eta(t)); y(t-\eta(t)))| \leq L_4|y(t-\eta(t)) - x(t-\eta(t))|^2 \leq L_4L^2|h|_\Gamma^2$$

for $t \in [0, \alpha]$. Then (4.14), together with (4.12) and (4.13) implies

$$\begin{aligned} |y(t) - x(t) - z(t)| \quad &\leq \quad 2L_4L^2|h|_\Gamma^2 + N_1|y(t-\eta(t)) - x(t-\eta(t)) - z(t-\eta(t))| + \int_0^t \Big(G_f(s;\bar{\gamma}, h) \\ &\quad + L_1|y_s - x_s - z_s|_C + L_1G_\Lambda(s;\bar{\gamma}, h) + L_1N_5|y_s - x_s - z_s|_C\Big)ds \qquad (4.15) \end{aligned}$$

for $t \in [0, \alpha]$, where

$$G_f(s;\bar{\gamma}, h) \quad = \quad |\omega_f(s, x_s, \Lambda(s, x_s, \bar{\sigma}), \bar{\theta}; y_s, \Lambda(s, y_s, \bar{\sigma} + h^\sigma), \bar{\theta} + h^\theta)|$$

and

$$G_\Lambda(s;\bar{\gamma}, h) = |\omega_\Lambda(s, x_s, \bar{\sigma}; y_s, \bar{\sigma} + h^\sigma)|.$$

Introduce $\mu(t;\bar{\gamma}, h) = \sup\{|y(s) - x(s) - z(s)| : \, -r \leq s \leq t\}$. Applying Lemma 2.1 for (4.15) we obtain

$$\mu(t;\bar{\gamma}, h) \leq A(h) + N_1\mu(t-\eta_0;\bar{\gamma}, h) + L_1(1 + N_5)\int_0^t \mu(s;\bar{\gamma}, h)\, ds, \qquad t \in [0, \alpha],$$

where

$$A(h) = 2L_4L^2|h|_\Gamma^2 + \int_0^\alpha \Big(G_f(s;\bar{\gamma}, h) + L_1G_\Lambda(s;\bar{\gamma}, h)\Big)ds.$$

Therefore Lemma 2.2 and $\mu(t;\bar{\gamma}, h) = 0$ for $t \in [-r, 0]$ imply

$$|y(t) - x(t) - z(t)| \leq \mu(t;\bar{\gamma}, h) \leq \frac{A(h)}{1 - N_1e^{-c\eta_0}}e^{c\alpha}, \qquad t \in [0, \alpha], \qquad (4.16)$$

where $c$ is the unique positive solution of $cN_1e^{-c\eta_0} + L_1(1 + N_5) = c$. Hence the claim of the theorem follows if we argue $A(h)/|h|_\Gamma \to 0$ as $|h|_\Gamma \to 0$, i.e., $\int_0^\alpha G_f(s;\bar{\gamma}, h)/|h|_\Gamma\, ds \to 0$ and $\int_0^\alpha G_\Lambda(s;\bar{\gamma}, h)/|h|_\Gamma\, ds \to 0$ as $|h|_\Gamma \to 0$.

First we show that $\int_0^\alpha G_\Lambda(s;\bar{\gamma}, h)/|h|_\Gamma\, ds \to 0$ as $|h|_\Gamma \to 0$ for $s \in [0, \alpha]$. Let $N_2$ be defined by (3.7) and $\Omega_{\dot{x}_s}$ be the modulus of continuity of $\dot{x}_s$. The definitions of $G_\Lambda$, $\omega_\Lambda$ and inequalities (3.2) and (4.8) yield

$$\begin{aligned} G_\Lambda(s;\bar{\gamma}, h) \;&=\; |\omega_\Lambda(s, x_s, \bar{\sigma}; y_s, \bar{\sigma} + h^\sigma)| \\ &\leq\; \Big[L_2\Omega_{\dot{x}_s}\Big(L_2(|y_s - x_s|_C + |h^\sigma|_\Sigma)\Big) + |x_s|_{W^{1,\infty}}\frac{|\omega_\tau(s, x_s, \bar{\sigma}; y_s, \bar{\sigma} + h^\sigma)|}{|y_s - x_s|_C + |h^\sigma|_\Sigma} \\ &\qquad + L_2|y_s - x_s|_{W^{1,\infty}}\Big](|y_s - x_s|_C + |h^\sigma|_\Sigma) \\ &\leq\; \Big[L_2\Omega_{\dot{x}_s}\Big(L_2(L + 1)|h|_\Gamma\Big) + N_2\frac{|\omega_\tau(s, x_s, \bar{\sigma}; y_s, \bar{\sigma} + h^\sigma)|}{|y_s - x_s|_C + |h^\sigma|_\Sigma} + L_2L|h|_\Gamma\Big](L + 1)|h|_\Gamma. \quad (4.17) \end{aligned}$$

Since $\Omega_{\dot{x}_s}\Big(L_2(L + 1)|h|_\Gamma\Big) \leq 2N_2$ and

$$|\omega_\tau(s, x_s, \bar{\sigma}; y_s, \bar{\sigma} + h^\sigma)| \leq 2L_2(|y_s - x_s|_C + |h^\sigma|_\Sigma)$$

by (A2) (ii), we get that $G_\Lambda(s;\bar\gamma,h)/|h|_\Gamma$ is bounded from above. On the other hand, for each fixed $s\in[0,\alpha]$ the function $\dot x_s$ is continuous, therefore $\Omega_{\dot x_s}\Big(L_2(L+1)|h|_\Gamma)\Big)\to 0$ as $|h|_\Gamma\to 0$, and similarly, $\frac{|\omega_\tau(s,x_s,\bar\sigma;y_s,\bar\sigma+h^\sigma)|}{|y_s-x_s|_C+|h^\sigma|_\Sigma}\to 0$ as $|h|_\Gamma\to 0$ by (A2) (iii) and $|y_s-x_s|_C\le L|h|_\Gamma$. Consequently, $\int_0^\alpha G_\Lambda(s;\bar\gamma,h)/|h|_\Gamma\,ds\to 0$ as $|h|_\Gamma\to 0$ by the Lebesgue's Dominated Convergence Theorem.

Next we show that $\int_0^\alpha G_f(s;\bar\gamma,h)/|h|_\Gamma\,ds\to 0$ as $|h|_\Gamma\to 0$. Combining (3.8), (4.2) and (4.12) we have

$$\begin{aligned} G_f(s;\bar\gamma,h) &= |\omega_f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta;y_s,\Lambda(s,y_s,\bar\sigma+h^\sigma),\bar\theta+h^\theta)|\\ &\le 2L_1(|y_s-x_s|_C+|\Lambda(s,y_s,\bar\sigma+h^\sigma)-\Lambda(s,x_s,\bar\sigma)|+|h^\theta|_\Theta)\\ &\le 2L_1K_1|h|_\Gamma \end{aligned}$$

for $s\in[0,\alpha]$ and $|h|_\Gamma<\delta$, where $K_1=2L+L_2N_2(L+1)+1$.

On the other hand, let $h_k=(h_k^\varphi,h_k^\sigma,h_k^\theta)\in\Gamma$ $(k=1,2,\ldots)$ be a sequence such that $|h_k|_\Gamma\le\delta$ and $|h_k|_\Gamma\to 0$, and let $y^k(t)=x(t;\bar\gamma+h_k)$. Define the set $M_3^*=\{\bar\theta+\nu h_k^\theta\colon\ \nu\in[0,1],\ k=1,2,\ldots\}$. Then $M_3^*\subset M_3$, and it is easy to check that $M_3^*$ is a compact subset of $\Theta$. Therefore the set $E:=M_1\times M_2\times M_3^*$ is a compact subset of $\Omega_1\times\Omega_2\times\Omega_3$,

$$(x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta),(y_s^k,\Lambda(s,y_s^k,\bar\sigma+h_k^\sigma),\bar\theta+h_k^\theta)\in E\qquad\text{for } s\in[0,\alpha],\ k=1,2\ldots,$$

and $E$ is a star domain with center at $(x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta)$ for each $s\in[0,\alpha]$. Then applying (3.8), (4.3) and the definition of $K_1$ we get

$$\begin{aligned} G_f(s;\bar\gamma,h_k) &= |\omega_f(s,x_s,\Lambda(s,x_s,\bar\sigma),\bar\theta;y_s^k,\Lambda(s,y_s^k,\bar\sigma+h_k^\sigma),\bar\theta+h_k^\theta)|\\ &\le \Omega_f\Big(|y_s^k-x_s|_C+|\Lambda(s,y_s^k,\bar\sigma+h^\sigma)-\Lambda(s,x_s,\bar\sigma)|+|h_k^\theta|_\Theta;\alpha,E\Big)\\ &\qquad\times\Big(|y_s^k-x_s|_C+|\Lambda(s,y_s^k,\bar\sigma+h_k^\sigma)-\Lambda(s,x_s,\bar\sigma)|+|h_k^\theta|_\Theta\Big)\\ &\le \Omega_f\Big(K_1|h_k|_\Gamma;\alpha,E\Big)K_1|h_k|_\Gamma,\qquad s\in[0,\alpha]. \end{aligned}\tag{4.18}$$

Since $D_2f$, $D_3f$ and $D_3f$ are continuous on $E$, they are uniformly continuous on $E$, as well, so $\Omega_f\Big(K_1|h_k|_\Gamma;\alpha,E\Big)\to 0$ as $k\to\infty$. Therefore $\int_0^\alpha G_f(s;\bar\gamma,h_k)/|h_k|_\Gamma\,ds\to 0$ as $k\to\infty$ by the Lebesgue's Dominated Convergence Theorem.

The proof of the theorem is complete. □

We note that in the the previous theorem the assumption $\bar\gamma\in\mathcal{M}$ was essential, since the proof relied on that $x$ is a $C^1$-function, therefore $D_2\Lambda$ and $D_3\Lambda$ exist.

The proof immediately implies differentiability of the parameter map in a $C$-norm:

**Corollary 4.5** *Assume the conditions of Theorem 4.4. Then the function*

$$\Big(\mathcal{B}_\Gamma(\bar\gamma;\,\delta)\subset\Gamma\Big)\to C,\qquad \gamma\mapsto x(\cdot;\gamma)_t$$

*is differentiable at $\bar\gamma$ for $t\in[0,\alpha]$, and its derivative is given by*

$$D_2x(\cdot;\bar\gamma)_th=z(\cdot;\bar\gamma,h)_t,\quad h\in\Gamma.$$

Next we show that, under slightly more smoothness on $\tau$ and $g$ than that in the previous theorem, $x(\cdot;\gamma)_t$ is differentiable wrt $\gamma$ if we use $W^{1,\infty}$ as the state-space of the solutions.

**Theorem 4.6** *Assume (A1) (i)–(iii), (A2) (i)–(iv), (A3), (A4) (i)–(iii) and (A5) (i)–(iv), and let $\bar\gamma \in \mathcal{M}$. Let $\delta > 0$ and $\alpha > 0$ be defined by Theorem 3.1, and $x(t;\gamma)$ be the solution of the IVP* (1.1)-(1.2) *on* $[0,\alpha]$ *for* $\gamma \in \mathcal{B}_\Gamma(\bar\gamma;\delta)$. *Then the function*

$$\Big(\mathcal{B}_\Gamma(\bar\gamma;\,\delta) \subset \Gamma\Big) \to W^{1,\infty}, \qquad \gamma \mapsto x(\cdot;\gamma)_t$$

*is differentiable at $\bar\gamma$ for $t \in [0,\alpha]$, and*

$$D_2 x(\cdot;\bar\gamma)_t h = z(\cdot;\bar\gamma,h)_t, \qquad h \in \Gamma,$$

*where $z$ is the solution of the IVP* (4.10)-(4.11).

**Proof** Note that (A4) (iii) yields using Schwarz's Theorem that $D_1 D_2 g$ exists and it is continuous on $[0,\alpha] \times \Omega_5$. We use all the notations introduced in the proof of Theorem 4.4. It follows from the proof of Theorem 4.4 that $|y_t - x_t - z_t|_C/|h|_\Gamma \to 0$ as $|h|_\Gamma \to 0$. It is easy to argue with the method of steps that $z$ is a.e. differentiable on $[-r,\alpha]$. Therefore we have

$$\dot y(t) - \dot x(t) - \dot z(t) = A(t;\bar\gamma,h) + B(t;\bar\gamma,h), \qquad \text{a.e. } \ t \in [0,\alpha], \tag{4.19}$$

where

$$\begin{aligned} A(t;\bar\gamma,h) \;=\;& D_1 g(t, y(t-\eta(t))) + D_2 g(t, y(t-\eta(t))\dot y(t-\eta(t))(1-\dot\eta(t)) \\ & - D_1 g(t, x(t-\eta(t))) - D_2 g(t, x(t-\eta(t)))\dot x(t-\eta(t))(1-\dot\eta(t)) \\ & - D_1 D_2 g(t, x(t-\eta(t)))z(t-\eta(t)) \\ & - D_2 D_2 g(t, x(t-\eta(t))\dot x(t-\eta(t))(1-\dot\eta(t))z(t-\eta(t)) \\ & - D_2 g(t, x(t-\eta(t)))\dot z(t-\eta(t))(1-\dot\eta(t)) \end{aligned}$$

and

$$\begin{aligned} B(t;\bar\gamma,h) \;=\;& f(t, y_t, \Lambda(t, y_t, \bar\sigma + h^\sigma), \bar\theta + h^\theta) - f(t, x_t, \Lambda(t, x_t, \bar\sigma), \bar\theta) - D_2 f(t, x_t, \Lambda(t, x_t, \bar\sigma), \bar\theta) z_t \\ & - D_3 f(t, x_t, \Lambda(t, x_t, \bar\sigma), \bar\theta)\Big(D_2\Lambda(t, x_t, \bar\sigma) z_t + D_3\Lambda(t, x_t, \bar\sigma) h^\sigma\Big) \\ & - D_4 f(t, x_t, \Lambda(t, x_t, \bar\sigma), \bar\theta) h^\theta. \end{aligned}$$

Simple computations show

$$\begin{aligned} & A(t;\bar\gamma,h) \\ &= D_1 g(t, y(t-\eta(t))) - D_1 g(t, x(t-\eta(t))) - D_1 D_2 g(t, x(t-\eta(t)))\Big(y(t-\eta(t)) - x(t-\eta(t))\Big) \\ &\quad + D_1 D_2 g(t, x(t-\eta(t)))\Big(y(t-\eta(t)) - x(t-\eta(t)) - z(t-\eta(t))\Big) \\ &\quad + \Big(D_2 g(t, y(t-\eta(t))) - D_2 g(t, x(t-\eta(t))) \\ &\qquad - D_2 D_2 g(t, x(t-\eta(t)))(y(t-\eta(t)) - x(t-\eta(t)))\Big)\dot y(t-\eta(t))(1-\dot\eta(t)) \\ &\quad + D_2 D_2 g(t, x(t-\eta(t)))\Big(y(t-\eta(t)) - x(t-\eta(t)) - z(t-\eta(t))\Big)\dot y(t-\eta(t))(1-\dot\eta(t)) \\ &\quad + D_2 D_2 g(t, x(t-\eta(t)))\Big(\dot y(t-\eta(t)) - \dot x(t-\eta(t))\Big)(1-\dot\eta(t))z(t-\eta(t)) \\ &\quad + D_2 g(t, x(t-\eta(t)))\Big(\dot y(t-\eta(t)) - \dot x(t-\eta(t)) - \dot z(t-\eta(t))\Big)(1-\dot\eta(t)). \end{aligned} \tag{4.20}$$

Let $N_1$, $N_2$ and $N_3$ be defined by (3.6), (3.7) and (3.13), respectively. Assumption (A4) (iii) yields that

$$N_6 = \max\{\max\{|D_2 D_i g(t,u)|\colon\ t\in[0,\alpha],\ u\in M_5\}\colon\ i=1,2\} \tag{4.21}$$

is finite. Lemma 2.3 implies for $i=1,2$

$$|D_i g(t,u) - D_i g(t,\bar u) - D_2 D_i g(t,\bar u)(u-\bar u)| \le \Omega_{D_2 D_i g}(|u-\bar u|;\alpha,M_5)|u-\bar u|, \qquad t\in[0,\alpha],\ u,\bar u\in M_5, \tag{4.22}$$

where $\Omega_{D_2 D_i g}$ is the modulus of continuity of $D_2 D_i g$ $(i=1,2)$, i.e.,

$$\Omega_{D_2 D_i g}(\varepsilon;\alpha,M_5) = \sup\{|D_2 D_i g(t,u) - D_2 D_i g(t,\bar u)|\colon\ |u-\bar u|\le\varepsilon,\ t\in[0,\alpha],\ u,\bar u\in M_5\}.$$

Let $\mu$ be defined as in the proof of Theorem 4.4 and $\zeta(t;\bar\gamma,h) := \operatorname{ess\,sup}\{|\dot y(s)-\dot x(s)-\dot z(s)|\colon\ s\in[0,t]\}$. Combining (4.20), (4.21), (4.22) and the estimate

$$|\dot y(t-\eta(t))| \le |\dot y(t-\eta(t)) - \dot x(t-\eta(t))| + |\dot x(t-\eta(t))| \le L|h|_\Gamma + N_2 \le L\delta + N_2, \qquad t\in[0,\alpha],$$

we get for a.e. $t\in[0,\alpha]$

$$\begin{aligned} |A(t;\bar\gamma,h)| \ \le\ & \Omega_{D_2 D_1 g}(|y(t-\eta(t)) - x(t-\eta(t))|;\alpha,M_5)|y(t-\eta(t)) - x(t-\eta(t))| \\ & + N_6\mu(t-\eta(t);\bar\gamma,h) \\ & + N_3(L\delta+N_2)\Omega_{D_2 D_2 g}(|y(t-\eta(t)) - x(t-\eta(t))|;\alpha,M_5)|y(t-\eta(t)) - x(t-\eta(t))| \\ & + N_3 N_6 (L\delta+N_2)\mu(t-\eta(t);\bar\gamma,h) \\ & + N_3 N_6 |\dot y(t-\eta(t)) - \dot x(t-\eta(t))||z(t-\eta(t))| + N_1 N_3 \zeta(t-\eta(t);\bar\gamma). \end{aligned} \tag{4.23}$$

Then, using (3.2), the monotonicity of $\Omega_{D_2 D_i g}$, $\mu$ and $\zeta$ and Theorem 4.3, (4.23) yields

$$\begin{aligned} |A(t;\bar\gamma,h)| \ \le\ & L\Omega_{D_2 D_1 g}(L|h|_\Gamma;\alpha,M_5)|h|_\Gamma + N_3(L\delta+N_2)L\Omega_{D_2 D_2 g}(L|h|_\Gamma;\alpha,M_5)|h|_\Gamma \\ & + N_6(1+N_3(L\delta+N_2))\mu(t;\bar\gamma,h) + LN_3N_4N_6|h|_\Gamma^2 + N_1N_3\zeta(t-\eta_0;\bar\gamma). \end{aligned} \tag{4.24}$$

Using the notations introduced in the proof of Theorem 4.4 we have

$$\begin{aligned} B(t;\bar\gamma,h) \ =\ & \omega_f(t,x_t,\Lambda(t,x_t,\bar\sigma),\bar\theta;y_t,\Lambda(t,y_t,\bar\sigma+h^\sigma),\bar\theta+h^\theta) \\ & + D_2 f(t,x_t,\Lambda(t,x_t,\bar\sigma),\bar\theta)(y_t-x_t-z_t) \\ & + D_3 f(t,x_t,\Lambda(t,x_t,\bar\sigma),\bar\theta)\Big(\omega_\Lambda(t,x_t,\bar\sigma;y_t,\bar\sigma+h^\sigma) + D_2\Lambda(t,x_t,\bar\sigma)(y_t-x_t-z_t)\Big) \end{aligned}$$

hence

$$|B(t;\bar\gamma,h)| \le G_f(t;\bar\gamma,h) + L_1 G_\Lambda(t;\bar\gamma,h) + L_1(N_5+1)\mu(t;\bar\gamma,h).$$

Therefore, combining it with (4.19) and (4.24) we get

$$|\dot y(t) - \dot x(t) - \dot z(t)| \le P_1(h) + P_2\zeta(t-\eta_0;\bar\gamma), \qquad \text{a.e. } t\in[0,\alpha],$$

where

$$\begin{aligned} P_1(h) \ =\ & L\Big(\Omega_{D_2 D_1 g}(L|h|_\Gamma;\alpha,M_5) + N_3(L\delta+N_2)\Omega_{D_2 D_2 g}(L|h|_\Gamma;\alpha,M_5)\Big)|h|_\Gamma \\ & + LN_3N_4N_6|h|_\Gamma^2 + \Big(N_6(1+N_3(L\delta+N_2)) + L_1(N_5+1)\Big)\mu(\alpha;\bar\gamma,h) \\ & + \max\{G_f(t;\bar\gamma,h) + L_1 G_\Lambda(t;\bar\gamma,h)\colon\ t\in[0,\alpha]\}, \end{aligned}$$

and $P_2 = N_1N_3$. Clearly, $\dot{y}(t) - \dot{x}(t) - \dot{z}(t) = 0$ for $t \in [-r, 0]$, hence

$$\zeta(t;\bar{\gamma}) \leq P_1(h) + P_2\zeta(t-\eta_0;\bar{\gamma}), \qquad t \in [0,\alpha].$$

This, using the method of steps, yields

$$|\dot{y}_t - \dot{x}_t - \dot{z}_t|_{L^\infty} \leq \zeta(t;\bar{\gamma}) \leq (1 + P_2 + \cdots + P_2^m)P_1(h), \qquad t \in [0,\alpha], \tag{4.25}$$

where $m = [\alpha/\eta_0]$. Therefore, in view of (4.25), it suffices to show that $P_1(h)/|h|_\Gamma \to 0$ as $|h|_\Gamma \to 0$. From the proof of Theorem 4.4 follows that $\mu(\alpha;\bar{\gamma},h)/|h|_\Gamma \to 0$ as $|h|_\Gamma \to 0$, the continuity of $D_2D_ig$ on the compact set $[0,\alpha] \times M_5$ yields $\Omega_{D_2D_ig}(L|h|_\Gamma;\alpha,M_5) \to 0$ as $|h|_\Gamma \to 0$ for $i = 1, 2$. Therefore we have to argue only that $G_f(t;\bar{\gamma},h)/|h|_\Gamma \to 0$ and $G_\Lambda(t;\bar{\gamma},h)/|h|_\Gamma \to 0$ uniformly in $t \in [0,\alpha]$ as $|h|_\Gamma \to 0$.

Consider a sequence $h_k \in \Gamma$ such that $|h_k|_\Gamma \to 0$ as $k \to \infty$. As in the proof of Theorem 4.4, we again use notations $y^k(t) = x(t;\bar{\gamma} + h_k)$ and $z(t) = z(t;\bar{\gamma},h_k)$. It follows from (4.18) that $\max\{G_f(t;\bar{\gamma},h_k)\colon t \in [0,\alpha]\} \to 0$ as $k \to \infty$.

Define

$$\bar{\Omega}(\varepsilon) = \sup\{|\dot{x}(u) - \dot{x}(\bar{u})|\colon |u - \bar{u}| \leq \varepsilon,\ u,\bar{u} \in [-r,\alpha]\}.$$

Then (4.17) combined with $\Omega_{\dot{x}_s}(\varepsilon) \leq \bar{\Omega}(\varepsilon)$ for $s \in [0,\alpha]$ and (4.9) yields

$$G_\Lambda(s;\bar{\gamma},h) \leq \Big(L_2\bar{\Omega}\Big(L_2(L+1)|h|_\Gamma\Big) + N_2L_3(L+1)|h|_\Gamma + L_2L|h|_\Gamma\Big)(L+1)|h|_\Gamma. \qquad s \in [0,\alpha].$$

This concludes the proof of the theorem. □

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