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On the Asymptotic Behavior of the Solutions of a State-Dependent Delay Equation

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Abstact: In this paper we establish an asymptotic formula for "small" solutions of the delay equation $\dot{x}(t) = ax(t - b|x(t)|)$, where a and b are positive constants.

1 Introduction

In a recent paper ([3]) we have studied the theoretical convergence properties of an approximation technique (using equations with piecewise constant arguments) for state-dependent delay equations and in a follow-up paper ([4]) we have done extensive numerical testing on the performance of our method. In particular, we have considered in [4] numerical solutions of the following initial value problem (IVP)

$$\dot{y}(t) = y(t - |y(t)|) + \sin 2t - \sin^2(t - \sin^2 t), \qquad t \geq 0, \tag{1.1}$$
$$y(t) = \Phi(t), \qquad t \leq 0. \tag{1.2}$$

It is easy to check that the function $y(t) = \sin^2 t,\ t \geq 0$ solves (1.1)-(1.2) with initial function $\Phi(t) = \sin^2 t$. As a matter of fact, from the initial function we only use the information that $y(0) = 0$. On Figure 1 we display the numerical solutions of (1.1)-(1.2) using various values for the discretization constant, h. The graph indicates convergence on finite intervals (in agreement with the theoretical predictions of [3]), but after some time (which of course depends on h) we can observe a "very regular divergence" (i.e., more or less linearly growing error) of the numerical solutions.

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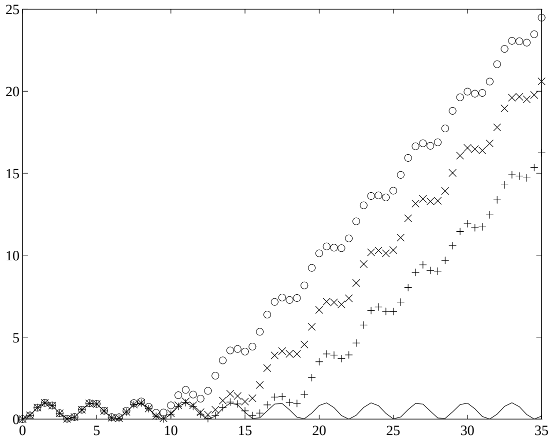

Figure 1. solid line: $\sin^2 t$, o: $h = 0.01$, x: $h = 0.001$, +: $h = 0.0001$

We have studied numerical solutions of IVP (1.1)-(1.2) with perturbed initial functions (see Figure 2), then we have considered the homogeneous equation corresponding to (1.1) with various initial functions (see Figure 3) and in all cases have observed the same type of asymptotics.

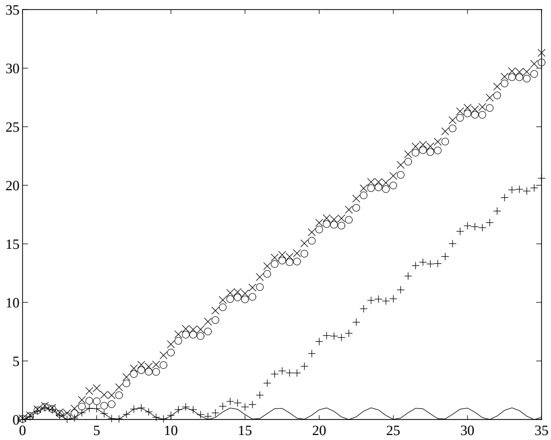

Figure 2. Numerical solutions of (1.1)-(1.2) with $h = 0.001$ using initial function
+: $\sin^2 t$, o: $\sin^2 t + 0.05$, x: $\sin^2 t + 0.1 \cos 5t$,
solid line: $\sin^2 t$

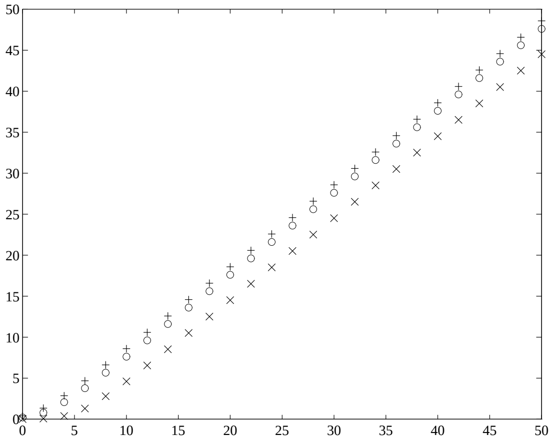

Figure 3. Numerical solutions of (1.3)-(1.4) with $h = 0.001$ using initial function
o: $t + 0.2$, x: $0.2 \sin 5t + 0.01$, +: $0.4 \cos 2t$

Motivated by these numerical findings, in this paper we consider "small" solutions, corresponding to small (in sup norm) initial functions, of the IVP

$$\dot{x}(t) = x(t - |x(t)|), \qquad t \geq 0, \tag{1.3}$$
$$x(t) = \Phi(t), \qquad t \leq 0, \tag{1.4}$$

where the delay equation (1.3) is the homogeneous counterpart of equation (1.1).

The investigation of "small" solutions is justified by the fact that the approximation error initially can be controlled by the proper selection of the discretization constant.

The asymptotic analysis, presented in the next section establishes the following relation on the unique solution, $x(t)$, of IVP (1.3)-(1.4)

$$x(t) = t + \alpha + \beta(t), \tag{1.5}$$

where α is a constant, $\beta(t)$ has the properties that $\lim_{t\to\infty}\beta(t) = 0$ and $\lim_{t\to\infty}\dot{\beta}(t) = 0$.

In Section 3 we present examples to illustrate applications and limitations of (1.5) and to indicate how the results could possibly be extended to the more general equation

$$\dot{x}(t) = ax\Big(t - r(x(t))\Big), \qquad a > 0, \tag{1.6}$$

where $r(t)$ is a given Lipschitz-continuous function.

We conclude this section by noting that a complete asymptotic theory for Equation (1.6), with $a < 0$ can be found in [1].

2 Main Results

Consider initial value problem

$$\dot{x}(t) = ax(t - b|x(t)|), \qquad t \geq 0, \tag{2.1}$$
$$x(t) = \Phi(t), \qquad t \leq 0, \tag{2.2}$$

where $a > 0$, $b > 0$.

We assume throughout the paper that the initial function, $\Phi(t)$, is continuous, which implies that IVP (2.1)-(2.2) has a solution, i.e., there exists a continuously differentiable function which satisfies (2.1) and (2.2). Moreover we also assume that the solution is unique, which is satisfied if the initial function is Lipschitz-continuous. (See [2], [3] or [5] for existence, uniqueness theorems.)

We introduce the simplifying notations $x_0 \equiv \Phi(0)$ and $\omega(t) \equiv t - b|x(t)|$, which are used throughout the paper.

It is easy to check the following two statements.

Proposition 2.1

(i) If $x(t)$ is the solution of (2.1)-(2.2) corresponding to the initial function $\Phi(t)$, then $-x(t)$ is the solution of (2.1)-(2.2) corresponding to the initial function $-\Phi(t)$.

(ii) If $x_0 = 0$, then the solution of (2.1)-(2.2), $x(t)$, is identically zero for $t \geq 0$.

By Proposition 2.1 it is enough to study the qualitative behavior of IVP (2.1)-(2.2) with initial function $\Phi(\cdot)$ satisfying $\Phi(0) = x_0 > 0$.

Proposition 2.2 *If $x_0 > 0$, then the solution of IVP (2.1)-(2.2) satisfies $x(t) > 0$ for $t \geq 0$.*

PROOF: The continuity of $\Phi(t)$ implies that there exists $\tilde{t} > 0$ such that $\Phi(t) > 0$ for $t \in (-\tilde{t}, 0]$. Suppose that there exists $t^* > 0$ such that the solution $x(t)$ of (2.1)-(2.2) satisfies $x(t^*) = 0$ and $x(t) > 0$ for $t \in (0, t^*)$. Then the mean value theorem implies that for any $0 < \delta < t^*$ there exists $\bar{t} \in (t^* - \delta, t^*)$ such that $\dot{x}(\bar{t}) < 0$. But $\dot{x}(\bar{t}) = ax(\bar{t} - bx(\bar{t}))$ implies that $\bar{t} - bx(\bar{t}) < -\tilde{t}$ for such $\bar{t}$, which yields $\bar{t} + \tilde{t} < bx(\bar{t})$. If $\delta \to 0$ we get $t^* + \tilde{t} \leq 0$ which is a contradiction. Therefore $x(t) > 0$ for all $t \geq 0$.

In the next proposition we reveal an interesting asymptotic property of the solutions of IVP (2.1)-(2.2).

Proposition 2.3 *If there exists a $T \geq 0$ such that the solution of (2.1)-(2.2) satisfies $\dot{x}(T) = \frac{1}{b}$, then the solution has the form $x(t) = \frac{1}{b}(t - T) + x(T)$ for $t \geq T$.*

PROOF: Equation (2.1) and the condition $\dot{x}(T) = \frac{1}{b}$ yield the relation

$$\frac{1}{b} = \dot{x}(T) = ax(T - bx(T)) = ax\left(t - b\left(\frac{1}{b}(t - T) + x(T)\right)\right),$$

which means that the function $\frac{1}{b}(t-T)+x(T)$ satisfies (2.1) for $t \geq T$.

This result immediately implies that $\omega(\cdot)$ is a monotone function, because if there is a $T \geq 0$ such $\dot{\omega}(T)=0$, i.e., $\dot{x}(T)=\frac{1}{b}$, then $\omega(t)=\omega(T)$ for $t \geq T$. But if $\dot{\omega}(t) \neq 0$ for all $t \geq 0$, then $\omega(\cdot)$ is a strictly monotone increasing or decreasing function. Therefore we have obtained the following proposition.

Proposition 2.4 *The time lag function, $\omega(t)$, is monotone for $t \geq 0$.*

Simple calculations and substitution into (2.1)-(2.2) show that for the case $\omega(t) \leq 0$, the time lag function, $\omega(\cdot)$, satisfies the following ordinary differential equation

$$\dot{\omega}(t)=1-ab\Phi(\omega(t)), \qquad \text{for } t \geq 0, \quad \omega(t) \leq 0, \tag{2.3}$$

with initial condition

$$\omega(0)=-bx_0. \tag{2.4}$$

Assuming that the initial interval is finite, i.e., there exists $K>0$ such that $\omega(t) \equiv t-b|x(t)| \geq -K$ for all $t \geq 0$, we can prove that solutions of IVP (2.1)-(2.2) are asymptotically approaching a line with slope $\frac{1}{b}$. In particular, we have

Theorem 2.5 *Assume that $x_0>0$. Then the following two statements are equivalent*

(i) There exists a $K>0$ such that the solution $x(t)$ of IVP (2.1)-(2.2) satisfies $t-b|x(t)| \geq -K$ for $t \geq 0$.

(ii) There exist a constant α and a function $\beta(t)$ such that the solution of IVP (2.1)-(2.2) has the form

$$x(t)=\frac{1}{b}(t+\alpha+\beta(t)), \qquad t \geq 0, \tag{2.5}$$

where $\lim_{t\to\infty}\beta(t)=0$ and $\lim_{t\to\infty}\dot{\beta}(t)=0$.

Proof: Trivially (ii) implies (i).

Assume that condition (i) holds.

By assumption $\omega(0)<0$. If there exists a $t>0$ such that $\omega(t)>0$, then $\omega(\cdot)$ is a monotone increasing function. Otherwise $\omega(t) \leq 0$ for $t \geq 0$. We want to show that the function $\omega(\cdot)$ is bounded from above.

The only interesting case is when there exists a $T>0$ such that $\omega(T)=0$, $\omega(t)<0$ for $t \in [0,T)$ and $\omega(t)>0$ for $t>T$. Using Proposition 2.2 and the definition of T we have that $\dot{x}(t)=ax(\omega(t))>0$ for $t>T$, hence the solution, $x(\cdot)$, is monotone increasing for $t>T$, therefore we have

$$\dot{x}(t) \geq \min_{t\in[0,T]} ax(t)>0 \qquad \text{for } t>T,$$

which implies that $x(t) \to \infty$ as $t \to \infty$.

Suppose that $\lim_{t\to\infty}\omega(t)=\infty$. Using the monotonicity of $\omega(\cdot)$, Equation (2.1) and the fact that $x(t)\to\infty$ as $t\to\infty$ we obtain

$$0\le\dot\omega(t)=1-b\dot x(t)=1-abx(t-bx(t))=1-abx(\omega(t))\to-\infty, \qquad (2.6)$$

which is a contradiction. This means that there exists a constant $L>0$ such that $0<\omega(t)\le L$ for $t>T$. Therefore in every case we have that $\omega(\cdot)$ is a bounded function from above, and by assumption (i) it is also bounded from below. It is a monotone function, therefore its limit at ∞ exists, so we can define $\alpha\equiv-\lim_{t\to\infty}\omega(t)$ and $\beta(t)\equiv-\omega(t)-\alpha$. With these definitions relation (2.5) and $\lim_{t\to\infty}\beta(t)=0$ are satisfied. Monotonicity and boundedness of $\omega(\cdot)$ imply that $\lim_{t\to\infty}\dot\beta(t)=-\lim_{t\to\infty}\dot\omega(t)=0$.
The proof of the theorem is complete.

The following proposition gives conditions, which guarantee that IVP (2.1)-(2.2) has finite initial interval, i.e., condition (i) holds in Theorem 2.5.

Proposition 2.6

(i) If $\Phi(-bx_0)\le\frac{1}{ab}$, then $\omega(t)\ge-bx_0$ for $t\ge0$.

(ii) If $\Phi(-bx_0)>\frac{1}{ab}$ and there exists a constant $L>bx_0$ such that $\Phi(-L)=\frac{1}{ab}$, then $-L<\omega(t)$ for $t\ge0$.

(iii) If $\Phi(t)>\frac{1}{ab}$ for $t\le-bx_0$, then $\lim_{t\to\infty}\omega(t)=-\infty$.

PROOF:
(i) If $\Phi(-bx_0)=\frac{1}{ab}$, then $-bx_0$ is an equilibrium of (2.3), therefore $\omega(t)=-bx_0$ for all $t\ge0$. Consider the case when $\Phi(-bx_0)<\frac{1}{ab}$. This implies that $\dot\omega(0)=1-ab\Phi(-bx_0)>0$, therefore using Proposition 2.4 the function $\omega(\cdot)$ is monotone increasing, hence $\omega(t)\ge-bx_0$ for all $t\ge0$.
(ii) By assumption $-L$ is an equilibrium of the autonomous equation (2.3), $\omega(0)=-bx_0>-L$, therefore we have that $\omega(t)>-L$ for all $t\ge0$.
(iii) The assumption, $\Phi(t)>\frac{1}{ab}$, $t\le-bx_0$, implies that $\dot\omega(t)<0$ for all $t\ge0$, hence $\omega(\cdot)$ is defined by (2.3)-(2.4) for all $t\ge0$, and is a monotone decreasing function. Suppose that $\omega(t)\to-L>-\infty$ as $t\to\infty$. Then $-L$ has to be an equilibrium point of (2.3), which is equivalent to that $\Phi(-L)=\frac{1}{ab}$, which contradicts to our assumption.

Summarizing our results we formulate the following corollary to Theorem 2.5.

Theorem 2.7 *The solution of IVP (2.1)-(2.2) is asymptotically a straight line with slope $\frac{1}{b}$, i.e., there exist a constant α and a function $\beta(t)$ such that the solution of IVP (2.1)-(2.2) has the form*

$$x(t)=\frac{1}{b}(t+\alpha+\beta(t)),\qquad t\ge0, \qquad (2.7)$$

where $\lim_{t\to\infty}\beta(t)=0$ and $\lim_{t\to\infty}\dot\beta(t)=0$ if and only if either

(i) $\Phi(-bx_0)\le\frac{1}{ab}$. In this case $t-b|x(t)|\ge-bx_0$ for $t\ge0$.
or

(ii) $\Phi(-bx_0)>\frac{1}{ab}$ and there exists a constant $L\ge bx_0$ such that $\Phi(-L)=\frac{1}{ab}$. In this case $-L<t-b|x(t)|\le-bx_0$ for $t\ge0$.

3 Examples, and a more general IVP

Consider the special case of (2.1)–(2.2)

$$\begin{aligned} \dot{x}(t) &= x(t-|x(t)|), \qquad t \geq 0, && (3.1) \\ x(t) &= \Phi(t), \qquad t \leq 0, && (3.2) \end{aligned}$$

with various initial functions.

Example 3.1 Let the initial function $\Phi(t) = 1 + t$. Then the solution of (3.1)-(3.2) is $x(t) = t + e^{-t}$, for $t \geq 0$. In this example we have that $-1 \leq \omega(t) < 0$ for $t \geq 0$.

Example 3.2 If the initial function

$$\Phi(t) = \begin{cases} 3+t, & -2 \leq t \leq -1 \\ 1-t, & -1 \leq t \leq 0, \end{cases}$$

then the solution of (3.1)-(3.2) is $x(t) = t + 2 - e^{-t}$. We have $-2 < \omega(t) \leq -1$.

Example 3.3 Let the initial function $\Phi(t) = 1 - t^2$. Then the solution of (3.1)-(3.2) is $x(t) = t + \frac{1}{t+1}$.

Example 3.4 If $\Phi(t) = 0.5$, then Theorem 2.7 yields, that the solution of (3.1)-(3.2) has form (2.7), where $\alpha < 0$, because the right hand side of (2.3) is positive for all t, therefore $-\alpha = \lim_{t\to\infty} \omega(t) > 0$.

These examples indicate that in (2.7) α and $\beta(t)$ can have any sign, and the order of convergence of $\beta(\cdot)$ to zero can be for example exponential or polynomial. The time lag function, $\omega(\cdot)$ can be both increasing and decreasing.

The next two examples show cases when the assumptions of Theorem 2.7 are not satisfied.

Example 3.5 Consider the initial function $\Phi(t) = p$, where $p > 1$. Then the solution of (3.1)-(3.2) is $x(t) = pt + p$, for $t \geq 0$. For this example $\omega(t) = (1-p)t - p$, so $\omega(t) \to -\infty$ as $t \to \infty$, and we do not have asymptotic formula (2.5).

Example 3.6 If the initial function $\Phi(t) = 1 - t$, then the solution of (3.1)-(3.2) is $x(t) = t + e^t$, for $t \geq 0$. In this example also $\omega(t) \to -\infty$ as $t \to \infty$, and the solution grows exponentially.

To conclude we consider the IVP

$$\begin{aligned} \dot{x}(t) &= ax\Big(t - r(x(t))\Big), \qquad t \geq 0, && (3.3) \\ x(t) &= \Phi(t), \qquad t \leq 0, && (3.4) \end{aligned}$$

where $r(\cdot)$ is monotone increasing, $r(0) = 0$, and $a > 0$.

Numerical studies indicate that solutions corresponding to “small” initial functions have similar asymptotic properties. For example on Figure 4 we show numerical solutions of (3.3)-(3.4) for the delay function $r(x) = x^2$ with parameter $a = 1$, corresponding to initial functions

$$\begin{aligned} \Phi_1(t) &= 0.2 \sin 5t + 0.01, \\ \Phi_2(t) &= 0.4 \cos 2t, \\ \Phi_3(t) &= 0.05. \end{aligned}$$

This and other numerical runings suggest the conjecture, that for "small" initial function, the solution of (3.3)-(3.4) is asymptotically a shift of the inverse of $r(\cdot)$, i.e., $r^{-1}(t+\alpha)$.

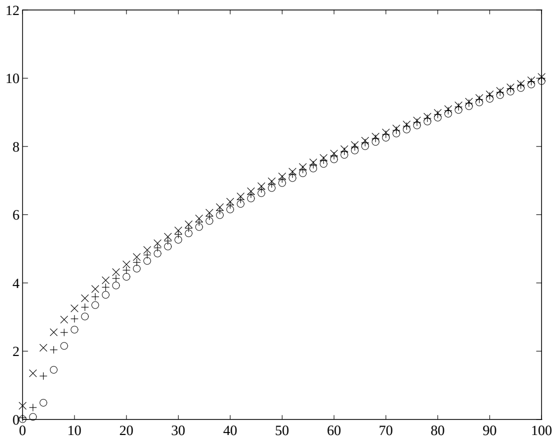

Figure 4. Numerical solutions of IVP (3.3)-(3.4) with $r(x) = x^2$, $a = 1$ and with initial functions o: $\Phi_1(t)$, x: $\Phi_2(t)$, +: $\Phi_3(t)$

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