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Stability Results for Cellular Neural Networks with Delays

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This paper is dedicated to Prof. László Hatvani on the occasion of his 60th birthday.

Abstract

In this paper we give a sufficient condition to imply global asymptotic stability of a delayed cellular neural network of the form

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_{ij})) + u_i, \qquad t \geq 0, \quad i = 1, \ldots, n,$$

where $f(t) = \frac{1}{2}(|t+1| - |t-1|)$. In order to prove this stability result we need a sufficient condition which guarantees that the trivial solution of the linear delay system

$$\dot{z}_i(t) = \sum_{j=1}^{n} a_{ij} z_j(t) + \sum_{j=1}^{n} b_{ij} z_j(t-\tau_{ij}), \qquad t \geq 0, \quad i = 1, \ldots, n$$

is asymptotically stable independently of the delays τ_{ij}.

keywords: delayed cellular neural networks, global asymptotic stability, M-matrix

1 Introduction

The notion of cellular neural networks (CNNs) was introduced by Chua and Yang ([5]), and since then, CNN models have been used in many engineering applications, e.g., in signal processing and especially in static image treatment [6]. As a generalization of CNNs, cellular neural networks with delays (DCNNs) were introduced by Roska and Chua [14].

In this paper we study the asymptotic stability of the DCNN model described by the system of nonlinear delay differential equations

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_{ij})) + u_i, \qquad t \geq 0, \quad i = 1, \ldots, n. \quad (1.1)$$

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Here n is the number of cells; $x_i(t)$ denotes the potential of the ith cell at time t; d_i represents the rate with which the ith unit resets its potential to the resting state when it is isolated from other cells and inputs; a_{ij} and b_{ij} denote the strengths of the jth unit on the ith unit at time t and $t-\tau_{ij}$, respectively; τ_{ij} corresponds to transmission delay between the ith and jth cells; f denotes an output function; u_i is an external input to the ith cell.

The stability of (1.1) and more general classes of DCNNs has been intensively studied, see, e.g., [2]–[4], [11]–[13], [15]–[18], and the references therein. We will assume throughout this paper that the output function $f\colon \mathbb{R}\to\mathbb{R}$ is defined by

$$f(t)=\frac{1}{2}(|t+1|-|t-1|)=\begin{cases}1, & t>1,\\ t, & -1\le t\le 1,\\ -1, & t<-1.\end{cases} \tag{1.2}$$

This function is widely used in CNN and DCNN models.

In a recent paper Mohamad and Gopalsamy ([13]) have shown using fixed point method that if f is defined by (1.2) and

$$d_i>\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|),\qquad i=1,2,\dots,n, \tag{1.3}$$

then (1.1) has a unique fixed point which is globally exponentially stable. In our Theorem 4 (see below) we show that the weaker assumption

$$d_i-a_{ii}>\sum_{\substack{j=1,\\ j\ne i}}^{n}|a_{ij}|+\sum_{j=1}^{n}|b_{ij}|,\qquad i=1,2,\dots,n, \tag{1.4}$$

together with another condition (see (3.11) below) implies the global asymptotic stability of the unique equilibrium of (1.1). We also conjecture (see Conjecture 1 below) that assumption (3.11) can be omitted, (1.4) itself, or even a weaker condition implies the global asymptotic stability of the equilibrium.

We remark that condition (1.4) is equivalent to saying that the matrix $K=(k_{ij})$ with elements

$$k_{ij}=\begin{cases}d_i-a_{ii}-|b_{ii}|, & \text{if } i=j,\\ -|a_{ij}|-|b_{ij}| & \text{otherwise}\end{cases}$$

is diagonally dominant and it has positive diagonal elements. We recall that an $n\times n$ matrix $K=(k_{ij})$ is (row) diagonally dominant, if

$$|k_{ii}|>\sum_{\substack{j=1,\\ j\ne i}}^{n}|k_{ij}|,\qquad i=1,\dots,n.$$

Our condition (1.4) is similar to that given by Takahashi in [15], where it was shown that if $d_1=d_2=\cdots=d_n=1$ and the $n\times n$ matrix $W=(w_{ij})$ with elements

$$w_{ij}=\begin{cases}a_{ii}-1-|b_{ii}|, & \text{if } i=j,\\ -|a_{ij}|-|b_{ij}| & \text{otherwise}\end{cases}$$

is a nonsingular M-matrix (see definition below), then every solution of (1.1) tends to a constant equilibrium, i.e., the system is completely stable. Clearly, condition (1.4) implies that $d_i - a_{ii} > |b_{ii}|$, so in this case W can not be an M-matrix. Similarly, if W is an M-matrix, then (1.4) can not hold, therefore the two conditions cover disjoint cases. We comment that despite the similarities of the two conditions, the proof of our result requires a different technique than that used in [15]. Our results were motivated by the monotone technique we used in [9], where we studied the scalar version of (1.1) with f defined by (1.2), and showed that the scalar version of (1.4) implies the global asymptotic stability of the unique equilibrium.

In Section 2 we give a sufficient condition which implies asymptotic stability of a linear delay system for all delays. Such stability is called absolute stability in the engineering literature. We extend a known result [3] for the case we use in Section 3 to prove our stability results for (1.1). In Section 4 we give an example to illustrate the main result and we formulate a conjecture to generalize the result.

First we introduce some notations. Let $\mathbb{R}_+$ be the set of positive real numbers. We use the relation $\mathbf{x} \le \mathbf{y}$ ($\mathbf{x} < \mathbf{y}$, respectively) for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $x_i \le y_i$ ($x_i < y_i$, respectively) for all $i = 1, \dots, n$, where $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$. We introduce the vectors $\mathbf{0} = (0, 0, \dots, 0)^T \in \mathbb{R}^n$ and $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$.

For an $n \times n$ matrix B the symbol $|B|$ denotes the corresponding $n \times n$ matrix with ijth element $|b_{ij}|$. Similarly, $|\mathbf{u}| = (|x_1|, \dots, |x_n|)^T$.

We say that an $n \times n$ matrix K is an M-matrix, if all of its diagonal elements are nonnegative, and its off-diagonal elements are nonpositive, and all of its principal minors are nonnegative (see, e.g., [1], [3] or [7]). It is known (see, e.g., [1]) that if K is a nonsingular M-matrix, then $\mathbf{x} \le \mathbf{y}$ implies $K^{-1}\mathbf{x} \le K^{-1}\mathbf{y}$.

Remark 1 Let K be a matrix such that the diagonal elements of K are all positive and the off-diagonal elements are all nonpositive. Then it is known (see, e.g., Theorem 2.3 in [1]) that if K is a diagonally dominant, then it is a nonsingular M-matrix, as well. Moreover, K is a nonsingular M-matrix, if and only if, there exists a positive diagonal matrix D such that KD is a diagonally dominant matrix. We note that there are 50 conditions listed in [1] which are all equivalent to that a matrix is a nonsingular M-matrix.

2 Absolute Stability of a Linear System

Consider the autonomous linear delay system

$$\dot{z}_i(t) = \sum_{j=1}^{n} a_{ij} z_j(t) + \sum_{j=1}^{n} b_{ij} z_j(t - \tau_{ij}), \qquad t \ge 0, \quad i = 1, \dots, n, \tag{2.1}$$

where $\tau_{ij} \ge 0$ for $i, j = 1, \dots, n$.

We put the coefficients to the $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$. For the matrix A we associate the $n \times n$ diagonal matrix $A_0 = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn})$, i.e., the diagonal part of

A, and let $A_1 = A - A_0$ be the off-diagonal part of A. Then with this notation, which we use throughout this paper, we can rewrite A as $A = A_0 + A_1$. Similarly, let B_0 be the diagonal part of B, and denote $B_1 = B - B_0$.

In the case when $A_1 = 0$ and $B_0 = 0$ the necessary and sufficient condition for the stability and asymptotic stability of (2.1) for all selection of the delays τ_{ij} was established in [10]. Following the methods of [10] this result was extended in [3] for the special case when only $A_1 = 0$, i.e., A is a diagonal matrix in (2.1), and B is an arbitrary matrix.

Theorem 1 (see Theorem 2.6 in [3]) *Suppose $A = A_0$. Then the trivial solution of* (2.1) *is asymptotically stable for all delays $\tau_{ij} \geq 0$, if and only if $-A - |B|$ is an M-matrix and $A + B$ is a nonsingular matrix.*

Note that in the case when B is a nonnegative matrix, this result follows from a more general theorem in [7], where such result was proved for quasilinear delay differential equations. In the case when B is a nonnegative matrix, Theorem 1 also follows from an other generalization of it given in [8], where it was shown that if $\tau_k \geq 0$, $(k = 1, \ldots, p)$, $D_k \geq 0$ are diagonal matrices for $k = 1, \ldots, p$ such that $\sum_{k=1}^{p} D_k$ is invertible, B_ℓ are nonnegative $n \times n$ matrices for $\ell = 1, \ldots, r$, and equation

$$\dot{\mathbf{u}}(t) = -\sum_{k=1}^{p} D_k \mathbf{u}(t - \tau_k)$$

has a positive fundamental solution, then the trivial solution of

$$\dot{\mathbf{x}}(t) = -\sum_{k=1}^{p} D_k \mathbf{x}(t - \tau_k) + \sum_{\ell=1}^{r} B_\ell \mathbf{x}(t - \sigma_\ell)$$

is asymptotically stable for all $\sigma_1, \ldots, \sigma_\ell \geq 0$, if and only if

$$\sum_{k=1}^{p} D_k - \sum_{\ell=1}^{r} B_\ell$$

is a nonsingular M-matrix.

We extend the sufficient part of Theorem 1 for the case which we will need later. We assume $A \neq A_0$, i.e., there are nonzero off-diagonal parts of A. The proof follows that of Theorem 1 (see [3]).

Theorem 2 *Suppose $-A_0 - |A_1| - |B|$ is a nonsingular M-matrix. Then the trivial solution of* (2.1) *is asymptotically stable for all delays $\tau_{ij} \geq 0$.*

Proof Finding the solution of (2.1) in the form $e^{\lambda t}\mathbf{v}$ ($\mathbf{v} \neq 0$) leads to the characteristic equation

$$\det\begin{pmatrix} a_{11}+b_{11}e^{-\lambda\tau_{11}}-\lambda & a_{12}+b_{12}e^{-\lambda\tau_{12}} & \cdots & a_{1n}+b_{1n}e^{-\lambda\tau_{1n}} \\ a_{21}+b_{21}e^{-\lambda\tau_{21}} & a_{22}+b_{22}e^{-\lambda\tau_{22}}-\lambda & \cdots & a_{2n}+b_{2n}e^{-\lambda\tau_{2n}} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1}e^{-\lambda\tau_{n1}} & a_{n2}+b_{n2}e^{-\lambda\tau_{n2}} & \cdots & a_{nn}+b_{nn}e^{-\lambda\tau_{nn}}-\lambda \end{pmatrix} = 0 \qquad (2.2)$$

of (2.1). It is known that the asymptotic stability of the trivial solution of (2.1) is equivalent to that all roots of (2.2) have negative real parts. Let λ be a root of (2.2), then λ is an eigenvalue of the matrix

$$G(\lambda) = \begin{pmatrix} a_{11}+b_{11}e^{-\lambda\tau_{11}} & a_{12}+b_{12}e^{-\lambda\tau_{12}} & \cdots & a_{1n}+b_{1n}e^{-\lambda\tau_{1n}} \\ a_{21}+b_{21}e^{-\lambda\tau_{21}} & a_{22}+b_{22}e^{-\lambda\tau_{22}} & \cdots & a_{2n}+b_{2n}e^{-\lambda\tau_{2n}} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1}e^{-\lambda\tau_{n1}} & a_{n2}+b_{n2}e^{-\lambda\tau_{n2}} & \cdots & a_{nn}+b_{nn}e^{-\lambda\tau_{nn}} \end{pmatrix}.$$

Since $-A_0 - |A_1| - |B|$ is a nonsingular M-matrix, it is known (see, e.g., Theorem 2.3 in [1]) there exist positive constants $\gamma_1, \ldots, \gamma_n > 0$ such that

$$(-a_{ii} - |b_{ii}|)\gamma_i > \sum_{\substack{j=1,\\ j\neq i}}^{n} (|a_{ij}| + |b_{ij}|)\gamma_j, \qquad i = 1, \ldots, n. \qquad (2.3)$$

Let $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$. Then Γ is nonsingular, therefore λ is an eigenvalue of the matrix $\Gamma^{-1}G(\lambda)\Gamma$, as well. Therefore an application of Gersgorin's theorem for the matrix $\Gamma^{-1}G(\lambda)\Gamma$ yields

$$|\lambda - a_{ii} - b_{ii}e^{-\lambda\tau_{ii}}| \leq \sum_{\substack{j=1,\\ j\neq i}}^{n} \gamma_i^{-1}(|a_{ij}| + |b_{ij}||e^{-\lambda\tau_{ij}}|)\gamma_j$$

for some i. Therefore for this fixed i

$$\operatorname{Re}(\lambda) \leq \operatorname{Re}(a_{ii} + b_{ii}e^{-\lambda\tau_{ii}}) + \sum_{\substack{j=1,\\ j\neq i}}^{n} \gamma_i^{-1}(|a_{ij}| + |b_{ij}|e^{-(\operatorname{Re}\lambda)\tau_{ij}})\gamma_j.$$

Suppose $\operatorname{Re}(\lambda) \geq 0$. Then (2.3) yields

$$\operatorname{Re}(\lambda)\gamma_i \leq (a_{ii} + |b_{ii}|)\gamma_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} (|a_{ij}| + |b_{ij}|)\gamma_j < 0,$$

which contradicts to the assumption, therefore $\operatorname{Re}(\lambda) < 0$ for all solutions of (2.2). $\square$

The proof implies immediately the next technical result.

Corollary 3 *If $-A_0 - |A_1| - |B|$ is a nonsingular M-matrix, then $A + B$ is nonsingular, as well.*

Proof Let A and B satisfy the assumption, pick any $\tau_{ij} \geq 0$ $(i, j = 1, \dots, n)$, and consider the corresponding system (2.1). The proof of Theorem 2 shows that $\mathbf{v}$ is a nonzero constant solution of system (2.1) if and only if $\lambda = 0$ is a solution of (2.2). But under this assumption all solutions of (2.2) satisfy $\mathrm{Re}(\lambda) < 0$, therefore the only constant solution of (2.1) is the zero solution. On the other hand, the constant $\mathbf{v}$ solutions of (2.1) satisfy $(A + B)\mathbf{v} = \mathbf{0}$, hence $A + B$ is nonsingular. $\square$

3 Stability of a Delayed Neural Network System

Suppose n is a fixed positive integer,

$$d_i > 0,\ \tau_{ij} \geq 0, \quad a_{ij}, b_{ij}, u_i \in \mathbb{R}\ (i, j = 1, \dots, n), \qquad \text{and} \quad f(t) = \frac{1}{2}(|t+1| - |t-1|). \tag{3.1}$$

We introduce the notations $D = \mathrm{diag}(d_1, \dots, d_n)$, $A = (a_{ij})$, $B = (b_{ij})$, $\mathbf{u} = (u_1, \dots, u_n)^T$. As in the previous section, we use the notation $A = A_0 + A_1$, where A_0 is the diagonal part, A_1 is the off-diagonal part of A.

Consider the DCNN model equations

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t - \tau_{ij})) + u_i, \qquad t \geq 0, \quad i = 1, \dots, n \tag{3.2}$$

with the initial conditions

$$x_i(t) = \varphi_i(t), \qquad t \in [-r, 0], \quad i = 1, \dots, n, \tag{3.3}$$

where $r = \max\{\tau_{ij} \colon i, j = 1, \dots, n\}$.

To (3.2) we associate an auxiliary system. For a given $\mathbf{c} > \mathbf{0}$ and $\psi_i : [-r, 0] \to \mathbb{R}_+$ $(i = 1, \dots, n)$ consider the system

$$\dot{y}_i(t) = -d_i y_i(t) + a_{ii} f(y_i(t)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(y_j(t)) + \sum_{j=1}^{n} |b_{ij}| f(y_j(t - \tau_{ij})) + c_i,\ \ t \geq 0,\ i = 1, \dots, n \tag{3.4}$$

associated to (3.2), and the initial condition

$$y_i(t) = \psi_i(t) \qquad t \in [-r, 0], \quad i = 1, \dots, n. \tag{3.5}$$

Lemma 1 *Suppose* (3.1)*. Let* $\psi_i \colon [-r,0] \to \mathbb{R}_+$ *(*$i = 1,\ldots,n$*),* $\mathbf{c} > \mathbf{0}$*, and let* $y_1,\ldots,y_n$ *be the corresponding solution of* (3.4)-(3.5)*. Then there exists* $M > 0$ *such that*

$$0 < y_i(t) < M, \qquad t \geq 0, \quad i = 1,\ldots,n.$$

Proof Since $y_i(0) > 0$ and y_i is continuous on $[0,\infty)$ for all $i = 1,\ldots,n$, $y_i(t) > 0$ for small enough $t \geq 0$. Suppose there exists i and $T > 0$ such that

$$y_j(t) > 0 \quad \text{for } t \in [-r,T),\ j = 1,\ldots,n, \quad \text{and} \quad y_i(T) = 0.$$

Then $\dot{y}_i(T-) \leq 0$. On the other hand, (3.4) implies

$$\dot{y}_i(T) = \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(y_j(T)) + \sum_{j=1}^{n} |b_{ij}| f(y_j(T-\tau_{ij})) + c_i > 0,$$

which is a contradiction. Therefore $y_i(t) > 0$ for all $t > 0$ and $i = 1,\ldots,n$.

Fix i. To prove that y_i is bounded from above, assume that $\limsup_{t\to\infty} y_i(t) = \infty$. Then there exists a monotone increasing sequence t_n such that

$$\lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} y_i(t_n) = \infty, \qquad \text{and} \qquad y_i(t_n) = \max\{y_i(t) \colon t \in [-r,t_n]\}.$$

Then $\dot{y}_i(t_n-) \geq 0$, which contradicts to the relations

$$\begin{aligned}
\dot{y}_i(t_n) &= -d_i y_i(t_n) + a_{ii} f(y_i(t_n)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(y_j(T)) + \sum_{j=1}^{n} |b_{ij}| f(y_j(t_n - \tau_i)) + c_i \\
&\leq -d_i y_i(t_n) + \sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}| + c_i \\
&< 0
\end{aligned}$$

for large enough n. $\square$

Remark 2 It is easy to check that the matrix $D - A_0 - |A_1| - |B|$ is a diagonally dominant matrix with positive diagonal elements, if and only if

$$\mathbf{0} < (D - A_0 - |A_1| - |B|)\mathbf{1}.$$

Lemma 3 *Assume* (3.1)*,* $D - A_0 - |A_1| - |B|$ *is a diagonally dominant matrix, and*

$$\mathbf{0} < \mathbf{c} < (D - A_0 - |A_1| - |B|)\mathbf{1}. \tag{3.6}$$

Let $\psi_i \colon [-r,0] \to \mathbb{R}_+$ *(*$i = 1,\ldots,n$*), and let* $\mathbf{y}(t) = (y_1(t),\ldots,y_n(t))^T$ *be the corresponding solution of* (3.4)-(3.5)*. Then*

$$\lim_{t\to\infty} \mathbf{y}(t) = (D - A_0 - |A_1| - |B|)^{-1}\mathbf{c} < \mathbf{1}. \tag{3.7}$$

Proof It follows from Lemma 1 that

$$M_i = \limsup_{t\to\infty} y_i(t) \qquad m_i = \liminf_{t\to\infty} y_i(t)$$

are finite and $m_i \geq 0$. For a fixed i there exists a sequence t_n such that

$$t_n \to \infty \quad \text{as } n \to \infty, \qquad \dot{y}_i(t_n) \geq 0,\ n = 1, 2 \ldots, \qquad \text{and} \qquad \lim_{n\to\infty} y_i(t_n) = M_i.$$

We may also assume that

$$\lim_{n\to\infty} y_j(t_n) = m_j^* \qquad \text{and} \qquad \lim_{n\to\infty} y_j(t_n - \tau_{ij}) = m_{ij}^{**}$$

for all $j = 1, \ldots, n$ for some $m_j^*, m_{ij}^{**} \in [m_j, M_j]$, since otherwise we can select a subsequence of t_n with this property. Then

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \dot{y}_i(t_n) \\
&= \lim_{n\to\infty} \Big(-d_i y_i(t_n) + a_{ii} f(y_i(t_n)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(y_j(t_n)) + \sum_{j=1}^{n} |b_{ij}| f(y_i(t_n - \tau_{ij})) + c_i \Big) \\
&= -d_i M_i + a_{ii} f(M_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(m_j^*) + \sum_{j=1}^{n} |b_{ij}| f(m_{ij}^{**}) + c_i \\
&\leq -d_i M_i + a_{ii} f(M_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(M_j) + \sum_{j=1}^{n} |b_{ij}| f(M_j) + c_i.
\end{aligned}$$

Therefore for all $i = 1, \ldots, n$

$$\begin{aligned}
c_i &\geq d_i M_i - a_{ii} f(M_i) - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(M_j) - \sum_{j=1}^{n} |b_{ij}| f(M_j) \\
&\geq d_i M_i - a_{ii} f(M_i) - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|. \qquad (3.8)
\end{aligned}$$

Suppose $M_i \geq 1$ for some i. Then (3.8) implies

$$c_i \geq d_i - a_{ii} - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|$$

which contradicts to assumption (3.6), which yields

$$0 < c_i < d_i - a_{ii} - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|.$$

Therefore $0 \le M_i < 1$ for all $i = 1, \dots, n$. This means there exists $t_1 > 0$ such that for $t \ge t_1$ (3.4) is equivalent to the linear system

$$\dot{y}_i(t) = (-d_i + a_{ii})y_i(t) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| y_j(t) + \sum_{j=1}^{n} |b_{ij}| y_j(t - \tau_{ij}) + c_i, \qquad t \ge t_1. \tag{3.9}$$

Define

$$\mathbf{e} = (D - A_0 - |A_1| - |B|)^{-1}\mathbf{c}.$$

Then $\mathbf{e} = (e_1, \dots, e_n)^T$ is the unique equilibrium of the system (3.9), and it follows from (3.6) that $0 \le e_i \le M_i < 1$, so $\mathbf{0} \le \mathbf{e} < \mathbf{1}$. Introducing $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{e}$ we can rewrite (3.9) as

$$\dot{z}_i(t) = (-d_i + a_{ii})z_i(t) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| z_j(t) + \sum_{j=1}^{n} |b_{ij}| z_j(t - \tau_{ij}), \qquad t \ge t_1. \tag{3.10}$$

Since $D - A_0 - |A_1| - |B|$ is a nonsingular M-matrix by Remark 1, Theorem 2 yields the trivial solution of (3.10) is asymptotically stable (independently of the size of the delays), therefore (3.7) holds. $\square$

Theorem 4 *Assume* (3.1), $D - A_0 - |A_1| - |B|$ *is a diagonally dominant matrix with positive diagonal elements, and* $\mathbf{u}$ *is such that*

$$|\mathbf{u}| < (D - A_0 - |A_1| - |B|)\mathbf{1}. \tag{3.11}$$

Then any solution x *of* (3.2)-(3.3) *satisfies*

$$\lim_{t\to\infty} \mathbf{x}(t) = (D - A - B)^{-1}\mathbf{u}. \tag{3.12}$$

Proof Fix any initial functions $\psi_i \colon [-r, 0] \to \mathbb{R}_+$ such that

$$\psi_i(s) > |\varphi_i(s)|, \qquad s \in [-r, 0], \quad i = 1, \dots, n,$$

and let $\mathbf{c} > |\mathbf{u}|$ be such that $\mathbf{c} < (D - A_0 - |A_1| - |B|)\mathbf{1}$. Let $\mathbf{y}$ denote the solution of the corresponding IVP (3.4)-(3.5). Since $\mathbf{y}(0) > |\mathbf{x}(0)|$, relation $|\mathbf{x}(t)| < \mathbf{y}(t)$ holds for sufficiently small $t > 0$. Suppose there exists i and $T > 0$ such that

$$|x_j(t)| < y_j(t), \qquad t \in [-\tau, T), \quad j = 1, \dots, n, \qquad \text{and} \quad |x_i(T)| = y_i(T). \tag{3.13}$$

It follows from Lemma 1 that $|x_i(T)| = y_i(T) \neq 0$, therefore $\frac{d}{dt}|x_i(t)|$ exists at T, and $\frac{d}{dt}(|x_i(t)|)_{|t=T} = \dot{x}_i(T) \operatorname{sign} x_i(T)$. Hence

$$\begin{aligned} &\frac{d}{dt}(|x_i(t)|)_{|t=T} \\ &\quad = \Big(-d_i x_i(T) + \sum_{j=1}^{n} a_{ij} f(x_j(T)) + \sum_{j=1}^{n} b_{ij} f(x_j(T - \tau_{ij})) + u_i\Big) \operatorname{sign} x_i(T) \end{aligned}$$

$$
\begin{aligned}
&= -d_i|x_i(T)| + a_{ii}f(|x_i(T)|) + \sum_{\substack{j=1,\\ j\neq i}}^{n} a_{ij}f(x_j(T))\,\mathrm{sign}\,x_i(T) \\
&\quad + \sum_{j=1}^{n} b_{ij}f(x_j(T-\tau_{ij}))\,\mathrm{sign}\,x_i(T) + u_i\,\mathrm{sign}\,x_i(T) \\
&< -d_i|x_i(T)| + a_{ii}f(|x_i(T)|) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|f(|x_j(T)|) + \sum_{j=1}^{n} |b_{ij}|f(|x_j(T-\tau_{ij})|) + c_i \\
&\leq -d_iy_i(T) + a_{ii}f(y_i(T)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|f(y_j(T)) + \sum_{j=1}^{n} |b_{ij}|f(y_j(T-\tau_{ij})) + c_i \\
&= \dot{y}_i(T).
\end{aligned}
$$

This contradicts to assumption (3.13), therefore $|x_i(t)| < y_i(t)$ holds for all $t > 0$ and $i = 1, \ldots, n$. Moreover, Lemma 3 yields

$$\lim_{t\to\infty} \mathbf{y}(t) = (D - A_0 - |A_1| - |B|)^{-1}\mathbf{c} < \mathbf{1}$$

holds, therefore there exists $t_1 > 0$ such that $|\mathbf{x}(t)| < \mathbf{1}$ for $t \geq t_1$. Then (3.2) is equivalent to

$$\dot{x}_i(t) = -d_ix_i(t) + \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-\tau_{ij}) + u_i, \qquad t \geq t_1.$$

This implies (3.12) using an argument similar to that in the proof of Lemma 3. □

4 Examples

To illustrate our results consider the two-dimensional DCNN model equations

$$\dot{x}_1(t) = -x_1(t) - 6f(x_1(t)) + f(x_2(t)) - 3f(x_1(t-1)) + f(x_2(t-2)) + u_1 \tag{4.1}$$

$$\dot{x}_2(t) = -x_2(t) - f(x_1(t)) - 3f(x_2(t)) - f(x_1(t-1)) + f(x_2(t-2)) + u_2, \tag{4.2}$$

where f is defined by (1.2). It is easy to see that

$$D - A_0 - |A_1| - |B| = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$$

is a diagonally dominant matrix. Therefore Theorem 4 yields that if $|u_1| < 2$ and $|u_2| < 1$ then the trivial solution of this system is asymptotically stable. In Figure 1 we have plotted the two components of the solutions corresponding to $u_1 = -1$ and $u_2 = 0.5$ and to the initial functions

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} t+1 \\ -t \end{pmatrix}, \quad \begin{pmatrix} \sin 2t \\ t^2-1 \end{pmatrix}, \quad \begin{pmatrix} \cos t + 1 \\ t+2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t^3-2 \\ -2\cos t \end{pmatrix}, \tag{4.3}$$

respectively.

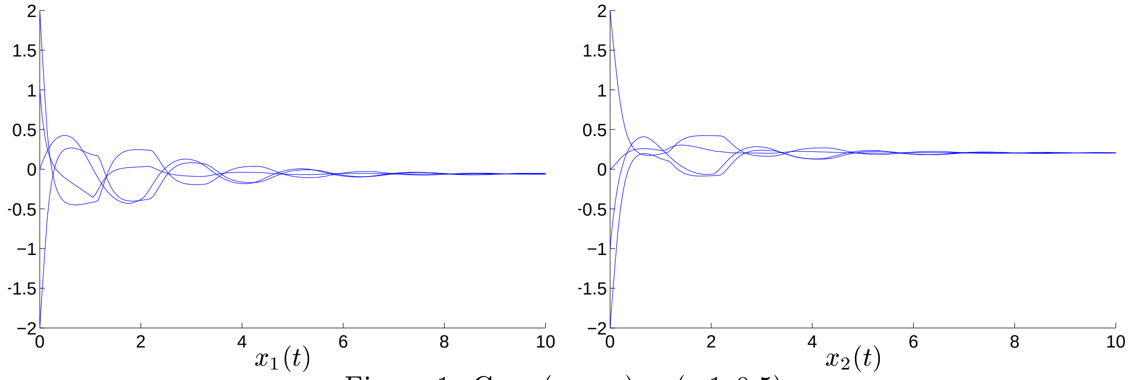

Figure 1. Case $(u_1, u_2) = (-1, 0.5)$.

We can observe that all solutions tend to the unique equilibrium $(-0.058824, 0.20588)^T$.

Note that the condition of Mohamad and Gopalsamy (1.3) is not satisfied for (4.1)-(4.2), and also the condition of Takahashi gives the matrix

$$W = \begin{pmatrix} -7 & -2 \\ -2 & -4 \end{pmatrix},$$

which is not an M-matrix. Therefore none of this two conditions can be applied for system (4.1)-(4.2).

By checking other input values outside the region $|u_1| < 2$ and $|u_2| < 1$ we observed in every cases we tried all solutions tended to the unique equilibrium $(v_1, v_2)^T$ of the system (not necessary satisfying $|v_1|, |v_2| < 1$). In Figure 2 we can see the graphs of solutions of (4.1)-(4.2) corresponding to $(u_1, u_2) = (3, 5)$ and to the initial functions (4.3). We can observe that all solutions tend to the unique equilibrium $(0.5, 2)^T$.

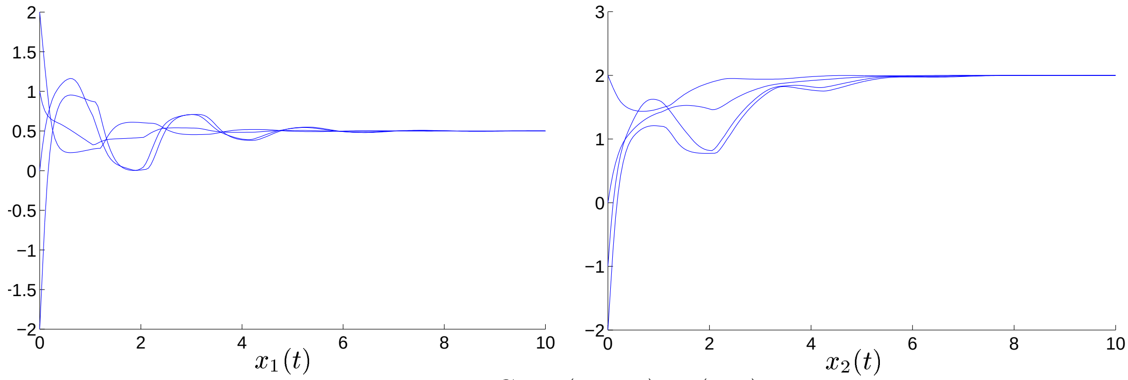

Figure 2. Case $(u_1, u_2) = (3, 5)$.

Next we plotted the solutions corresponding to $(u_1, u_2) = (-8.5, -5.5)$ and to the initial functions (4.3) in Figure 3. Again, all solutions tend to the unique equilibrium $(-1.5, -1.5)^T$.

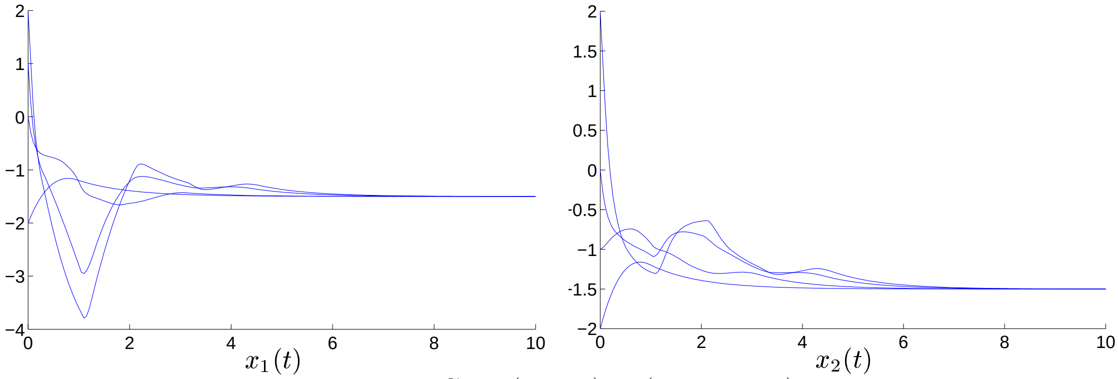

Figure 3. Case $(u_1, u_2) = (-8.5, -5.5)$.

Now change the coefficient of $f(x_2(t-2))$ in (4.1) to 4, i.e., consider the system

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - 6f(x_1(t)) + f(x_2(t)) - 3f(x_1(t-1)) + 4f(x_2(t-2)) + u_1 \qquad (4.4)\\ \dot{x}_2(t) &= -x_2(t) - f(x_1(t)) - 3f(x_2(t)) - f(x_1(t-1)) + f(x_2(t-2)) + u_2. \qquad (4.5)\end{aligned}$$

We plotted the solutions corresponding to $(u_1, u_2) = (-6, 4)$ and to the initial functions (4.3) in Figure 4. As before, all solutions tend to the unique equilibrium, which is $(-0.1, 2.2)^T$ in this case. On the other hand,

$$D - A_0 - |A_1| - |B| = \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}$$

is no longer a diagonally dominant matrix, but it is a nonsingular M-matrix.

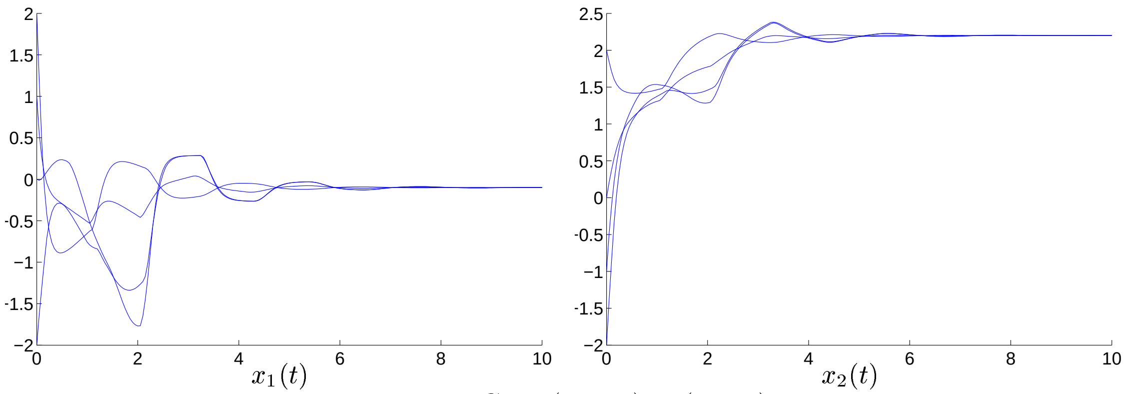

Figure 4. Case $(u_1, u_2) = (-6, 4)$.

Therefore our numerical experiments on these and other systems suggest the following conjecture.

Conjecture 1 *Assume* (3.1) *and* $D - A_0 - |A_1| - |B|$ *is a nonsingular M-matrix. Then* (3.2) *has a unique equilibrium for any input vector* $\mathbf{u}$, *and any solution of* (3.2) *tends to this equilibrium.*

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