Linearized Stability in Periodic Functional Differential Equations with State-Dependent Delays

Ferenc Hartung
Department of Mathematics and Computing, University of Veszprém
H-8201 Veszprém, P.O.Box 158, Hungary

ABSTRACT: In this paper we study stability of periodic solutions of a class of nonlinear functional differential equations (FDEs) with state-dependent delays using the method of linearization. We show that a periodic solution of the nonlinear FDE is exponentially stable, if the zero solution of an associated linear periodic linear homogeneous FDE is exponentially stable.
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1. INTRODUCTION

Functional differential equation with state-dependent delays (sd-FDEs) appear frequently in applications as model equations (see, e.g., Aiello, Freedman & Wu (1992), Arino, Hbid & Bravo del la Parra (1998), Cao, Fan & Gard (1992), and Mahaffy, Bélair & Mackey (1998), and the study of such equations is an active research area (see the references in this paper). Stability of the solution is one of the most important qualitative property of a model. There are many papers which give sufficient conditions for the stability of the trivial (zero) solution in sd-FDEs (see, e.g., Győri & Hartung (2000), Yoneyama (1987) and (1991)).

For nonlinear equations the method of linearization is a standard tool in stability investigations, but for sd-FDEs there are many technical problems with it. (See, e.g., Brokate & Colonius (1990), Hartung (1997), Hartung & Turi (1997), and Krishnan (2002).) Linearization theorems for obtaining stability of the zero or constant equilibriums were given in Cooke & Huang (1996), Hartung & Turi (1995) and (2000) for various classes of sd-FDEs. In this paper we extend these results for periodic solutions of a class of nonlinear sd-FDEs (see Theorem 2.5 below). Our results were motivated by Luzyanina, Engelborghs & Rose (2001), where the existence of such result was conjectured, and extensive numerical investigation of stability of constant and periodic solutions of sd-FDEs was given.

For results concerning the existence of a periodic solutions of sd-FDEs we refer the interested reader to Domoshnitsky & Drakhlin (1997), Magal & Arino (2000), Mallet-Paret, Nussbaum & Paraskevopoulos (1994), and Smith & Kuang (1992).

2. MAIN RESULTS

Consider the nonlinear state-dependent delay system

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t, x_t))), \qquad t \geq 0 \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.2}$$

Here and later on x_t denotes the solution segment function, i.e., $x_t(s) = x(t+s)$ for $s \in [-r, 0]$. The Banach-space of continuous functions $\psi\colon [-r, 0] \to \mathbb{R}^n$ with the supremum norm $\|\psi\| = \max\{|\psi(s)|\colon s \in [-r, 0]\}$ is denoted by C. A closed neighborhood with radius ϱ of a set A in a Banach-space X is denoted by $B_X(A; \varrho) = \{x \in X\colon |x-a|_X \le \varrho$ for some $a \in A\}$. We use $|\cdot|$ for any fixed norm on $\mathbb{R}^n$ and for the corresponding induced matrix norm on $\mathbb{R}^{n\times n}$, as well. $\mathcal{L}(C, \mathbb{R})$ denotes the Banach-space of bounded linear functionals on C with the norm $|\cdot|_{\mathcal{L}(C,\mathbb{R})}$.

We assume the following conditions throughout the paper:

(H1) $f : [0, \infty) \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuously differentiable, where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$, and let f be T-periodic, i.e.,

$$f(t, u, v) = f(t+T, u, v), \qquad t \ge 0, \quad u \in \Omega_1, \quad v \in \Omega_2,$$

(H2) (i) $\tau\colon [0, \infty) \times \Omega_3 \to [0, r]$ is continuously differentiable, where Ω_3 is an open subset of C, and τ is T-periodic, i.e.,

$$\tau(t, \psi) = \tau(t+T, \psi), \qquad t \ge 0, \quad \psi \in \Omega_3,$$

(ii) τ is locally Lipschitz-continuous in the following sense: for every bounded and closed subset M of C there exists a constant $L_1 = L_1(M) \ge 0$ such that

$$|\tau(t, \psi) - \tau(t, \tilde{\psi})| \le L_1\|\psi - \tilde{\psi}\|, \qquad t \in [0, T], \quad \psi, \tilde{\psi} \in M,$$

(iii) $D_2\tau$ is locally Lipschitz-continuous in the following sense: for every bounded and closed subset M of C there exists a constant $L_2 = L_2(M) \ge 0$ such that

$$|D_2\tau(t, \psi) - D_2\tau(t, \tilde{\psi})|_{\mathcal{L}(C,\mathbb{R})} \le L_2\|\psi - \tilde{\psi}\|, \qquad t \in [0, T], \quad \psi, \tilde{\psi} \in M,$$

Let $\bar{x} : [-r, \infty) \to \mathbb{R}^n$ be a T-periodic solution of (2.1). The restriction of $\bar{x}$ to the interval $[-r, 0]$ is denoted by $\bar{\varphi}$, i.e., $\bar{x}$ is the solution of (2.1)-(2.2) corresponding to initial function $\bar{\varphi}$. It is assumed that $\bar{\varphi}$ and $\bar{x}$ are fixed throughout this paper. Since $\bar{x}$ is a solution of (2.1), the continuity of f and τ imply that $\dot{\bar{x}}$ is continuous on $[0, \infty)$, therefore $\bar{x}$ is continuously differentiable on $[-r, \infty)$, as well.

We note that, in general, assumptions (H1)–(H2) together with $\varphi \in C$ imply the existence, but not the uniqueness of the solution of (2.1)-(2.2). But the stronger assumption $\bar{\varphi} \in C^1$ is sufficient in order the solution $\bar{x}$ be unique. (See, e.g., [11] or [12].) Throughout this paper $x(t; \varphi)$ will denote any solution of (2.1)-(2.2) corresponding to initial function $\varphi \in C$.

To simplify notation we introduce

$$\xi(t) \equiv x(t - \tau(t, x_t)) \qquad \text{and} \qquad \bar{\xi}(t) \equiv \bar{x}(t - \tau(t, \bar{x}_t)).$$

The dependence of ξ on x is omitted from the notation for simplicity, but it should always be kept in mind. We define the following sets associated to $\bar{x}$:

$$A_1 \equiv \{\bar{x}(t)\colon t \in [0, T]\}, \qquad A_2 \equiv \{\bar{\xi}(t)\colon t \in [0, T]\}, \quad \text{and} \quad A_3 \equiv \{\bar{x}_t\colon t \in [0, T]\}.$$

Then $A_1 \subset \mathbb{R}^n$, $A_2 \subset \mathbb{R}^n$ and $A_3 \subset C$ are compact subsets of the respective spaces, since $\bar{x}$ is continuous. The sets Ω_1, Ω_2 and Ω_3 are opens subsets of the respective spaces, therefore there exist positive constants ϱ_1, ϱ_2 and ϱ_3 such that

$$B_{\mathbb{R}^n}(A_1;\, \varrho_1) \subset \Omega_1, \quad B_{\mathbb{R}^n}(A_2;\, \varrho_2) \subset \Omega_2 \quad \text{and} \quad B_C(A_3;\, \varrho_3) \subset \Omega_3.$$

Since f is T-periodic and continuously differentiable with respect to its second and third arguments, there exists a constant $N_1 > 0$ such that

$$|D_2 f(t,u,v)| \le N_1 \qquad \text{and} \qquad |D_3 f(t,u,v)| \le N_1 \tag{2.3}$$

for $t \ge 0$, $u \in B_{\mathbb{R}^n}(A_1;\, \varrho_1)$, and $v \in B_{\mathbb{R}^n}(A_2;\, \varrho_2)$.

We shall need the following estimate.

Lemma 2.1 *Assume (H2), and let $\bar{x}\colon [-r,\infty) \to \mathbb{R}^n$ be a continuously differentiable and T-periodic function. There exists a constant $N_2 \ge 1$ such that for any $S > 0$*

$$|\xi(t) - \bar{\xi}(t)| \le N_2 \|x_t - \bar{x}_t\|, \qquad t \in [0,S]$$

for any continuous function $x\colon [-r,\infty) \to \mathbb{R}^n$ satisfying

$$x_t \in B_C(A_3;\, \varrho_3), \qquad t \in [0,S]. \tag{2.4}$$

Proof Let L_1 be the constant from (H2) (ii) associated to the set $B_C(A_3;\, \varrho_3)$. The definition of ξ and $\bar{\xi}$, and the Mean Value Theorem yield

$$\begin{aligned} |\xi(t) - \bar{\xi}(t)| &\le |x(t-\tau(t,x_t)) - \bar{x}(t-\tau(t,x_t))| + |\bar{x}(t-\tau(t,x_t)) - \bar{x}(t-\tau(t,\bar{x}_t))| \\ &\le \|x_t - \bar{x}_t\| + \|\dot{\bar{x}}_t\| |\tau(t,x_t) - \tau(t,\bar{x}_t)|, \end{aligned}$$

which proves the statement using (H2) (ii) and $N_2 \equiv 1 + L_1 \max\{|\dot{\bar{x}}(t)|\colon t \in [0,T]\}$. $\square$

For $\bar{x}$ and for any fixed $t \ge 0$ we define the linear operator $F(t)\colon C \to \mathbb{R}^n$ by

$$\begin{aligned} F(t)\psi \;\equiv\; & D_2 f(t,\bar{x}(t),\bar{\xi}(t))\psi(0) + D_3 f(t,\bar{x}(t),\bar{\xi}(t))\psi(-\tau(t,\bar{x}_t)) \\ & - D_3 f(t,\bar{x}(t),\bar{\xi}(t))\dot{\bar{x}}(t-\tau(t,\bar{x}_t)) D_2\tau(t,\bar{x}_t)\psi, \end{aligned} \tag{2.5}$$

and the function

$$g \;:\; [0,\infty) \times \Omega_3 \to \mathbb{R}^n, \qquad g(t,\psi) \equiv f(t,\psi(0),\psi(-\tau(t,\psi))) - F(t)\psi. \tag{2.6}$$

Note that for each $t \ge 0$ the linear operator $F(t)$ is bounded, since by (H2) it satisfies

$$\begin{aligned} |F(t)\psi| \;\le\; & \Big(\max_{t\in[0,T]} \big|D_2 f(t,\bar{x}(t),\bar{\xi}(t))\big| \\ & + \max_{t\in[0,T]} \big|D_3 f(t,\bar{x}(t),\bar{\xi}(t))\big| \, (1 + \max_{t\in[0,T]} |\dot{\bar{x}}(t)| \max_{t\in[0,T]} |D_2\tau(t,\bar{x}_t)|_{\mathcal{L}(C,\mathbb{R})}) \Big) \|\psi\|. \end{aligned}$$

By these notations we can rewrite (2.1) as

$$\dot{x}(t) = F(t)x_t + g(t,x_t), \qquad t \ge 0, \tag{2.7}$$

and therefore we can consider it as a perturbation of the homogeneous linear T-periodic FDE

$$\dot{y}(t) = F(t)y_t, \qquad t \geq 0. \tag{2.8}$$

We denote the fundamental solution of (2.8) by $U(t,s)$, i.e., it is a matrix valued solution of the initial value problem

$$\frac{\partial}{\partial t}U(t,s) = F(t)U(\cdot,s)_t, \qquad t \geq s, \tag{2.9}$$

$$U(t,s) = \begin{cases} I, & t = s, \\ 0, & t < s. \end{cases} \tag{2.10}$$

It is known (see, e.g., [9]) that the asymptotic stability of the trivial solution of (2.8) is equivalent to its exponential stability, and to that there exist constants $K_0 \geq 1$ and $\alpha_0 > 0$ such that

$$|U(t,s)| \leq K_0 e^{-\alpha_0(t-s)}, \qquad t \geq s. \tag{2.11}$$

The proof of our main theorem will be based on the following series of lemmas.

Lemma 2.2 *Assume (H2), and let $\bar{x}\colon [-r,\infty) \to \mathbb{R}^n$ be a continuously differentiable and T-periodic function. Then there exists a constant $N_3 \geq 0$ such that for every $\eta > 0$ there exists $\theta_2 > 0$ such that for any $S > 0$*

$$|\bar{x}(t-\tau(t,x_t)) - \bar{x}(t-\tau(t,\bar{x}_t)) + \dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)(x_t-\bar{x}_t)| \leq N_3(\eta + \|x_t-\bar{x}_t\|)\|x_t-\bar{x}_t\|$$

for $t \in [0,S]$ and for any continuous function $x\colon [-r,\infty) \to \mathbb{R}^n$ satisfying

$$x_t \in B_C(A_3;\, \theta_2), \qquad t \in [0,S]. \tag{2.12}$$

Proof For a fixed $t \in [0,S]$ we introduce the real function

$$p(s) = \bar{x}(t - \tau(t, \bar{x}_t + s(x_t - \bar{x}_t))) + s\dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)(x_t - \bar{x}_t).$$

The Chain Rule implies that the real function p is continuously differentiable, and

$$\dot{p}(s) = \Big(-\dot{\bar{x}}(t-\tau(t,\bar{x}_t+s(x_t-\bar{x}_t)))D_2\tau(t,\bar{x}_t+s(x_t-\bar{x}_t)) + \dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)\Big)(x_t-\bar{x}_t).$$

Then the definition of p and the Mean Value Theorem yield that there exists $\nu \in [0,1]$ such that

$$\begin{aligned}
&|\bar{x}(t-\tau(t,x_t)) - \bar{x}(t-\tau(t,\bar{x}_t)) + \dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)(x_t-\bar{x}_t)| \\
&\quad = |p(1) - p(0)| \\
&\quad = |\dot{p}(\nu)| \\
&\quad \leq \Big|\dot{\bar{x}}(t-\tau(t,\bar{x}_t)) - \dot{\bar{x}}(t-\tau(t,\bar{x}_t+\nu(x_t-\bar{x}_t)))\Big|\Big|D_2\tau(t,\bar{x}_t+\nu(x_t-\bar{x}_t))\Big|_{\mathcal{L}(C,\mathbb{R})}\|x_t-\bar{x}_t\| \\
&\qquad |\dot{\bar{x}}(t-\tau(t,\bar{x}_t))|\Big|D_2\tau(t,\bar{x}_t) - D_2\tau(t,\bar{x}_t+\nu(x_t-\bar{x}_t))\Big|_{\mathcal{L}(C,\mathbb{R})}\|x_t-\bar{x}_t\|.
\end{aligned} \tag{2.13}$$

Let L_1 and L_2 be the constants from (H2) (ii) and (iii), respectively, corresponding to the set $B_C(A_3;\, \varrho_3)$. Fix an arbitrary $\eta > 0$. Since $\dot{\bar{x}}$ is T-periodic, and so it is uniformly continuous, for any $\eta > 0$ there exists $\delta > 0$ such that $|\dot{\bar{x}}(u) - \dot{\bar{x}}(\tilde{u})| \le \eta |u - \tilde{u}|$ for $u, \tilde{u} \ge -r$ satisfying $|u - \tilde{u}| \le \delta$, and there exists a constant $R_1 \ge 0$ such that $|\dot{\bar{x}}(t)| \le R_1$ for $t \ge -r$. Let $\theta_2 \equiv \min(\varrho_3, \delta/L_1)$, and suppose x satisfies (2.12). Then (H2) (ii) implies

$$|\tau(t, \bar{x}_t) - \tau(t, \bar{x}_t + \nu(x_t - \bar{x}_t))| \le L_1 \|x_t - \bar{x}_t\| \le \delta,$$

and hence

$$\left| \dot{\bar{x}}(t - \tau(t, \bar{x}_t)) - \dot{\bar{x}}(t - \tau(t, \bar{x}_t + \nu(x_t - \bar{x}_t))) \right| \le \eta.$$

Using (H2) (iii) and the periodicity of τ there exists a constant $R_2 \ge 0$ such that

$$\left| D_2 \tau(t, \bar{x}_t + \nu(x_t - \bar{x}_t)) \right|_{\mathcal{L}(C,\mathbb{R})} \le \left| D_2 \tau(t, \bar{x}_t) \right|_{\mathcal{L}(C,\mathbb{R})} + L_2 \|x_t - \bar{x}_t\| \le R_2$$

for any function satisfying (2.12). Then the statement of the lemma follows from (2.13) with $N_3 = \max(R_2, R_1 L_2)$. □

Lemma 2.3 *Assume (H1), (H2), and let* $\bar{x} \colon [-r, \infty) \to \mathbb{R}^n$ *be a continuously differentiable and* T*-periodic solution of* (2.1) *corresponding to initial function* $\bar{\varphi}$*, and let* x *be a solution of* (2.1)-(2.2) *satisfying* (2.4)*. Then there exists a constant* $N_4 > 0$ *such that for any* $S > 0$

$$|\dot{x}(t) - \dot{\bar{x}}(t)| \le N_4 \|x_t - \bar{x}_t\|, \qquad t \in [0, S] \tag{2.14}$$

and

$$\|x_t - \bar{x}_t\| \le e^{N_4 t} \|\varphi - \bar{\varphi}\|, \qquad t \in [0, S] \tag{2.15}$$

for any solution x *of* (2.1) *satisfying*

$$x_t \in B_C(A_3; \varrho_3), \qquad t \in [0, S]. \tag{2.16}$$

Proof Let N_2 be the constant from Lemma 2.1, fix $S > 0$, and suppose x satisfies (2.16). Then the Mean Value Theorem, Lemma 2.1 and (2.3) yield for $t \in [0, S]$

$$\begin{aligned} |\dot{x}(t) - \dot{\bar{x}}(t)| &= |f(t, x(t), \xi(t)) - f(t, \bar{x}(t), \bar{\xi}(t))| \\ &\le N_1(|x(t) - \bar{x}(t)| + |\xi(t) - \bar{\xi}(t)|) \\ &\le N_1(1 + N_2)\|x_t - \bar{x}_t\|. \end{aligned}$$

Therefore (2.14) holds with $N_4 = N_1(1 + N_2)$.

To prove (2.15), consider the inequalities

$$|x(t) - \bar{x}(t)| \le |\varphi(0) - \bar{\varphi}(0)| + \int_0^t |\dot{x}(s) - \dot{\bar{x}}(s)|\, ds \le \|\varphi - \bar{\varphi}\| + N_4 \int_0^t \|x_s - \bar{x}_s\|\, ds.$$

Let $v(t) \equiv \max\{|x(s) - \bar{x}(s)| \colon -r \le s \le t\}$. Then

$$|x(t) - \bar{x}(t)| \le \|\varphi - \bar{\varphi}\| + N_4 \int_0^t v(s)\, ds, \qquad t \in [0, S],$$

and since the right-hand-side is monotone increasing in t, it implies

$$v(t) \le \|\varphi - \bar{\varphi}\| + N_4 \int_0^t v(s)\, ds, \qquad t \in [0, T].$$

Therefore Gronwall's inequality proves (2.15), since $\|x_t - \bar{x}_t\| \le v(t)$. □

We will need the following estimate of g.

Lemma 2.4 *Assume (H1), (H2), and let* $\bar{x}\colon [-r,\infty)\to\mathbb{R}^n$ *be a continuously differentiable and* T*-periodic solution of* (2.1) *corresponding to initial function* $\bar{\varphi}$*, and let* x *be a solution of* (2.1)-(2.2)*. Then there exists a constant* $N_5\geq 0$ *such that for every* $\eta>0$ *there exists* $\theta_3>0$ *such that for any* $S>r$

$$|g(t,x_t)-g(t,\bar{x}_t)|\leq\begin{cases} N_5(\eta+1)\|x_t-\bar{x}_t\| & 0\leq t\leq r,\\ N_5\left(\eta+\max\limits_{t-r\leq s\leq t}\|x_s-\bar{x}_s\|\right)\|x_t-\bar{x}_t\|, & t\in[r,S]\end{cases}$$

for x *satisfying*

$$x_t\in B_C(A_3;\,\theta_3),\qquad t\in[0,S]. \tag{2.17}$$

Proof The definition of g and F, and the Mean Value Theorem imply

$$\begin{aligned}
|g(t,x_t)-g(t,\bar{x}_t)| &= |f(t,x(t),\xi(t))-f(t,\bar{x}(t),\bar{\xi}(t))-F(t)(x_t-\bar{x}_t)|\\
&= \Big|f(t,x(t),\xi(t))-f(t,\bar{x}(t),\bar{\xi}(t))-D_2f(t,\bar{x}(t),\bar{\xi}(t))(x(t)-\bar{x}(t))\\
&\quad -D_3f(t,\bar{x}(t),\bar{\xi}(t))\Big(x(t-\tau(t,\bar{x}_t))-\bar{\xi}(t)\Big)\\
&\quad +D_3f(t,\bar{x}(t),\bar{\xi}(t))\dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)(x_t-\bar{x}_t)\Big|\\
&\leq \Big|f(t,x(t),\xi(t))-f(t,\bar{x}(t),\bar{\xi}(t))-D_2f(t,\bar{x}(t),\bar{\xi}(t))(x(t)-\bar{x}(t))\\
&\quad -D_3f(t,\bar{x}(t),\bar{\xi}(t))(\xi(t)-\bar{\xi}(t))\Big|\\
&\quad +\Big|D_3f(t,\bar{x}(t),\bar{\xi}(t))\Big(x(t-\tau(t,x_t))-x(t-\tau(t,\bar{x}_t))\\
&\qquad +\dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)(x_t-\bar{x}_t)\Big)\Big|\\
&\leq \sup_{0<\nu<1}\Big|D_2f\Big(t,\bar{x}(t)+\nu(x(t)-\bar{x}(t)),\bar{\xi}(t)+\nu(\xi(t)-\bar{\xi}(t))\Big)\\
&\qquad -D_2f(t,\bar{x}(t),\bar{\xi}(t))\Big||x(t)-\bar{x}(t)|\\
&\quad +\sup_{0<\nu<1}\Big|D_3f\Big(t,\bar{x}(t)+\nu(x(t)-\bar{x}(t)),\bar{\xi}(t)+\nu(\xi(t)-\bar{\xi}(t))\Big)\\
&\qquad -D_3f(t,\bar{x}(t),\bar{\xi}(t))\Big||\xi(t)-\bar{\xi}(t)|\\
&\quad +\Big|D_3f(t,\bar{x}(t),\bar{\xi}(t))\Big(x(t-\tau(t,x_t))-x(t-\tau(t,\bar{x}_t))\\
&\qquad +\dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)(x_t-\bar{x}_t)\Big)\Big|.
\end{aligned}\tag{2.18}$$

Fix $\eta>0$ and $S>r$. By the continuous differentiability and T-periodicity of f guaranteed by (H1) there exists $\delta>0$ such that

$$|D_2f(t,u,v)-D_2f(t,\tilde{u},\tilde{v})|\leq\eta\quad\text{and}\quad |D_3f(t,u,v)-D_3f(t,\tilde{u},\tilde{v})|\leq\eta$$

hold for $t\geq 0$, $u,\tilde{u}\in B_{\mathbb{R}^n}(A_1;\,\varrho_1)$, $v,\tilde{v}\in B_{\mathbb{R}^n}(A_2;\,\varrho_2)$, and $|u-\tilde{u}|\leq\delta$ and $|v-\tilde{v}|\leq\delta$. Let L_1 and L_2 be the constants from (H2) (ii) and (iii), respectively, corresponding to the set $B_C(A_3;\,\varrho_3)$, let N_2, N_3 and N_4 be the constants from Lemmas 2.1, 2.2 and 2.3, respectively, and let θ_2 be the constant from Lemma 2.2 corresponding to η. Define

$$\theta_3\equiv\min(\delta/N_2,\varrho_1,\varrho_2/N_2,\theta_2,\varrho_3),$$

and suppose x satisfies (2.17). Then $|x(t)-\bar{x}(t)|\le\delta$ and $|\xi(t)-\bar{\xi}(t)|\le\delta$ for $t\in[0,S]$. Therefore it follows from (2.3), (2.18), Lemma 2.1 and 2.2 for $t\in[0,S]$

$$
\begin{aligned}
&|g(t,x_t)-g(t,\bar{x}_t)|\\
&\quad\le\ \eta|x(t)-\bar{x}(t)|+\eta|\xi(t)-\bar{\xi}(t)|\\
&\qquad+N_1\Big|\bar{x}(t-\tau(t,x_t))-\bar{x}(t-\tau(t,\bar{x}_t))+\dot{\bar{x}}(t-\tau(t,\bar{x}_t))D_2\tau(t,\bar{x}_t)(x_t-\bar{x}_t)\Big|\\
&\qquad+N_1|x(t-\tau(t,x_t))-x(t-\tau(t,\bar{x}_t))-\bar{x}(t-\tau(t,x_t))+\bar{x}(t-\tau(t,\bar{x}_t))|\\
&\quad\le\ \eta(1+N_2+N_1N_3)\|x_t-\bar{x}_t\|+N_1N_3\|x_t-\bar{x}_t\|^2\\
&\qquad+N_1|x(t-\tau(t,x_t))-x(t-\tau(t,\bar{x}_t))-\bar{x}(t-\tau(t,x_t))+\bar{x}(t-\tau(t,\bar{x}_t))|. \qquad (2.19)
\end{aligned}
$$

For $t\in[0,r]$ we estimate the last term of (2.19) simply as

$$|x(t-\tau(t,x_t))-x(t-\tau(t,\bar{x}_t))-\bar{x}(t-\tau(t,x_t))+\bar{x}(t-\tau(t,\bar{x}_t))|\le 2\|x_t-\bar{x}_t\|.$$

For $t\ge r$ we use a different estimate: the Mean Value Theorem, (H1) (ii), and relations $t-\tau(t,x_t)\ge 0$ and $t-\tau(t,\bar{x}_t)\ge 0$ yield

$$
\begin{aligned}
&|x(t-\tau(t,x_t))-x(t-\tau(t,\bar{x}_t))-\bar{x}(t-\tau(t,x_t))+\bar{x}(t-\tau(t,\bar{x}_t))|\\
&\qquad\le\ L_1\max_{s\in[t-r,t]}|\dot{x}(s)-\dot{\bar{x}}(s)|\cdot\|x_t-\bar{x}_t\|.
\end{aligned}
$$

Therefore Lemma 2.3 implies for $t\in[r,S]$

$$
\begin{aligned}
&|x(t-\tau(t,x_t))-x(t-\tau(t,\bar{x}_t))-\bar{x}(t-\tau(t,x_t))+\bar{x}(t-\tau(t,\bar{x}_t))|\\
&\qquad\le\ L_1N_4\max_{s\in[t-r,t]}\|x_s-\bar{x}_s\|\cdot\|x_t-\bar{x}_t\|.
\end{aligned}
$$

Hence the statement of the lemma follows from (2.19) with

$$N_5\equiv\max\left(1+N_2+N_1N_3,\ N_1N_3+L_1N_1N_4,\ N_1N_3\varrho_3+2N_1\right).$$

□

We show that the exponential stability of the periodic steady-state solution $\bar{x}$ of the nonlinear sd-FDE (2.1) can be obtained by that of the homogeneous linear FDE (2.8).

Theorem 2.5 *Assume (H1), (H2), and let $\bar{x}:[-r,\infty)\to\mathbb{R}^n$ be a continuously differentiable and T-periodic solution of (2.1) corresponding to initial function $\bar{\varphi}$. Suppose the trivial solution of (2.8) is exponentially stable, i.e., there exist $K_0\ge 1$ and $\alpha_0>0$ such that (2.11) holds. Then for every $0<\alpha<\alpha_0$ there exists $\delta>0$ and $K\ge 1$ such that if $\|\varphi-\bar{\varphi}\|<\delta$, then any corresponding solution $x(t)=x(t;\varphi)$ of (2.1) satisfies*

$$|x(t)-\bar{x}(t)|\le Ke^{-\alpha t}\|\varphi-\bar{\varphi}\|,\qquad t\ge 0,$$

i.e., $\bar{x}$ is an exponentially stable periodic steady-state of (2.1).

Proof Fix $\varepsilon>0$ and $0<\alpha<\alpha_0$. Let N_4 and N_5 be the constants defined by Lemma 2.3 and 2.4, respectively, and let

$$\eta\equiv\frac{\varepsilon(\alpha_0-\alpha)}{2(1+\varepsilon)K_0N_5}e^{-\alpha r}.$$

Define $\theta_3 = \theta_3(\eta)$ by Lemma 2.4, let

$$\delta_1 \equiv \min\left(\varrho_1,\, \varrho_3,\, \theta_3,\, \eta,\, \frac{\eta}{N_4}\right), \qquad K_1 \equiv K_0\left(1 + rN_5 e^{(\alpha_0+N_4)r}\right), \quad \text{and} \quad K \equiv (1+\varepsilon)K_1,$$

and finally, let

$$\delta \equiv \frac{\delta_1}{K}.$$

Let $\varphi \in C$ be such that $\|\varphi - \bar\varphi\| < \delta$, and let $x(t) = x(t;\varphi)$ be a corresponding solution of (2.1)-(2.2). Then $|x(t) - \bar x(t)| < \delta_1$ for small $t > 0$ since $\delta < \delta_1$. Suppose there exists $S > 0$ such that

$$|x(t) - \bar x(t)| < \delta_1, \qquad \text{for } t \in [0, S), \quad \text{and} \quad |x(S) - \bar x(S)| = \delta_1. \tag{2.20}$$

The variation-of-constants formula (see, e.g., [9]) implies

$$x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)g(s,x_s)\,ds, \qquad t \geq 0.$$

Similarly,

$$\bar x(t) = U(t,0)\bar\varphi(0) + \int_0^t U(t,s)g(s,\bar x_s)\,ds, \qquad t \geq 0.$$

Therefore

$$|x(t) - \bar x(t)| \leq |U(t,0)||\varphi(0) - \bar\varphi(0)| + \int_0^t |U(t,s)||g(s,x_s) - g(s,\bar x_s)|\,ds, \qquad t \geq 0. \tag{2.21}$$

Suppose $S > r$. Relations $\delta_1 \leq \theta_3$, (2.11) and Lemma 2.4 imply for $t \in [r, S]$

$$\begin{aligned} |x(t) - \bar x(t)| \;&\leq\; K_0 e^{-\alpha_0 t}\|\varphi - \bar\varphi\| + K_0 \int_0^r e^{-\alpha_0(t-s)} N_5(\eta + 1)\|x_s - \bar x_s\|\,ds \\ &\quad + K_0 \int_r^t e^{-\alpha_0(t-s)} N_5(\eta + \max_{s-r\leq u\leq s}\|x_u - \bar x_u\|)\|x_s - \bar x_s\|\,ds. \end{aligned}$$

Since

$$\max_{s-r\leq u\leq s}\|x_u - \bar x_u\| \leq \delta_1 \leq \eta, \qquad s \in [0, S],$$

it follows

$$\begin{aligned} |x(t) - \bar x(t)| \;&\leq\; K_0 e^{-\alpha_0 t}\|\varphi - \bar\varphi\| + K_0 N_5(\eta+1) \int_0^r e^{-\alpha_0(t-s)}\|x_s - \bar x_s\|\,ds \\ &\quad + K_0 N_5 2\eta \int_r^t e^{-\alpha_0(t-s)}\|x_s - \bar x_s\|\,ds \\ &\leq\; K_0 e^{-\alpha_0 t}\|\varphi - \bar\varphi\| + K_0 N_5 e^{-\alpha_0 t} \int_0^r e^{\alpha_0 s}\|x_s - \bar x_s\|\,ds \\ &\quad + K_0 N_5 2\eta e^{-\alpha_0 t} \int_0^t e^{\alpha_0 s}\|x_s - \bar x_s\|\,ds. \end{aligned}$$

Note that the last inequality holds for $t \in [0, r]$ and for $S \leq r$, as well. Multiplying both sides of this inequality by $e^{\alpha t}$, and using relation (2.15) and the definition of K_1 we get

$$e^{\alpha t}|x(t) - \bar x(t)| \;\leq\; K_1\|\varphi - \bar\varphi\| + K_0 N_5 2\eta e^{(\alpha-\alpha_0)t} \int_0^t e^{\alpha_0 s}\|x_s - \bar x_s\|\,ds.$$

Let $v(t) = \max\{e^{\alpha s}|x(s) - \bar{x}(s)|\colon -r \le s \le t\}$. Then we have

$$\begin{aligned} e^{\alpha t}|x(t) - \bar{x}(t)| &\le K_1\|\varphi - \bar{\varphi}\| + K_0 N_5 2\eta v(t) e^{(\alpha-\alpha_0)t+\alpha r}\int_0^t e^{(\alpha_0-\alpha)s}\,ds \\ &\le K_1\|\varphi - \bar{\varphi}\| + \frac{K_0 N_5 2\eta}{\alpha_0 - \alpha} e^{\alpha r} v(t), \qquad t \in [0, S]. \end{aligned}$$

Since the right-hand-side is monotone increasing in t, it implies

$$v(t) \le K_1\|\varphi - \bar{\varphi}\| + \frac{K_0 N_5 2\eta}{\alpha_0 - \alpha} e^{\alpha r} v(t), \qquad t \in [0, S].$$

Then, using the definition of η, we get

$$v(t) \le K_1\|\varphi - \bar{\varphi}\| + \frac{\varepsilon}{1+\varepsilon} v(t), \qquad \text{for } t \in [0, S],$$

and hence

$$|x(t) - \bar{x}(t)| \le e^{-\alpha t} v(t) \le K e^{-\alpha t}\|\varphi - \bar{\varphi}\|, \qquad t \in [0, S].$$

But this yields

$$\delta_1 = |x(S) - \bar{x}(S)| < K\delta = \delta_1,$$

which contradicts to the definition of S. Therefore $S = \infty$, and

$$|x(t) - \bar{x}(t)| \le K e^{-\alpha t}\|\varphi - \bar{\varphi}\|$$

holds for all $t \ge 0$. The proof of the theorem is completed. □

We note that in the proofs of our results it was very important that the solution $\bar{x}$ around which we linearized the nonlinear sd-FDE is continuously differentiable. Linearization of a nonlinear (not necessary periodic) sd-FDE around an absolutely continuous solution is much more technical (see Brokate & Colonius (1990), Hartung (1997), and Hartung & Turi (1997)), so it is an interesting open question to investigate linearized stability of such solutions.

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