Proceedings of the conference Science, Education and Society, Zilina, Slovak Republic, September 17-19, 2003, Section No. 7., Mathematics in Interdisciplinary Context, University of Zilina, 2003,p. 15-18.

On Stability of Neural Networks with Delays

István Győri – Ferenc Hartung

Abstract

In this paper to continue our previous work for the scalar case, we study the asymptotic stability of the neural network system of the form

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_{ij})) + u_i, \quad t \geq 0, \quad i = 1, \ldots, n.$$

1 Introduction

Cellular neural networks (CNNs), introduced by Chua and Yang in 1988 ([4]), have been successfully applied in various engineering and scientific applications. In a standard CNN model the model equations are ordinary differential equations (ODEs) assuming that the interactions in the system are instantaneous. On the other hand it is known that in the real models of electronic networks time delays are likely to be present, due to the finite switching speed of amplifiers. So in the so-called delayed CNNs (DCNNs) the model equations are delay differential equations, which have much more complicated dynamics than the ODEs. In the applications DCNNs are usually required to be globally asymptotically stable, completely stable, absolutely stable or stable independently of the delays. These different types of stability of DCNNs have been rigorously done and many criteria have been obtained so far (see, e.g.,[2], [3], [6]–[10]). Most of these methods and results are devoted to the case when a non-delayed, linear terms dominate the others.

In [5] we studied the single neuron model equation described by the scalar equation

$$\dot{x}(t) = -dx(t) + af(x(t)) + bf(x(t-\tau)) + u, \qquad t \geq 0, \tag{1.1}$$

in the case when the feedback function f is a Hopfield activation function defined by

$$f(t) = \frac{1}{2}(|t+1| - |t-1|) = \begin{cases} 1, & t > 1, \\ t, & -1 \leq t \leq 1, \\ -1, & t < -1. \end{cases}$$

István Győri, Ferenc Hartung, Department of Mathematics and Computing, University of Veszprém, H-8201 Veszprém, P.O.Box 158, Hungary, email: gyori@almos.vein.hu, hartung@szt.vein.hu
This research was partially supported by Hungarian National Foundation for Scientific Research Grant No. T031935.

We proved that condition

$$d > a + |b| + |u| \tag{1.2}$$

implies the global asymptotic stability of the unique equilibrium point of (1.1). In the case when $b > 0$ and $a+b-|u| < d \leq a+b+|u|$ we have a complete understanding of the dynamics of (1.1) (see [5]), but in the remaining cases we have only partial theoretical results. In [5] we made numerical studies, and based on those experiments we conjecture that if $b > 0$, then every solution of (1.1) tends to a constant equilibrium, i.e., (1.1) is completely stable. In the case when $b < 0$ and $a+b+|u| < d \leq a+|b|+|u|$ we presented numerical studies and conjectured cases when the solutions of (1.1) are asymptotically periodic.

In this paper we generalize condition (1.2) to the system case of (1.1), which extends the results of [2]. Using numerical schemes introduced in [5] we illustrate our theoretical findings on a numerical example.

2 Stability Results

We consider the system version of (1.1), i.e., consider the neuron model equation

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_{ij})) + u_i, \qquad t \geq 0, \quad i = 1, \ldots, n, \tag{2.1}$$

where

$$d_i > 0,\ \tau_{ij} \geq 0, \quad a_{ij}, b_{ij}, u_i \in \mathbb{R}\ (i,j = 1, \ldots, n), \qquad \text{and} \quad f(t) = \frac{1}{2}(|t+1| - |t-1|). \tag{2.2}$$

Let $r = \max\{\tau_{ij}\colon\ i,j = 1, \ldots, n\}$. We associate the initial conditions

$$x_i(t) = \varphi_i(t), \qquad t \in [-r, 0], \quad i = 1, \ldots, n \tag{2.3}$$

to (2.1).

To simplify notation we introduce the $n \times n$ matrices $D = \operatorname{diag}(d_1, \ldots, d_n)$, $A = (a_{ij})$ and $B = (b_{ij})$, and the vectors $\mathbf{u} = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$ and $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$. We use the relation $\mathbf{x} < \mathbf{y}$ for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $x_i < y_i$ for all $i = 1, \ldots, n$, where $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$.

For the matrix A we associate the $n \times n$ diagonal matrix $A_0 = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn})$, i.e., the diagonal part of A, and let $A_1 = A - A_0$ be the off-diagonal part of A. Then with this notation, which we use throughout this paper, we can rewrite A as $A = A_0 + A_1$.

For an $n \times n$ matrix B the symbol $|B|$ denotes the corresponding $n \times n$ matrix with ijth element $|b_{ij}|$.

We say that the $n \times n$ matrix $K = (k_{ij})$ is diagonally dominant, if

$$|k_{ii}| > \sum_{\substack{j=1,\\ j \neq i}}^{m} |k_{ij}|, \qquad i = 1, \ldots, n.$$

We say that an $n \times n$ matrix K is an M-matrix, if all of its diagonal elements are nonnegative, and its off-diagonal elements are nonpositive, and all of its principal minors are positive (see, e.g., [1] or [2]).

We can formulate the generalization of condition (1.2) for the stability of the scalar equation (1.1) to neural system (2.1) as follows.

Theorem 2.1 *Assume* (2.2), $D - A_0 - |A_1| - |B|$ *is a diagonally dominant M-matrix, and* $\mathbf{u}$ *is such that*

$$|\mathbf{u}| < (D - A_0 - |A_1| - |B|)\mathbf{1}. \tag{2.4}$$

Then any solution $\mathbf{x}$ *of* (2.1)-(2.3) *satisfies*

$$\lim_{t\to\infty} \mathbf{x}(t) = (D - A - B)^{-1}\mathbf{u}, \tag{2.5}$$

i.e., equilibrium $(D - A - B)^{-1}\mathbf{u}$ *of* (2.1)-(2.3) *is globally asymptotically stable.*

One can show that under the conditions of the previous theorem the system has solutions satisfying $|x_i(t)| < 1$ $(i = 1, \dots, n)$ for large t, therefore it is equivalent to the linearized version of (2.1):

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_{ij}) + u_i, \qquad t \ge 0, \quad i = 1, \dots, n. \tag{2.6}$$

It is possible to show that under this condition system (2.6) has a globally stable unique equilibrium solution. The above idea of the proof of Theorem 2.1 follows that of Theorem 2.3 in [5], the details will be given elsewhere.

To illustrate this theorem consider the two-dimensional system

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) - f(x_1(t)) + f(x_2(t)) + f(x_2(t-2)) + u_1 & (2.7)\\ \dot{x}_2(t) &= -3x_2(t) + f(x_1(t)) - 2f(x_2(t)) - 2f(x_1(t-1)) + u_2, & (2.8) \end{aligned}$$

where f is defined by (2.2). Here

$$D - A_0 - |A_1| - B_0 - |B_1| = \begin{pmatrix} 3 & -2 \\ -3 & 5 \end{pmatrix},$$

which is a diagonally dominant M-matrix. Applying condition (2.4) and Theorem 2.1, we get if $|u_1| < 1$ and $|u_2| < 2$, then the system has a unique equilibrium, which is globally asymptotically stable. For example, if $u_1 = -0.5$ and $u_2 = 1$, then system (2.7)-(2.8) has equilibrium $(e_1, e_2)^T = (-0.029412, 0.20588)^T$. Solutions $x_1(t)$ and $x_2(t)$ of (2.7)-(2.8) corresponding to initial conditions

$$(\varphi_1(t), \varphi_2(t))^T = (t^2 + 2, \cos t - 3)^T, \quad (t - 3, 2 - t)^T \quad \text{and} \quad (0, 0)^T, \quad t \in [-2, 0]$$

can be seen on Figure 1 and 2, respectively. We can observe the corresponding solutions tend to equilibrium $(e_1, e_2)^T$.

We can easily find u_1 and u_2 in (2.7)-(2.8) so that condition (2.4) fails and for some large t either $|x_1(t)| > 1$ or $|x_2(t)| > 1$, so in such case (2.1) is not equivalent to the linear system (2.6). For system (2.7)-(2.8) with a given u_1 and u_2 it is easy to compute the equilibrium solutions of (2.1). In each case we tried we always got unique equilibrium solutions, and observed that the numerically generated solutions tend to the equilibrium. It is also easy to construct example when the conditions of Theorem 2.1 fail and the corresponding system has periodic solutions. The analytical study of the asymptotic behaviour of solutions in such cases is an interesting and difficult open problem.

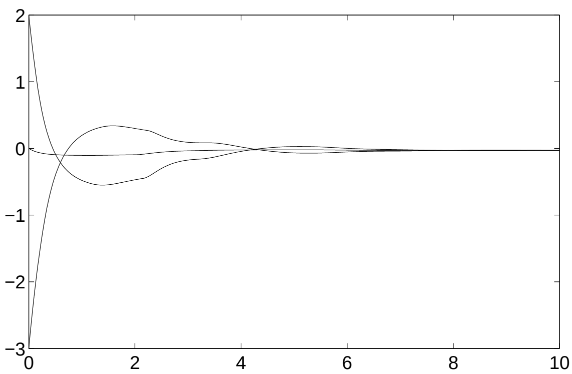

Figure 1: $x_1(t)$

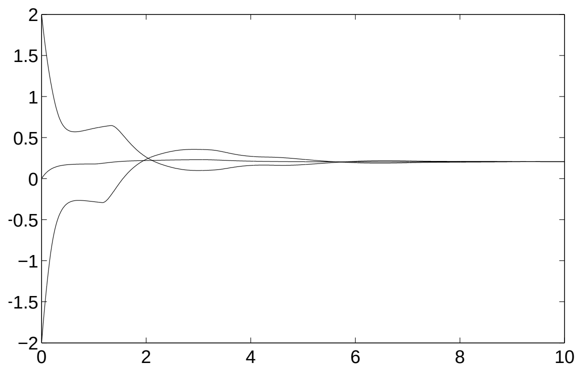

Figure 2: $x_2(t)$

References

[1] A. Berman and R. J. Plemmons, "Nonnegative Matrices in the Mathematical Sciences", Academic Press, New York, 1979.

[2] S. A. Campbell, Delay independent stability for additive neural networks, *Diff. Equat. Dyn. Syst.*, **9:3-4** (2001) 115–138.

[3] J. Cao, Global exponential stability and periodic solutions of delayed cellular neural networks, *J. Comput. System Sci.* **60** (2000) 38–46.

[4] L. O. Chua and L. Yang, Cellular neural networks: Theory, *IEEE Trans. Circuits and Systems I* **35** (1988) 1257–1272.

[5] I. Győri, F. Hartung, Stability Analysis of a Single Neuron Model with Delay, to appear in *J. Comput. Appl. Math.*

[6] M. Joy, Results concerning the absolute stability of delayed neural networks, *Neural Networks* **13** (2000) 613–616.

[7] X. Liao, Z. Wu and J. Yu, Stability analyses of cellular neural networks with continuous time delay, *J. Comput. Appl. Math.* **143** (2002) 29–47.

[8] N. Takahashi, A new sufficient condition for complete stability of cellular neural networks with delay, *IEEE Trans. Circuits Systems I. Fund. Theory Appl.* **47:6** (2000) 793–799.

[9] D. Xu, H. Zhao and H. Zhu, Global dynamics of Hopfield neural networks involving variable delays, *Comput. Math. Appl.* **42** (2001) 39–45.

[10] J. Zhang and X. Jin, Global stability analysis in delayed Hopfield neural network models, *Neural Networks* **13** (2000) 745–753.