PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 18 – 21, 2000, Atlanta, USA pp. **000–000**

LINEARIZED STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS

Ferenc Hartung

Department of Mathematics and Computing, University of Veszprém
H-8201 Veszprém, P.O.Box 158, Hungary

Janos Turi

Programs in Mathematical Sciences, University of Texas At Dallas
Richardson, TX 75083, USA

Abstract. In this paper we prove that a constant steady-state of an autonomous state-dependent delay equation is exponentially stable if a zero solution of a corresponding linear autonomous equation is exponentially stable.

1. **Introduction and notations.** Functional differential equations (FDEs) with state-dependent delays appear frequently in applications as model equations (see, e.g., [1]–[3], [17]), and the study of such equations is an active research area (see, e.g., [4]–[7], [10]–[16], [18]–[21]). Stability of the solution is one of the most important qualitative property of a model. There are many papers which give sufficient conditions for the stability of the trivial ($x = 0$) solution in state-dependent FDEs (see, e.g., [7], [8], [20], [21]).

In [4] and [12] linearized stability results were proved for certain classes of state-dependent FDEs concerning the asymptotic stability of the trivial solution. It was shown that the asymptotic stability of the trivial solution of the equation is implied by that of the trivial solution of an associated linear delay equation, the so-called linearized equation. Note the results of [4] and [12] are equivalent in the sense that they both provide the same associated linear equation for nonlinear equations which can be rewritten in both forms, but the classes of the equations studied were different. Cooke and Huang [4] investigated the nonlinear FDE with state-dependent delays of the form

$$\dot{x}(t) = g\left(x_t, \int_{-r_0}^{0} d\eta(s) g\Big(x(t+s-\tau(x_t))\Big)\right), \tag{1.1}$$

where $\tau : C \to [0, r_1]$, η is a matrix valued function of bounded variation, $r_0 > 0$, and r is such that $r \geq r_0 + r_1$. Motivated by the form of the delayed term in (1.1), the nonlinear FDE with state-dependent distributed delays

$$\dot{x}(t) = f\left(t, x(t), \int_{-r}^{0} d_s\mu(s,t,x_t)\, x(t+s)\right), \qquad t \geq 0, \tag{1.2}$$

was investigated in [10]. The term

$$\int_{-r}^{0} d_s\mu(s,t,x_t)x(t+s) \tag{1.3}$$

1991 *Mathematics Subject Classification.* 34K20.

This research was partially supported by Hungarian National Foundation for Scientific Research Grant No. T031935 and Janos Bolyai Research Grant of the Hungarian Academy of Sciences.

describing the delay dependence is a Stieltjes-integral of the solution segment $x(t+\cdot)$ with respect to $\mu(\cdot,t,x_t)$, which is a matrix valued function of bounded variations depending on time t and the state of the equation x_t. Here and throughout this paper $r>0$ is fixed and $x_t : [-r,0]\to\mathbb{R}^n$ is defined by $x_t(s)\equiv x(t+s)$. For well-posedness results for (1.2) we refer to [10], the appendix of [12], and see also Theorem 2.2 below.

Consider the linear time-dependent FDE of the form $\dot x(t) = L(t)x_t$, where $L(t)$ is a bounded linear operator on the space of continuous functions. Then the Riesz Representation Theorem yields that $L(t)x_t$ has the form (1.3) with $\mu=\mu(s,t)$. Therefore it seems like a natural extension to assume the structure described by (1.3) for the state-dependent case. Moreover, representation (1.3) includes discrete and distributed constant and time-dependent delays, and the "usual" state-dependent delays, $x(t-\tau(t,x(t))$ or $x(t-\tau(t,x_t))$ as well. A nice feature of this form is that it also allows delayed terms of the form

$$\sum_{i=1}^{\infty} A_i(t,x_t)x(t-\tau_i(t,x_t)) + \int_{-\tau_0}^{0} G(s,t,x_t)x(t+s)\,ds. \tag{1.4}$$

In [12] a linearization result was obtained for the autonomous version of (1.2). In this paper we extend the results of [12], and, under slightly more restrictive conditions, we prove stability theorems using linearization about any non-zero constant solution, not only the trivial solution. This generalization was motivated by the paper [15], where numerical experiments showed that the asymptotic stability of the trivial solution of the linearized equation implies the asymptotic stability of the constant or periodic steady-state of the state-dependent FDE. We also note that the result does not follow immediately by translating the constant solution to the origin, in this case we need stronger conditions that those for the trivial solution. The technique of the proof we present here is also a slightly different from that of [12].

The main problem to obtain linearization results for state-dependent FDEs is that it is difficult to differentiate the delayed term in the presence of state-dependent delays (see a detailed discussion of linearization and differentiability of solutions with respect to parameters for state-dependent delay equations in [5], [18], [11] and [12]). We shall define a bounded linear operator, $F : C\to\mathbb{R}^n$, and propose $\dot x(t) = Fx_t$ as a candidate for the linearized equation about the trivial solution. This is not the "true" linearization at zero, since the delayed term is not necessarily differentiable at zero (in the space C), but using assumption (H2) (ii) below, we can get an estimate on the error replacing the right hand side of the equation by Fx_t (see Lemma 3.2 below), which turns out to be sufficient to prove that the asymptotic stability of the corresponding linearized equation implies that of the nonlinear equation.

The class of autonomous FDEs with state-dependent delays we investigate is described in detailes in Section 2, and Section 3 contains our main theorem concerning linearized stability about a constant steady-state (see Theorem 3.3 below).

We close this section by introducing some notations which will be used throughout this paper. Let $|\cdot|$ denote a fixed vector norm on $\mathbb{R}^n$, and the corresponding induced matrix norm on $\mathbb{R}^{n\times n}$ is denoted by $|\cdot|$, as well. The partial derivatives of a function f of two variables with respect to its first and second variables is denoted by $D_1 f$ and $D_2 f$, respectively.

We denote the space of continuous functions $\psi\colon [-r,0]\to\mathbb{R}^n$ equipped with the supremum norm $\|\psi\|\equiv\max\{|\psi(s)|\colon s\in[-r,0]\}$ by C. Let L^∞ denote the Banach-space of Lebesgue-measurable, essentially bounded functions $\psi : [-r,0]\to\mathbb{R}^n$ with norm $|\psi|_{L^\infty}\equiv \operatorname{ess\,sup}\{|\psi(s)|\colon s\in[-r,0]\}$. $W^{1,\infty}$ is the Sobolev space of absolutely continuous functions $\psi : [-r,0]\to\mathbb{R}^n$ with essentially bounded derivatives. The norm in this Banach-space is defined by $|\psi|_{W^{1,\infty}}\equiv\max(\|\psi\|,|\dot\psi|_{L^\infty})$.

Let X be any normed linear space. We denote the closed ball with radius ϱ centered at the point $a\in X$ by $B_X(a;\,\varrho)$, i.e., $B_X(a;\,\varrho)=\{x\in X\colon |a-x|_X\le\varrho\}$.

2. **A Class of State-Dependent Delay Equations.** Consider the nonlinear state-dependent delay system

$$\dot{x}(t) = f(x(t), \lambda(x_t, x_t)), \qquad t \geq 0 \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.2}$$

We assume the following conditions throughout the paper:

(H1) $f : \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuously differentiable, where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$,

(H2) λ: $\Omega_3 \times C \to \mathbb{R}^n$, where Ω_3 is an open subset of C, and

(i) λ is linear in its second argument, and there exists $L_1 \geq 0$ such that

$$|\lambda(\psi, \xi)| \leq L_1 \|\xi\|, \qquad \psi \in C,$$

(ii) λ is locally Lipschitz-continuous in its first argument, i.e., for every $M \subset \Omega_3$ compact subset of C there exists a constant $L_2 = L_2(M)$ such that

$$|\lambda(\psi, \xi) - \lambda(\tilde{\psi}, \xi)| \leq L_2 |\dot{\xi}|_{L^\infty} \|\psi - \tilde{\psi}\| \tag{2.3}$$

for all $\xi \in W^{1,\infty}$, $\psi, \tilde{\psi} \in M$,

(H3) $\varphi \in C$.

It follows from property (H2) (i) and the Riesz Representation Theorem that the function λ has the representation

$$\lambda(\psi, \xi) \equiv \int_{-r}^{0} d_s \mu(s, \psi) \xi(s),$$

where $\mu(\cdot, \psi)$ is a matrix valued function of bounded variation. Therefore (2.1) is an autonomous version of (1.2).

Remark 2.1. We note that in [10] and [12] assumption (H2) (ii) was replaced by (H2) (ii'), where (2.3) was changed to

$$|\lambda(\psi, \xi) - \lambda(\tilde{\psi}, \xi)| \leq L_2 |\xi|_{W^{1,\infty}} \|\psi - \tilde{\psi}\|.$$

The difference between the two conditions is that this latter condition allows a larger class of delayed terms, including the autonomous version of (1.4). But if we assume the stronger condition (2.3), that excludes the "distributed" delay terms of (1.4), and includes only point state-dependent terms of the form

$$\lambda(\psi, \xi) = \sum_{i=1}^{\infty} A_i \xi(-\tau_i(\psi)), \tag{2.4}$$

where it is easy to formulate conditions on A_i and τ_i which imply (H2) (ii). Note that the class of state-dependent terms satisfying this condition still includes the "usual" point state-dependent delays

$$\lambda(\psi, \xi) = \xi(-\tau(\psi)) \qquad \text{or} \qquad \lambda(\psi, \xi) = \xi(-\tau(\psi(0))),$$

where (2.3) can be satisfied naturally assuming Lipschitz-continuity of τ.

We introduce the simplifying notation

$$\Lambda(\psi) \equiv \lambda(\psi, \psi).$$

Then the right-hand-side of (2.1) can be written shortly as $f(x(t), \Lambda(x_t))$.

It is easy to see that in order have a well-posed problem, the initial function φ has to satisfy

$$\varphi(0) \in \Omega_1, \quad \Lambda(\varphi) \in \Omega_2 \quad \text{and} \quad \varphi \in \Omega_3. \tag{2.5}$$

We recall the following result from [10] concerning the well-posedness of IVP (2.1)-(2.2). Note again that this result was proved in [10] under the weaker condition (H2) (ii') instead of (H2) (ii).

Theorem 2.2. *Assume (H1)–(H3), and* (2.5) *holds for* $\tilde\varphi$. *Then there exist* $\alpha > 0$ *and* $\delta > 0$ *such that IVP* (2.1)-(2.2) *has a solution* $x(t;\varphi)$ *on* $[0,\alpha]$ *for all* $\varphi \in B_C(\tilde\varphi;\delta)$. *Moreover, if we assume that* $\varphi \in W^{1,\infty}$, *i.e.,* φ *is Lipschitz-continuous, then the solution is unique, and there exists* $L > 0$ *such that* $|x(\cdot;\varphi)_t - x(\cdot;\tilde\varphi)_t|_{W^{1,\infty}} \le L|\varphi - \tilde\varphi|_{W^{1,\infty}}$ *for all* $t \in [0,\alpha]$.

Note that uniqueness is not required when we study stability, therefore we will assume (H3) instead of $\varphi \in W^{1,\infty}$.

We shall need the following estimate of Λ.

Lemma 2.3. *Assume (H2). Let* M *be a compact subset of* Ω_3, L_1 *be the constant from (H2) (i), and* L_2 *be the constant corresponding to* M *from (H2) (ii). Then*

$$|\Lambda(\psi) - \Lambda(\tilde\psi)| \le (L_1 + L_2|\dot{\tilde\psi}|_{L^\infty})\|\psi - \tilde\psi\|$$

for $\psi, \tilde\psi \in M$ *and* $\tilde\psi \in W^{1,\infty}$.

Proof. The statement follows immediately from the relation

$$\Lambda(\psi) - \Lambda(\tilde\psi) = \lambda(\psi, \psi - \tilde\psi) + \lambda(\psi, \tilde\psi) - \lambda(\tilde\psi, \tilde\psi)$$

and assumption (H2) (i) and (ii). □

The above lemma has immediately the following corollary.

Corollary 2.4. *Assume (H2). Let* L_1 *be the constant from (H2) (i), and* $c \in C$ *be a constant function. Then*

$$|\Lambda(\psi) - \Lambda(c)| \le L_1\|\psi - c\|$$

for any $\psi \in \Omega_3$.

3. **Linearized Stability.**

Consider again the state-dependent delay equation

$$\dot x(t) = f(x(t), \Lambda(x_t)), \qquad t \ge 0. \tag{3.1}$$

Let $\bar x$ be a constant function defined on $[-r,\infty)$. For simplicity, both its value $\bar x(t)$ and its segment function $\bar x_t$ at any t will be denoted as $\bar x$. Therefore we will write $f(\bar x, \Lambda(\bar x))$ for substituting it to the right-hand-side of (3.1). It should always be clear from the context whether $\bar x$ denotes a constant vector or a constant function.

We assume that $\bar x$ is a solution of (3.1), i.e.,

$$f(\bar x, \Lambda(\bar x)) = 0. \tag{3.2}$$

The sets Ω_1, Ω_2 and Ω_3 are opens subsets of the respective space, therefore there exist positive constants ϱ_1, ϱ_2 and ϱ_3 such that

$$B_{\mathbb{R}^n}(\bar x;\varrho_1) \subset \Omega_1, \quad B_{\mathbb{R}^n}(\Lambda(\bar x);\varrho_2) \subset \Omega_2 \quad \text{and} \quad B_C(\bar x;\varrho_3) \subset \Omega_3.$$

We define the linear operator $F\colon C \to \mathbb{R}^n$ associated to $\bar x$ by

$$F\psi \equiv D_1 f(\bar x, \Lambda(\bar x))\psi(0) + D_2 f(\bar x, \Lambda(\bar x))\lambda(\bar x, \psi) \tag{3.3}$$

and the function

$$g : C \to \mathbb{R}^n, \qquad g(\psi) \equiv f(\psi(0), \Lambda(\psi)) - F\psi. \tag{3.4}$$

Note that the linear operator F is a bounded operator, since by (H2) (i) it satisfies

$$|F\psi| \le \Big(|D_1 f(\bar x, \Lambda(\bar x))| + |D_2 f(\bar x, \Lambda(\bar x))|\, L_1\Big)\|\psi\|.$$

By this notation we can rewrite (2.1) as

$$\dot{x}(t) = Fx_t + g(x_t), \qquad t \geq 0, \tag{3.5}$$

and therefore we can consider it as a perturbation of the autonomous linear delay equation

$$\dot{x}(t) = Fx_t, \qquad t \geq 0. \tag{3.6}$$

We denote the fundamental solution of (3.6) by $U(t)$, i.e., it is a matrix valued solution of the initial value problem

$$\dot{U}(t) = FU_t, \qquad t \geq 0, \tag{3.7}$$

$$U(t) = \begin{cases} I, & t = 0, \\ 0, & t < 0. \end{cases} \tag{3.8}$$

It is known (see, e.g., [9]) that the trivial solution of (3.6) is exponentially stable, if and only if there exist constants $K_0 \geq 1$ and $\alpha_0 > 0$ such that

$$|U(t)| \leq K_0 e^{-\alpha_0 t}, \qquad t \geq 0. \tag{3.9}$$

We also recall (see, e.g., [9]) that the trivial solution of (3.6) is asymptotically stable, if and only if it is exponentially stable.

The proof of our main theorem will be based on the following two lemmas.

Lemma 3.1. *Assume (H1)–(H3), and let $\bar{x}$ be a constant function satisfying* (3.2). *Let $T > 0$ be given, and let x be a solution of* (3.1) *satisfying*

$$|x(t) - \bar{x}| \leq \varrho_3 \qquad \text{for } t \in [-r, T]. \tag{3.10}$$

Then there exists a constant $N_1 > 0$ independent of T such that

$$|\dot{x}(t)| \leq N_1 \|x_t - \bar{x}\|, \qquad t \in [0, T] \tag{3.11}$$

and

$$\|x_t - \bar{x}\| \leq e^{N_1 t} \|\varphi - \bar{x}\|, \qquad t \in [0, T]. \tag{3.12}$$

Proof. Le L_1 be the constant from (H2) (i), and suppose (3.10). Then, by Corollary 2.4,

$$|\Lambda(x_t) - \Lambda(\bar{x})| \leq L_1 \|x_t - \bar{x}\| \leq L_1 \varrho_3,$$

and therefore $x(t) \in B_{\mathbb{R}^n}(\bar{x}; \varrho_3)$ and $\Lambda(x_t) \in B_{\mathbb{R}^n}(\Lambda(\bar{x}); L_1\varrho_3)$ for $t \in [0, T]$. Hence assumption (H1) yields that there exists a constant $L_0 \geq 0$ such that

$$|\dot{x}(t)| = |f(x(t), \Lambda(x_t)) - f(\bar{x}, \Lambda(\bar{x}))| \leq L_0 \Big(|x(t) - \bar{x}| + |\Lambda(x_t) - \Lambda(\bar{x})| \Big).$$

Then Lemma 2.3 implies (3.11) with $N_1 = L_0(1 + L_1)$.

To prove (3.12), consider the inequalities

$$|x(t) - \bar{x}| \leq |\varphi(0) - \bar{x}| + \int_0^t |\dot{x}(s)|\, ds \leq \|\varphi - \bar{x}\| + N_1 \int_0^t \|x_s - \bar{x}\|\, ds.$$

Let $v(t) \equiv \max\{|x(s) - \bar{x}| \colon -r \leq s \leq t\}$. Then

$$|x(t) - \bar{x}| \leq \|\varphi - \bar{x}\| + N_1 \int_0^t v(s)\, ds, \qquad t \in [0, T],$$

and since the right-hand-side is monotone increasing in t, it implies

$$v(t) \leq \|\varphi - \bar{x}\| + N_1 \int_0^t v(s)\, ds, \qquad t \in [0, T].$$

Therefore Gronwall's inequality proves (3.12), since $\|x_t - \bar{x}\| \leq v(t)$. □

We will need the following estimate of g.

Lemma 3.2. *Assume (H1)–(H3), and let $\bar{x}$ be a constant function satisfying* (3.2). *Then there exists a constant $N_2 \geq 1$ such that for every $\eta > 0$ there exists a constant $\theta = \theta(\eta) > 0$ such that $\theta \leq \varrho_3$, and*

$$|g(x_t) - g(\bar{x})| \leq N_2(\eta + 1)\|x_t - \bar{x}\|, \qquad t \in [0, r], \tag{3.13}$$

and

$$|g(x_t) - g(\bar{x})| \leq N_2\Big(\eta + \|\dot{x}_t\|\Big)\|x_t - \bar{x}\|, \qquad t \geq r, \tag{3.14}$$

for all solution x of (3.1) *satisfying*

$$|x(t) - \bar{x}| < \theta \qquad \text{for } t \geq -r. \tag{3.15}$$

Proof. The definition of g and F, and the linearity of λ in its second argument imply

$$\begin{aligned}
&|g(x_t) - g(\bar{x})| \\
&\quad = \Big|f(x(t), \Lambda(x_t)) - f(\bar{x}, \Lambda(\bar{x})) \\
&\qquad\quad - D_1 f(\bar{x}, \Lambda(\bar{x}))(x(t) - \bar{x}) - D_2 f(\bar{x}, \Lambda(\bar{x}))(\lambda(\bar{x}, x_t) - \lambda(\bar{x}, \bar{x}))\Big| \\
&\quad \leq \Big|f(x(t), \Lambda(x_t)) - f(\bar{x}, \Lambda(\bar{x})) \\
&\qquad\quad - D_1 f(\bar{x}, \Lambda(\bar{x}))(x(t) - \bar{x}) - D_2 f(\bar{x}, \Lambda(\bar{x}))(\Lambda(x_t) - \Lambda(\bar{x}))\Big| \\
&\qquad\quad + \Big|D_2 f(\bar{x}, \Lambda(\bar{x}))(\lambda(x_t, x_t) - \lambda(\bar{x}, x_t))\Big| \\
&\quad \leq \sup_{0<\nu<1}\Big|D_1 f\Big(\bar{x} + \nu(x(t) - \bar{x}), \Lambda(\bar{x}) + \nu(\Lambda(x_t) - \Lambda(\bar{x}))\Big) - D_1 f(\bar{x}, \Lambda(\bar{x}))\Big||x(t) - \bar{x}| \\
&\qquad + \sup_{0<\nu<1}\Big|D_2 f\Big(\bar{x} + \nu(x(t) - \bar{x}), \Lambda(\bar{x}) + \nu(\Lambda(x_t) - \Lambda(\bar{x}))\Big) - D_2 f(\bar{x}, \Lambda(\bar{x}))\Big| \\
&\qquad\quad \cdot |\Lambda(x_t) - \Lambda(\bar{x})| + \Big|D_2 f(\bar{x}, \Lambda(\bar{x}))(\lambda(x_t, x_t) - \lambda(\bar{x}, x_t))\Big|.
\end{aligned} \tag{3.16}$$

Fix any $\eta > 0$. By the continuous differentiability of f guaranteed by (H1), there exists $\theta_1 > 0$ such that if $|u - \tilde{u}|, |v - \tilde{v}| \leq \theta_1$, $u, \tilde{u} \in B_{\mathbb{R}^n}(\bar{x};\, \varrho_1)$ and $v, \tilde{v} \in B_{\mathbb{R}^n}(\Lambda(\bar{x});\, \varrho_2)$, then

$$|D_1 f(u, v) - D_1 f(\tilde{u}, \tilde{v})| < \eta \quad \text{and} \quad |D_2 f(u, v) - D_2 f(\tilde{u}, \tilde{v})| < \eta. \tag{3.17}$$

Let L_1 be the constant from (H2) (i). Corollary 2.4 yields

$$|\Lambda(x_t) - \Lambda(\bar{x})| \leq L_1\|x_t - \bar{x}\|, \qquad t \geq 0$$

for all solution x of (3.1). Let $\theta \equiv \min(\theta_1,\, \varrho_1,\, \theta_1/L_1,\, \varrho_2/L_1)$. Then

$$|\Lambda(x_t) - \Lambda(\bar{x})| \leq \min(\theta_1,\, \varrho_2), \qquad t \geq 0,$$

for all solution x satisfying (3.15).

Therefore, for such x, inequality

$$\Big|D_i f\Big(\bar{x} + \nu(x(t) - \bar{x}), \Lambda(\bar{x}) + \nu(\Lambda(x_t) - \Lambda(\bar{x}))\Big) - D_i f(\bar{x}, \Lambda(\bar{x}))\Big| < \eta \tag{3.18}$$

holds for all $t \geq 0$, $0 \leq \nu \leq 1$ and $i = 1, 2$. It follows from (H2) (ii) that $\lambda(x_t, \bar{x}) = \lambda(\bar{x}, \bar{x})$ for $t \geq 0$, therefore the linearity of λ in its second argument yields

$$\lambda(x_t, x_t) - \lambda(\bar{x}, x_t) = \lambda(x_t, x_t - \bar{x}) - \lambda(\bar{x}, x_t - \bar{x}), \qquad t \geq 0. \tag{3.19}$$

Then combining (3.16), (3.18) and (3.19) we get

$$\begin{aligned} |g(x_t) - g(\bar{x})| \quad \le \quad & \eta|x(t) - \bar{x}| + \eta|\Lambda(x_t) - \Lambda(\bar{x})| \\ & + |D_2 f(\bar{x}, \Lambda(\bar{x}))||\lambda(x_t, x_t - \bar{x}) - \lambda(\bar{x}, x_t - \bar{x})| \end{aligned} \tag{3.20}$$

for any x satisfying (3.15).

For $t \in [0, r]$, using (H2) (i), we have

$$|\lambda(x_t, x_t - \bar{x}) - \lambda(\bar{x}, x_t - \bar{x})| \le 2L_1\|x_t - \bar{x}\|.$$

Since $\dot{x}(t) = f(x(t), \Lambda(x_t))$ for $t \ge 0$, and f is continuous on $B_{\mathbb{R}^n}(\bar{x}; \varrho_1) \times B_{\mathbb{R}^n}(\Lambda(\bar{x}); \varrho_2)$, there exists $m > 0$ such that $|\dot{x}_t|_{L^\infty} \le m$ for any x satisfying (3.15) and $t \ge r$. Moreover, $\dot{x}(t)$ is continuous for $t \ge 0$. Let M be the closure of the set $B_C(\bar{x}; \theta) \cap \{y \in W^{1,\infty} \colon |\dot{y}|_{L^\infty} \le m\}$ in C. Then $M \subset \Omega_3$, and by the Arsela-Ascoli Lemma, it is a compact subset of C. Let L_2 be the constant corresponding to M from (H2) (ii). Then we get

$$|\lambda(x_t, x_t - \bar{x}) - \lambda(\bar{x}, x_t - \bar{x})| \le L_2\|\dot{x}_t\|\|x_t - \bar{x}\|, \qquad t \ge r.$$

Let $N_2 \equiv \max(1 + L_1, |D_2 f(\bar{x}, \Lambda(\bar{x}))|L_2, |D_2 f(\bar{x}, \Lambda(\bar{x}))|2L_1)$, then (3.13) and (3.14) follow from (3.20). $\square$

We show that the exponential stability of the constant steady-state solution $\bar{x}$ of the nonlinear state-dependent autonomous FDE (3.1) can be obtained by that of the linear autonomous FDE (3.6).

Theorem 3.3. *Assume (H1)–(H3), and let $\bar{x}$ be a constant function satisfying* (3.2)*. Let F be the linear operator defined by* (3.3)*. Suppose the trivial solution of* (3.6) *is exponentially stable, i.e., there exist $K_0 \ge 1$ and $\alpha_0 > 0$ such that* (3.9) *holds. Then for every $0 < \alpha < \alpha_0$ there exists $\delta > 0$ and $K \ge 1$ such that if $\|\varphi - \bar{x}\| < \delta$, then any corresponding solution $x(t) = x(t; \varphi)$ of* (3.1) *satisfies*

$$|x(t) - \bar{x}| \le Ke^{-\alpha t}\|\varphi - \bar{x}\|, \qquad t \ge 0,$$

i.e., $\bar{x}$ is an exponentially stable steady-state of (3.1).

Proof. Fix $\varepsilon_0 > 0$ and let $0 < \varepsilon < \varepsilon_0$, and let $0 < \alpha < \alpha_0$ be fixed. Let $N_2 \ge 1$ be the constant from Lemma 3.2, and let

$$\eta \equiv \frac{\varepsilon(\alpha_0 - \alpha)}{2(1 + \varepsilon)K_0 N_2}.$$

Define $\theta = \theta(\eta)$ by Lemma 3.2. Let N_1 be the constant defined by Lemma 3.1, and let

$$\delta_1 \equiv \min\left(\varrho_1, \varrho_3, \theta, \frac{\eta}{N_1}\right) \quad \text{and} \quad K \equiv K_0\left(1 + r\eta N_2 e^{(\alpha_0 + N_1)r}\right),$$

and finally, let

$$\delta \equiv \frac{\delta_1}{(1 + \varepsilon_0)K}.$$

Let $\varphi \in C$ be such that $\|\varphi - \bar{x}\| < \delta$. Since $\delta < \delta_1$, there exists a neighborhood of 0 such that $|x(t) - \bar{x}| < \delta_1$ for t within this neighborhood. Suppose there exists $T > 0$ such that

$$|x(t) - \bar{x}| < \delta_1, \qquad \text{for } t \in [0, T), \quad \text{and} \quad |x(T) - \bar{x}| = \delta_1. \tag{3.21}$$

The definition of δ_1, (3.21) and Lemma 3.1 imply that

$$|\dot{x}(t)| \le N_1\|x_t - \bar{x}\| \le N_1\delta_1 \le \eta, \qquad \text{for } t \in [0, T]. \tag{3.22}$$

The variation-of-constants formula (see, e.g., [9]) implies

$$x(t) = U(t)\varphi(0) + \int_0^t U(t - s)g(x_s)\, ds, \qquad t \ge 0.$$

Similarly,

$$\bar{x} = U(t)\bar{x} + \int_0^t U(t-s)g(\bar{x})\,ds, \qquad t \geq 0.$$

Therefore

$$|x(t) - \bar{x}| \leq |U(t)||\varphi(0) - \bar{x}| + \int_0^t |U(t-s)||g(x_s) - g(\bar{x})|\,ds, \qquad t \geq 0. \tag{3.23}$$

Suppose $T > r$. Relations $\delta_1 \leq \theta$, (3.9), (3.21), (3.23) and Lemma 3.2 imply for $t \in [r, T]$

$$\begin{aligned} |x(t) - \bar{x}| \quad &\leq \quad K_0 e^{-\alpha_0 t}\|\varphi - \bar{x}\| + K_0 \int_0^t e^{-\alpha_0(t-s)} N_2 \eta \|x_s - \bar{x}\|\,ds \\ &\quad + K_0 \int_0^r e^{-\alpha_0(t-s)} N_2 \|x_s - \bar{x}\|\,ds + K_0 \int_r^t e^{-\alpha_0(t-s)} N_2 \|\dot{x}_s\| \|x_s - \bar{x}\|\,ds. \end{aligned}$$

Multiplying both sides of this inequality by $e^{\alpha t}$, and using the definition of K and estimates (3.12) and (3.22), we get

$$\begin{aligned} e^{\alpha t}|x(t) - \bar{x}| \quad &\leq \quad K_0 e^{(\alpha - \alpha_0)t}\|\varphi - \bar{x}\| + K_0 e^{(\alpha-\alpha_0)t} \int_0^r e^{\alpha_0 s} N_2 \eta e^{N_1 r} \|\varphi - \bar{x}\|\,ds \\ &\quad + K_0 e^{(\alpha-\alpha_0)t} \int_0^t e^{\alpha_0 s} N_2 2\eta \|x_s - \bar{x}\|\,ds \\ &\leq \quad K\|\varphi - \bar{x}\| + K_0 N_2 2\eta e^{(\alpha-\alpha_0)t} \int_0^t e^{\alpha_0 s}\|x_s - \bar{x}\|\,ds. \end{aligned}$$

Note that the last inequality holds for $t \in [0, r]$ and for $T \leq r$, as well. Let $v(t) = \max\{e^{\alpha s}|x(s) - \bar{x}| : -r \leq s \leq t\}$. Then we have

$$\begin{aligned} e^{\alpha t}|x(t) - \bar{x}| \quad &\leq \quad K\|\varphi - \bar{x}\| + K_0 N_2 2\eta e^{(\alpha-\alpha_0)t} v(t) \int_0^t e^{(\alpha_0 - \alpha)s}\,ds \\ &= \quad K\|\varphi - \bar{x}\| + K_0 N_2 2\eta v(t) \int_0^t e^{(\alpha-\alpha_0)s}\,ds, \qquad t \in [0, T]. \end{aligned}$$

Since the right-hand-side is monotone increasing in t, it implies

$$\begin{aligned} v(t) \quad &\leq \quad K\|\varphi - \bar{x}\| + K_0 N_2 2\eta v(t) \int_0^t e^{(\alpha-\alpha_0)s}\,ds \\ &\leq \quad K\|\varphi - \bar{x}\| + \frac{K_0 N_2 2\eta}{\alpha_0 - \alpha} v(t), \qquad t \in [0, T]. \end{aligned}$$

Then, using the definition of η, we get

$$v(t) \leq K\|\varphi - \bar{x}\| + \frac{\varepsilon}{1+\varepsilon} v(t), \qquad \text{for } t \in [0, T],$$

and hence

$$|x(t) - \bar{x}| \leq e^{-\alpha t} v(t) \leq (1+\varepsilon) K e^{-\alpha t}\|\varphi - \bar{x}\|, \qquad t \in [0, T].$$

But this yields

$$\delta_1 = |x(T) - \bar{x}| < (1+\varepsilon_0) K\delta = \delta_1,$$

which contradicts to the definition of T. Therefore $T = \infty$, and

$$|x(t) - \bar{x}| \leq (1+\varepsilon) K e^{-\alpha t}\|\varphi - \bar{x}\|$$

holds for all $t \geq 0$. This implies the statement of the theorem, since ε was arbitrary small positive number. $\square$

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E-mail address: `hartung@szt.vein.hu`
E-mail address: `turi@utdallas.edu`