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Preservation of Stability in Delay Equations under Delay Perturbations*

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Abstract

We consider a class of linear delay equations with perturbed time lags and present conditions which guarantee that the asymptotic stability of the trivial solution of the equation at hand is preserved under these perturbations. As an example we show how our results can be used to obtain an estimate on the maximum allowable sampling interval in the stabilization of a hybrid system with feedback delays. We also present applications of our perturbation theorem to obtain stability conditions for delay equations with multiple delays.

Key words: delay equations, asymptotic stability, delay perturbations

1 Introduction

In this paper we study the effects of perturbations of time delays to the stability of a class of linear delay systems. Our goal is to obtain a "practical" condition, i.e., a norm bound on the perturbations corresponding to the particular system under consideration, which guarantees the preservation of asymptotic stability under perturbations. It turns out that such condition can be formulated assuming that we know the fundamental solution of the unperturbed system (see Theorem 2.3 below). Since asymptotic stability of the unperturbed system implies that the components of its fundamental solution go to zero at infinity, it is possible to get "good" numerical estimates of these components, and consequently obtain norm bounds on the allowable perturbations.

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We present our main results in Section 2, and in Section 3 we consider numerical examples. Example 3.3 demonstrates how our results can be used to obtain an estimate on the maximum allowable sampling interval while preserving stability of a hybrid system with feedback delay. Note that this study was motivated by [5] where stabilization of a hybrid feedback control system was studied in the case when the plant is described by an ordinary differential equation. In Section 4, as an application of our perturbation results, we derive sufficient conditions for asymptotic stability for classes of linear scalar and vector delay differential equations with multiple time-dependent delays.

To put our current work into proper perspective for the reader, in the remaining part of this section, we recall some relevant developments from [6]. Theorems E and G in Chapter 34 in [6] give general perturbations results for the preservation of asymptotic stability of general linear delay systems. More specifically, Example 3 on page 397 in [6] addresses the question of the effects of delay perturbations on the stability of linear differential systems, and a sufficient condition guaranteeing the asymptotic stability of perturbed systems is stated (see also [10]) as follows: The trivial ($x(t) = 0$) solution of the perturbed system

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t - r_i), \qquad t \geq 0$$

is uniformly asymptotically stable, if the trivial solution of the unperturbed system

$$\dot{y}(t) = \sum_{i=0}^{m} A_i(t)y(t), \qquad t \geq 0$$

is uniformly asymptotically stable, i.e., there exist constants $M > 0$ and $\gamma > 0$ s.t. for $t \geq 0$ we have

$$\|y(t;\xi)\| \leq M\|\xi\|e^{-\gamma t}, \tag{1.1}$$

where ξ is the given initial condition, and if

$$\tau \sup_{t\geq 0} \sum_{i=1}^{m} \|A_i(t)\| \cdot \sum_{i=0}^{m} \sup_{t\geq 0} \|A_i(t)\| < \frac{\gamma}{M}, \tag{1.2}$$

where $r_1, r_2, \ldots, r_m$ represent constant delay perturbations and $\tau \equiv \max\{r_1, r_2, \ldots, r_m\}$.

Note that condition (1.2) has a straightforward generalization for continuous time-delay perturbations, i.e., when $r_i = r_i(t)$ are continuous functions. In this case in condition (1.2) the constant τ is replaced by $\tau^* \equiv \max_{i=1,\ldots,m} \sup_{t\geq 0} |r_i(t)|$. It is somewhat inconvenient that in order to apply condition (1.2), one has to assume: i) smallness of delays for all $t \geq 0$, ii) explicit knowledge of the constants M and γ.

In Section 2 below we derive a condition for the preservation of stability for a large class of equations assuming: i) smallness of perturbations only for sufficiently large times, (hence we allow perturbation which are not "small" initially), ii) the knowledge of the integral of the absolute value of the fundamental solution over $[0, \infty)$. Note that for asymptotically stable systems, it is relatively easy to obtain good estimates for the above integral, and therefore our condition may provide useful tool for applications. Furthermore, in the special case, when the fundamental solution is positive, the condition for preservation of stability can be formulated in terms of the coefficient matrices of the given system.

Investigating stability properties of perturbed delay equations, uncertain delay equations, or robust stability of delay equations is a reasonably active research area. Without claiming completeness we refer the reader to [1], [4], [6]–[7], [13]–[16], [21] and the references therein for related articles on these topics.

2 Main Results

Consider the delay differential equation

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - r_i - \eta_i(t)), \qquad t \geq 0 \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad -r \leq t \leq 0, \tag{2.2}$$

where A_i $(i = 0, \ldots, m)$ denote constant $n \times n$ matrices, $0 \leq r_0 \leq r_1 \leq \ldots \leq r_m$, $\varphi : [-r, 0] \to \mathbb{R}^n$ is a continuous function, and we shall assume that the piecewise continuous delay perturbations, $\eta_i(\cdot)$ $(i = 0, \ldots, m)$, satisfy

$$t - r \leq t - r_i - \eta_i(t) \leq t \qquad \text{for} \quad t \geq 0 \quad (i = 0, \ldots, m). \tag{2.3}$$

The solution of initial value problem (2.1)-(2.2) is an absolutely continuous function, which satisfies (2.2) for all $t \in [-r, 0]$, and satisfies (2.1) a.e. $t \geq 0$. Under our assumptions initial value problem (2.1)-(2.2) is a delay differential equation and has a unique solution, which is continuously differentiable at the points where $\tau_i(t)$ $(i = 0, \ldots, m)$ are continuous.

We consider the corresponding unperturbed system with constant delays, i.e.,

$$\dot{y}(t) = \sum_{i=0}^{m} A_i y(t - r_i), \qquad t \geq 0, \tag{2.4}$$

and we assume that

(H) the trivial solution of (2.4) is asymptotically stable.

The fundamental solution of (2.4), $V(t)$, is defined as the solution of the following system

$$\dot{V}(t) = \sum_{i=0}^{m} A_i V(t - r_i), \qquad t \geq T \tag{2.5}$$

and

$$V(t) = \begin{cases} I, & t = T, \\ 0, & t < T, \end{cases} \tag{2.6}$$

where $I, 0 \in \mathbb{R}^{n \times n}$ are the identity and the zero matrix, respectively, and $T \geq 0$.

Remark 2.1 *To emphasize the dependence of* $V(\cdot)$ *on* T *we use the notation* $V(t;\, T)$*. Note that* $V(t;\, T) = V(t-T;\, 0)$ *for* $t \geq T \geq 0$ *because* (2.4) *is autonomous (see e.g. [11]), hence*

$$\int_0^\infty V(t;\, T)\, dt = \int_0^\infty V(t;\, 0)\, dt.$$

We can rewrite (2.1) in the form

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - r_i) + f(t), \tag{2.7}$$

where

$$f(t) \equiv \sum_{i=0}^{m} A_i \Big(x(t - r_i - \eta_i(t)) - x(t - r_i) \Big). \tag{2.8}$$

In this setting (2.4) can be considered as the homogeneous equation corresponding to (2.7). The variation-of-constants formula (see, e.g., [11], p. 145) gives the following expression for the solution of initial value problem (2.1)-(2.2):

$$x(t) = y(t) + \int_T^t V(t-s) f(s)\, ds, \qquad t \geq T, \tag{2.9}$$

where $T > 0$, and y is the solution of (2.4) with initial function $y(t) = x(t)$ for $T - r \leq t \leq T$ and $V(\cdot) = V(\cdot; T)$ is the fundamental solution of (2.4).

For future convenience, we introduce the ˜ operation on vectors and on matrices, which means taking the absolute value of the vector or matrix componentwise, i.e., if $x = (x_1, x_2, \ldots, x_n)^T$, then by definition $\tilde{x} \equiv (|x_1|, |x_2|, \ldots, |x_n|)^T$, and similarly if $A = (a_{ij})_{n\times n}$, then $\tilde{A} \equiv (|a_{ij}|)_{n\times n}$. The relation $\leq$ between vectors means a componentwise comparison, i.e., $(x_1, x_2, \ldots, x_n)^T \leq (y_1, y_2, \ldots, y_n)^T$ if for all the components $x_i \leq y_i$.

Remark 2.2 *Hypothesis (H) implies (see e.g. [11]) that the trivial solution of* (2.4) *is exponentially stable, and there exist constants* $K > 0$ *and* $\alpha > 0$*, such that* $\|V(t)\| \leq K e^{-\alpha t}$ *for* $t \geq 0$*, (where* $\|\cdot\|$ *is the matrix norm induced by the vector norm* $\|(x_1, x_2, \ldots, x_n)\| \equiv max\{|x_1|, |x_2|, \ldots, |x_n|\}$*), and then every element of the matrix*

$$\int_0^\infty \tilde{V}(s)\, ds$$

is finite.

The next theorem shows that if the perturbations of the delays in (2.1) are small enough for large t, then the equation remains asymptotically stable.

Theorem 2.3 *Assume (H) and that the matrix*

$$M \equiv \int_0^\infty \tilde{V}(s)\, ds \left(\sum_{i=0}^{m} \overline{\lim_{t\to\infty}} |\eta_i(t)| \cdot \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \tag{2.10}$$

has spectral radius less than 1, *i.e.,* $\rho(M) < 1$. *Then the trivial solution of* (2.1) *is asymptotically stable.*

Proof: We prove the theorem in three steps. First we give an estimate of $\tilde{f}(t)$ for large t. Next we show that $\tilde{x}(t)$ is bounded, i.e., $\overline{\lim}_{t\to\infty} \tilde{x}(t)$ is finite, and then we show that $\overline{\lim}_{t\to\infty} \tilde{x}(t) = 0$, which proves the theorem.

(i) We will need an estimate of $f(t)$ for large t. Fix a constant $T > r$, then (2.3) implies that

$$t - r_i - \eta_i(t) \geq 0 \qquad \text{for} \qquad t > T, \quad i = 0, \ldots, m. \tag{2.11}$$

It is easy to see that for $t > r$ the solution of (2.1) is piecewise continuously differentiable and we can write

$$f(t) = \sum_{i=0}^{m} A_i \int_{t-r_i}^{t-r_i-\eta_i(t)} \dot{x}(s)\, ds.$$

Using (2.1) we get

$$f(t) = \sum_{i=0}^{m} A_i \int_{t-r_i}^{t-r_i-\eta_i(t)} \sum_{j=0}^{m} A_j x(s - r_j - \eta_j(s))\, ds. \tag{2.12}$$

This relation and the definition of the ˜ operation imply the inequality

$$\tilde{f}(t) \leq \sum_{i=0}^{m} \tilde{A}_i \left| \int_{t-r_i}^{t-r_i-\eta_i(t)} \sum_{j=0}^{m} \tilde{A}_j \tilde{x}(s - r_j - \eta_j(s))\, ds \right|. \tag{2.13}$$

Introduce the simplifying notation

$$\max_{0\leq s\leq t} \tilde{x}(s) \equiv \left(\max_{0\leq s\leq t} |x_1(s)|, \max_{0\leq s\leq t} |x_2(s)|, \ldots, \max_{0\leq s\leq t} |x_n(s)| \right)^T.$$

In addition to (2.11), we choose T large enough that all the arguments of $\tilde{x}(\cdot)$ in the integrals in (2.13) are positive. Then we can estimate all $\tilde{x}(\cdot)$ by $\max_{0\leq s\leq t} \tilde{x}(s)$, therefore we obtain from (2.13)

$$\tilde{f}(t) \leq \left(\sum_{i=0}^{m} |\eta_i(t)| \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq s\leq t} \tilde{x}(s), \qquad t \geq T. \tag{2.14}$$

Define the matrix

$$M_0 \equiv \int_0^\infty \tilde{V}(s)\, ds \left(\sum_{i=0}^{m} \tilde{A}_i \right)^2. \tag{2.15}$$

(We note that according to Remark 2.1, the matrices M and M_0 are independent of the choice of T.) It is easy to see that $\rho(M) < 1$ implies that there exists $\delta > 0$ such that

$$\rho(M + \delta M_0) < 1. \tag{2.16}$$

With this δ we can choose T such that (2.14) holds and furthermore, we have the following relations

$$|\eta_i(t)| < \overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta, \qquad t \geq T, \quad i = 0, \dots, m. \tag{2.17}$$

Then (2.14) yields the following estimate

$$\tilde{f}(t) \leq \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq s\leq t} \tilde{x}(s), \qquad t \geq T. \tag{2.18}$$

(ii) Next we prove that the solution of (2.1) is bounded for all initial functions. Choose $T > 0$ such that (2.18) holds. For such T, formula (2.9) and standard estimates yield the inequality

$$\tilde{x}(t) \leq \tilde{y}(t) + \int_T^t \tilde{V}(t-s)\, \tilde{f}(s)\, ds, \qquad t \geq T. \tag{2.19}$$

Combining (2.18) and (2.19) we get

$$\begin{aligned} \tilde{x}(t) &\leq \tilde{y}(t) + \int_T^t \tilde{V}(t-s) \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq u\leq s} \tilde{x}(u)\, ds \\ &\leq \tilde{y}(t) + \int_T^t \tilde{V}(t-s)\, ds \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq u\leq t} \tilde{x}(u). \end{aligned}$$

A change of variables gives the inequality

$$\begin{aligned} \tilde{x}(t) &\leq \tilde{y}(t) + \int_0^{t-T} \tilde{V}(s)\, ds \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq u\leq t} \tilde{x}(u) \\ &\leq \tilde{y}(t) + \int_0^{\infty} \tilde{V}(s)\, ds \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq u\leq t} \tilde{x}(u). \end{aligned}$$

Using the definition of M and M_0, we have

$$\begin{aligned} \tilde{x}(t) &\leq \tilde{y}(t) + (M + \delta M_0) \max_{0\leq u\leq t} \tilde{x}(u) \\ &\leq \max_{0\leq u\leq t} \tilde{y}(u) + (M + \delta M_0) \max_{0\leq u\leq t} \tilde{x}(u). \end{aligned} \tag{2.20}$$

The right hand side of inequality (2.20) is monotone in t, therefore (2.20) yields that

$$\max_{0\leq u\leq t} \tilde{x}(u) \leq \max_{0\leq u\leq t} \tilde{y}(u) + (M + \delta M_0) \max_{0\leq u\leq t} \tilde{x}(u). \tag{2.21}$$

Rearranging (2.21) and using that $y(t)$ is bounded by hypothesis (H), we have that there exists a constant vector $z \geq 0$ such that

$$(I - (M + \delta M_0)) \max_{0\leq u\leq t} \tilde{x}(u) \leq \max_{0\leq u\leq t} \tilde{y}(u) \leq z, \qquad t \geq T. \tag{2.22}$$

Inequality (2.16) and the fact that $M + \delta M_0$ has nonnegative components imply that $I - (M + \delta M_0)$ is a nonsingular M-matrix, therefore an application of Theorem 6.2.3 in [3] yields that $I - (M + \delta M_0)$ is a monotone matrix, hence

$$\max_{0\le u\le t} \tilde{x}(u) \le (I - (M + \delta M_0))^{-1} z, \qquad t \ge T,$$

i.e., $x(t)$ is bounded for $t \ge 0$.

(iii) Next we show that $x(t)$ tends to 0 as $t \to \infty$, i.e., $\overline{\lim}_{t\to\infty} \tilde{x}(t) = 0$. Inequality (2.19) yields

$$\overline{\lim_{t\to\infty}}\, \tilde{x}(t) \le \overline{\lim_{t\to\infty}}\, \tilde{y}(t) + \overline{\lim_{t\to\infty}} \int_T^t \tilde{V}(t-s)\, \tilde{f}(s)\, ds.$$

By (H) we have $\overline{\lim}_{t\to\infty} \tilde{y}(t) = 0$, hence

$$\overline{\lim_{t\to\infty}}\, \tilde{x}(t) \le \overline{\lim_{t\to\infty}} \int_T^t \tilde{V}(t-s)\, \tilde{f}(s)\, ds. \tag{2.23}$$

For any $\delta > 0$ we can choose T such that (2.17) is satisfied and moreover, in (2.13) all arguments of $\tilde{x}(\cdot)$ in the integrals are large enough, i.e., we can estimate $\tilde{x}(\cdot)$ by $\overline{\lim}_{t\to\infty} \tilde{x}(t) + \delta \underline{1}$, where $\underline{1} = (1, 1, \ldots, 1)^T$, and consequently, for $t \ge T$, relation (2.13) implies the inequality

$$\tilde{f}(t) \le \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) (\overline{\lim_{u\to\infty}}\, \tilde{x}(u) + \delta \underline{1}). \tag{2.24}$$

Combining (2.23) and (2.24) we have

$$\begin{aligned}
\overline{\lim_{t\to\infty}}\, \tilde{x}(t) &\le \overline{\lim_{t\to\infty}} \int_T^t \tilde{V}(t-s)\, ds \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) (\overline{\lim_{u\to\infty}}\, \tilde{x}(u) + \delta \underline{1}) \\
&\le \int_0^\infty \tilde{V}(s)\, ds \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) (\overline{\lim_{t\to\infty}}\, \tilde{x}(t) + \delta \underline{1}).
\end{aligned} \tag{2.25}$$

Since (2.25) holds for arbitrary δ, we have

$$\overline{\lim_{t\to\infty}}\, \tilde{x}(t) \le M\, \overline{\lim_{t\to\infty}}\, \tilde{x}(t). \tag{2.26}$$

Hence

$$(I - M)\, \overline{\lim_{t\to\infty}}\, \tilde{x}(t) \le 0. \tag{2.27}$$

By assumption $\rho(M) < 1$, M has nonnegative components, and therefore $I - M$ is a nonsingular M-matrix, therefore by Theorem 6.2.3 in [3] $I - M$ is monotone, hence (2.27) yields that $\overline{\lim}_{t\to\infty} \tilde{x}(t) \le 0$. On the other hand $\underline{\lim}_{t\to\infty} \tilde{x}(t) \ge 0$, therefore $\lim_{t\to\infty} \tilde{x}(t) = 0$.
The proof of the theorem is complete.

The following corollary is an easy consequence of the theorem.

Corollary 2.4 *Let M_0 be defined by* (2.15). *If*

$$\overline{\lim_{t\to\infty}}|\eta_i(t)| < \frac{1}{\rho(M_0)}, \qquad i = 0,\dots,m,$$

then the trivial solution of (2.1) *is asymptotically stable.*

If the fundamental solution $V(t)$ of (2.4) is nonnegative, (i.e., each component $v_{ij}(t)$ of $V(t)$ is nonnegative and therefore $V(t) = \tilde{V}(t)$), then it is easy to compute the integral in (2.15). In particular, we have the following result.

Proposition 2.5 *If the trivial solution of* (2.4) *is asymptotically stable, then the fundamental solution of* (2.4) *satisfies*

$$\left(\sum_{i=0}^{m} A_i\right) \int_0^\infty V(s)\,ds = -I,$$

where I is the identity matrix.

Proof: Let $V(t)$ be the fundamental solution of (2.4) corresponding to $T = 0$. By integrating (2.5) from 0 to $t > 0$ we get

$$V(t) - V(0) = \sum_{i=0}^{m} A_i \int_0^t V(s - r_i)\,ds.$$

A change of variables in the integrals and the assumed initial condition $V(t) = 0$ for $t < 0$ yield

$$\begin{aligned} V(t) - V(0) &= \sum_{i=0}^{m} A_i \int_{-r_i}^{t-r_i} V(s)\,ds \\ &= \sum_{i=0}^{m} A_i \int_0^{t-r_i} V(s)\,ds. \end{aligned}$$

Using $V(0) = I$ and the fact $V(t) \to 0$ as $t \to \infty$ we obtain the equality

$$-I = \left(\sum_{i=0}^{m} A_i\right) \int_0^\infty V(s)\,ds,$$

which proves the proposition.

Remark 2.6 *In the case when $V(t)$ is nonnegative, and $\sum_{i=0}^m A_i$ is nonsingular, Proposition 2.5 implies that*

$$M_0 = -\left(\sum_{i=0}^{m} A_i\right)^{-1} \left(\sum_{i=0}^{m} \tilde{A}_i\right)^2, \tag{2.28}$$

therefore our stability condition in Corollary 2.4 is given in terms of the coefficient matrices.

To conclude the section in the next Proposition we give a sufficient condition for positivity of the fundamental solution of (2.4). We shall need the following notations. Let $A_i = \left[a_{jk}^{(i)}\right]$, $V(t) = \left[v_{jk}(t)\right]$, $\alpha_{jj}^{(i)} \equiv \max\left\{-a_{jj}^{(i)}, 0\right\}$, $\beta_{jj}^{(i)} \equiv \max\left\{a_{jj}^{(i)}, 0\right\}$. Then we can rewrite initial value problem (2.5)-(2.6) in terms of the components:

$$\dot{v}_{jk}(t) = -\sum_{i=0}^{m} \alpha_{jj}^{(i)} v_{jk}(t-r_i) + \sum_{i=0}^{m} \sum_{\substack{l=1 \\ l \neq j}}^{n} a_{jl}^{(i)} v_{lk}(t-r_i) + \sum_{i=0}^{m} \beta_{jj}^{(i)} v_{jk}(t-r_i), \quad t \geq 0, \tag{2.29}$$

$$v_{jk}(t) = \begin{cases} \delta_{jk}, & t = 0, \\ 0, & t < 0, \end{cases} \tag{2.30}$$

(where δ_{jk} is the Kronecker-delta), $j, k = 1, 2, \ldots, n$. Consider the following two initial value problems associated to the negative parts of the components in the main diagonals of A_i, i.e., to the "homogeneous part" of (2.29):

$$\dot{w}_{jk}(t) = -\sum_{i=0}^{m} \alpha_{jj}^{(i)} w_{jk}(t-r_i), \qquad t \geq 0, \tag{2.31}$$

$$w_{jk}(t) = \begin{cases} \delta_{jk}, & t = 0, \\ 0, & t < 0, \end{cases} \tag{2.32}$$

and

$$\dot{u}_j(t) = -\sum_{i=0}^{m} \alpha_{jj}^{(i)} u_j(t-r_i), \qquad t \geq 0, \tag{2.33}$$

$$u_j(t) = \begin{cases} 1, & t = 0, \\ 0, & t < 0, \end{cases} \tag{2.34}$$

$j, k = 1, 2, \ldots, n$. Clearly, we have that for all $t \geq 0$

$$w_{jk}(t) = \begin{cases} 0, & j \neq k, \\ u_j(t), & j = k. \end{cases} \tag{2.35}$$

Proposition 2.7 *Assume that*

(i) $a_{jk}^{(i)} \geq 0$ *for all* $j, k = 1, 2, \ldots, n$, $j \neq k$.

(ii) $\sum_{i=0}^{m} \alpha_{jj}^{(i)} r_i \leq \frac{1}{e}$ *for all* $j = 1, 2, \ldots, n$.

Then $v_{jk}(t) \geq 0$ *for all* $t \geq 0$ *and* $j, k = 1, 2, \ldots, n$.

Proof: Let $w_{jk}(t)$ and $u_j(t)$ be the solutions of initial value problems (2.31)-(2.32) and (2.33)-(2.34), respectively, $(j, k = 1, 2, \ldots, n)$. By Theorem 3.31 in [9] it follows that $u_j(t) \geq 0$ for all $j = 1, 2, \ldots, n$. (The above theorem applies for solutions corresponding to continuous initial functions. To use that result for IVP (2.33)-(2.34) we approximate the initial function in (2.34) by appropriate continuous initial functions, $u_j^l(t)$, $t \leq 0$,

$l = 1, 2, \ldots$, and by arguing that the corresponding solutions $u_j^l(t)$, $t \geq 0$ approximate $u_j(t)$ uniformly on compact time intervals we get that the limit $u_j(t) = \lim_{l\to\infty} u_j^l(t)$, is also nonnegative.) Nonnegativeness of $u_j(t)$ and relation (2.35) yield that $w_{jk}(t) \geq 0$ for $t \geq 0$, $j, k = 1, 2, \ldots, n$ as well. The variation-of-constant formula implies the relation

$$v_{jk}(t) = w_{jk}(t) + \sum_{i=0}^{m} \sum_{\substack{l=1 \\ l\neq j}}^{n} a_{jl}^{(i)} \int_0^t u_j(t-s) v_{lk}(s-r_i)\, ds + \sum_{i=0}^{m} \beta_{jj}^{(i)} \int_0^t u_j(t-s) v_{jk}(s-r_i)\, ds.$$

Using the nonnegativeness of $w_{jk}(t)$, $u_j(t)$, $\beta_{jj}^{(i)}$, $a_{jl}^{(i)}$ $(l \neq j)$, and (2.30) it is easy to see the nonnegativeness of $v_{jk}(t)$.

Note that in the ODE case, i.e., when $m = 0$, $r_0 = 0$, condition (ii) of the previous proposition is satisfied automatically, and then condition (i) is also necessary for the positivity of $v_{jk}(t)$. (See Theorem 3 in Chapter 10 of [2].)

3 Examples and Applications

Consider the scalar version of (2.1).

$$\dot{x}(t) = \sum_{i=0}^{m} a_i x(t - r_i - \eta_i(t)), \qquad t \geq 0 \tag{3.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad -r \leq t \leq 0, \tag{3.2}$$

where $\varphi : [-r, 0] \to \mathbb{R}$ is a continuous function. The corresponding equation with unperturbed delays is

$$\dot{y}(t) = \sum_{i=0}^{m} a_i y(t - r_i), \qquad t \geq 0. \tag{3.3}$$

Let $v(t)$ be the fundamental solution of (3.3), i.e.

$$\dot{v}(t) = \sum_{i=0}^{m} a_i v(t - r_i), \qquad t \geq 0 \tag{3.4}$$

$$v(t) = \begin{cases} 1, & t = 0, \\ 0, & t < 0. \end{cases} \tag{3.5}$$

The scalar version of Theorem 2.3 is the following.

Theorem 3.1 *Assume that the trivial solution of* (3.3) *is asymptotically stable and the functions* $\eta_i(\cdot)$ $(i = 0, \ldots, m)$ *satisfy*

$$\sum_{i=0}^{m} |a_i| \overline{\lim_{t\to\infty}} |\eta_i(t)| < \frac{1}{\left(\sum_{i=0}^{m} |a_i|\right) \int_0^\infty |v(t)|\, dt}. \tag{3.6}$$

Then the trivial solution of (3.1) *is asymptotically stable.*

Note that if the fundamental solution is nonnegative, then Remark 2.6 yields that condition (3.6) is equivalent to

$$\sum_{i=0}^{m} |a_i| \overline{\lim_{t\to\infty}} |\eta_i(t)| < \frac{-\sum_{i=0}^{m} a_i}{\sum_{i=0}^{m} |a_i|}. \tag{3.7}$$

In the general case we would need an upper estimate of $\int_0^\infty |v(t)|\,dt$ to get an easily verifiable condition on the allowable perturbation. Such an estimate at this time is known (see [8]) only for the single-delay equation of the form

$$\dot{x}(t) = -bx(t-\tau), \tag{3.8}$$

where $b > 0$ and $b\tau < \pi/2$ (hence the trivial solution is asymptotically stable). For this equation it can be shown (see [8]) that there exists a unique characteristic root $\lambda_0 = \alpha_0 + \beta_0 i$ of equation (3.8) , i.e., a solution of $\lambda = -be^{-\lambda\tau}$, satisfying $\beta_0 \in [0, \frac{\pi}{2\tau})$. Then the fundamental solution of (3.8) satisfies

$$\int_0^\infty |v(t)|\,dt \le \frac{1}{b}\frac{\alpha_0^2+\beta_0^2}{\alpha_0^2}. \tag{3.9}$$

In the general case, the practical importance of our result can be argued as follows: i) it is easy to obtain numerical approximation of the fundamental solution, ii) using the fact that the fundamental solution exponentially converges to 0 if the trivial solution is asymptotically stable, it is easy to obtain good numerical approximation of the integral $\int_0^\infty |v(t)|\,dt$, and iii) using the numerical value of the integral and condition (3.6) get approximate bounds for the allowable perturbations.

The following examples show applications of this method.

Example 3.2 Consider the scalar equation

$$\dot{x}(t) = -x(t - r_0 - \eta(t)). \tag{3.10}$$

We know, (see e.g. [11]), that the trivial solution of

$$\dot{y}(t) = -y(t-r_0) \tag{3.11}$$

is asymptotically stable if and only if $0 \le r_0 < \pi/2$. Also we know, (see e.g. [8]), that the fundamental solution of (3.11) is positive if $0 \le r_0 \le 1/e$, and oscillates if $1/e < r_0 < \pi/2$. Therefore if we pick e.g. $r_0 = 0.3$, (i.e., $r_0 < 1/e$), then an application of Proposition 2.5 yields that $\int_0^\infty |v(s)|\,ds = 1$. Therefore by using our condition in Theorem 3.1 we have that if $\overline{\lim_{t\to\infty}} |\eta(t)| < 1$, then the trivial solution of (3.10) is asymptotically stable. On the other hand, if we pick, e.g., $r_0 = 1$, then we obtain $\alpha_0 + \beta_0 i = -0.3181 + 1.3372i$ as the numerical value of the characteristic root of (3.11) satisfying $\beta_0 \in [0, \frac{\pi}{2})$. Inequality (3.9) yields the estimate $\int_0^\infty |v(s)|\,ds \le 18.6687$, and therefore if $\overline{\lim_{t\to\infty}} |\eta(t)| \le 1/18.6687 = 0.0536$, then the trivial solution of (3.10) is asymptotically stable. By numerical integration we get $\int_0^\infty |v(s)|\,ds = 2.9302$, hence the allowable perturbation is $\overline{\lim_{t\to\infty}} |\eta(t)| \le 0.3413$ by Theorem

3.1, which is much better than that of obtained by using estimate (3.9). Figure 1 and 2 contain the graph of the fundamental solution of (3.10) corresponding to $r_0 = 1$, and $\eta(t) = 0$ and $\eta(t) = \frac{4}{t+1} + 0.3$, respectively.

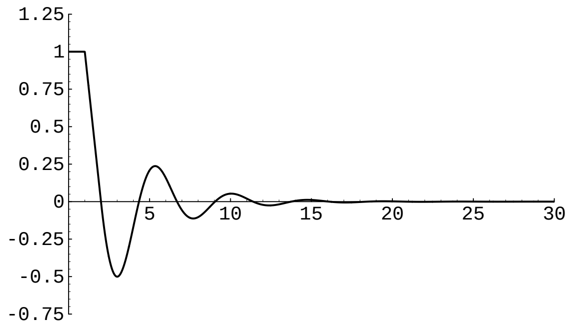

Figure 1

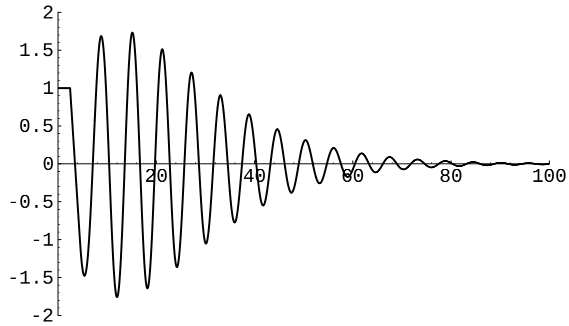

Figure 2

Example 3.3 Consider the one-dimensional control system

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) + Ku(t). \tag{3.12}$$

One can check that for $K = 0$ the trivial solution of (3.12) is unstable. Let $K = -2$ and $u(t) = x(t - 1.3)$ in (3.12). Numerical approximation of the fundamental solution of the corresponding equation

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x(t-1.3). \tag{3.13}$$

is shown on Figure 3. This picture indicates that the fundamental solution exponentially tends to zero, i.e., the trivial solution of (3.13) is asymptotically stable. Therefore the feedback law $Ku(t) = -2x(t-1.3)$ stabilizes (3.12). The term $t - 1.3$ represents a time delay in the control mechanism. Suppose that we sample the system only at the points $h, 2h, 3h, \ldots$, and use a piecewise constant feedback control $u(t) = x([(t-1.3)/h]h)$ instead of $u_h(t) = x(t-1.3)$. Here $[\cdot]$ denotes the greatest integer function and $h > 0$ is the sampling period. The question we are interested in is to find a bound on the sampling period h, which guarantees that the trivial solution of the resulting hybrid feedback system

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x\left(\left[\frac{t-1.3}{h}\right]h\right). \tag{3.14}$$

remains asymptotically stable. The piecewise constant delay in the last term in (3.14) can be considered as a perturbation of $t - 1.3$ in (3.13) with

$$\eta(t) = t - 1.3 - \left[\frac{t-1.3}{h}\right]h.$$

Then we have that $|\eta(t)| \le h$ for all $t \ge 0$. Numerical approximation gives that the fundamental solution of (3.13) satisfies $\int_0^\infty |v(t)|\,dt = 10.5914$. Therefore by Theorem 3.1

we have as a sufficient condition that $h < \frac{1}{10.5914 \cdot 8.2} = 0.0115$ guarantees that the trivial solution of (3.14) is asymptotically stable.

Example 3.4 Consider system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - 1 - \eta_1(t)) + A_2 x(t - 1.4 - \eta_2(t)), \tag{3.15}$$

where

$$A_0 = \begin{pmatrix} -0.1 & 0.2 \\ 0.0 & -0.3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.0 & 0.1 \\ 0.0 & -0.2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -0.2 & 0.0 \\ 0.2 & 0.0 \end{pmatrix}.$$

The corresponding unperturbed equation is

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_2 x(t - 1.4). \tag{3.16}$$

On Figure 4 we display the components of the numerical solutions of the fundamental matrix solution of (3.16). By Proposition 2.7 the fundamental solution is nonnegative, and Figure 4 shows that each components of it tends to zero exponentially as $t \to \infty$, i.e., the trivial solution of (3.16) is asymptotically stable. Nonnegativeness of the components of $V(\cdot)$ and Remark 2.6 yield that

$$M_0 = \begin{pmatrix} 0.778 & 1.144 \\ 0.511 & 0.778 \end{pmatrix},$$

hence $\rho(M_0) = 1.543$. By using Corollary 2.4, if the perturbations of the delays satisfy $\overline{\lim}_{t\to\infty} |\eta_i(t)| < 0.648$ $(i = 1, 2)$, then the trivial solution of (3.15) is asymptotically stable.

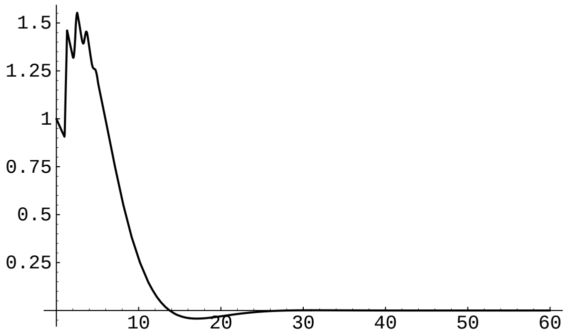

Figure 3

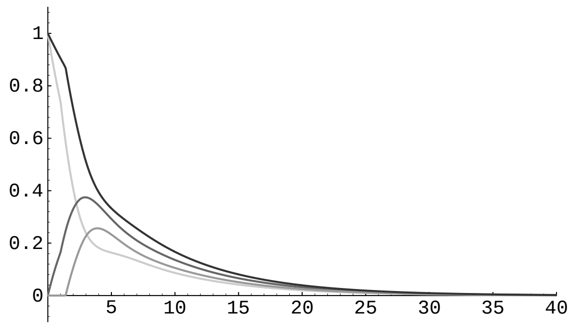

Figure 4

Example 3.5 Consider the following system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - 1 - \eta_1(t)) + A_2 x(t - 1.5 - \eta_2(t)), \tag{3.17}$$

where $x(t) \in \mathbb{R}^2$,

$$A_0 = \begin{pmatrix} -0.1 & 0.3 \\ -0.5 & 0.0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.7 & -0.4 \\ 0.5 & -0.8 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -1.0 & 0.1 \\ 0.1 & 0.4 \end{pmatrix}.$$

The corresponding unperturbed equation is

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_2 x(t-1.5). \tag{3.18}$$

On Figure 5 we display the components of the numerical solutions of the fundamental matrix solution. This picture indicates that every component function tends to zero exponentially as $t \to \infty$, therefore the trivial solution of (3.18) is asymptotically stable. Numerical approximation of the components of $\int_0^\infty \tilde{V}(t)\,dt$ gives the following numerical values for the matrix M_0

$$M_0 = \begin{pmatrix} 18.699 & 10.800 \\ 16.441 & 10.641 \end{pmatrix},$$

therefore $\rho(M_0) = 28.591$. Corollary 2.4 implies that if the perturbations of the delays satisfy $\overline{\lim_{t\to\infty}} |\eta_i(t)| < 0.035$ $(i = 1, 2)$, then the trivial solution of (3.17) is asymptotically stable. On Figure 6 we present the components of the solution of (3.17) corresponding to $\eta_1(t) = 0$ and $\eta_2(t) = \frac{5}{t+1} + 0.03$ and the initial condition $x(0) = I$, and $x(t) = 0$, $t < 0$.

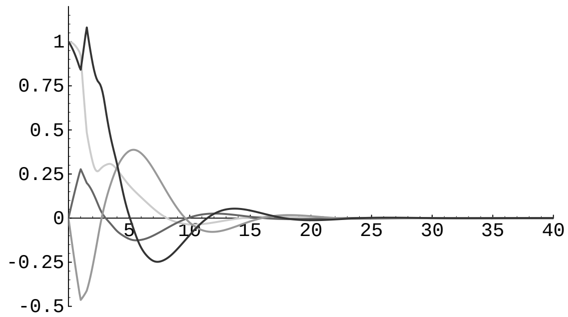

Figure 5

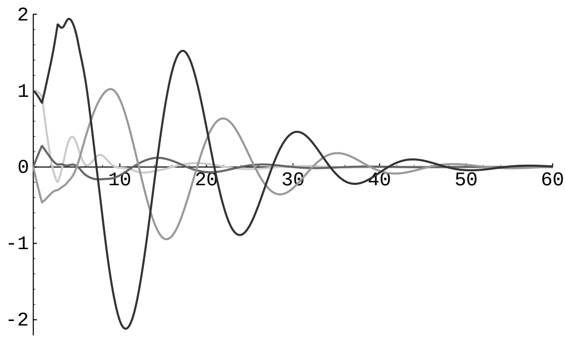

Figure 6

4 Stability results

In this section we show an application of our perturbation theorem to obtain stability conditions for vector and scalar linear delay equations with multiple delays.

Consider first a vector equation of the form

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - \sigma_i(t)), \qquad t \geq 0, \tag{4.1}$$

and the corresponding ordinary differential equation

$$\dot{y}(t) = \left(\sum_{i=0}^{m} A_i \right) y(t), \qquad t \geq 0, \tag{4.2}$$

where $x(\cdot), y(\cdot) \in \mathbb{R}^n$, and $\sigma_i(t) \geq 0$ are piecewise continuous, bounded functions. We can think of $\sigma_i(t)$ $(i = 0, 1, \ldots, m)$ in (4.1) as perturbations of the (zero) delays in (4.2). Recall that if all the eigenvalues of the matrix $\sum_{i=0}^m A_i$ have negative real parts, then the trivial solution of (4.2) is asymptotically stable. Let $A_i = \left(a_{jk}^{(i)}\right)$. By Theorem 3 in Chapter 10 of [2] it follows that the fundamental solution of (4.2) is positive if and only if $\sum_{i=0}^m a_{jk}^{(i)} \geq 0$, for $j,k = 1,2,\ldots,n$, $j \neq k$. Assuming the asymptotic stability and the positiveness of the fundamental solution of (4.2) we can apply Theorem 2.3 and Proposition 2.5 and get that the trivial solution of (4.1) is asymptotically stable, if the matrix

$$M \equiv -\left(\sum_{i=0}^m A_i\right)^{-1}\left(\sum_{i=0}^m \overline{\lim_{t\to\infty}}\,\sigma_i(t)\tilde{A}_i\right)\left(\sum_{i=0}^m \tilde{A}_i\right)$$

has spectral radius less than 1. In the sequel we shall give condition yielding that $\|M\| < 1$, which clearly implies that $\rho(M) < 1$. Here $\|\cdot\|$ is the matrix norm generated by either the $\|\cdot\|_1$ or the $\|\cdot\|_\infty$ vector norm. Note that condition $\|M\| < 1$ is satisfied if we require that

$$\sum_{i=0}^m \overline{\lim_{t\to\infty}}\,\sigma_i(t)\|A_i\| < \frac{1}{\|\left(\sum_{i=0}^m A_i\right)^{-1}\|\cdot\|\sum_{i=0}^m \tilde{A}_i\|}$$

be satisfied. Here we used that $\|A_i\| = \|\tilde{A}_i\|$, and

$$\begin{aligned}\|M\| &\leq \left\|-\left(\sum_{i=0}^m A_i\right)^{-1}\right\|\left\|\sum_{i=0}^m \overline{\lim_{t\to\infty}}\,\sigma_i(t)\tilde{A}_i\right\|\left\|\sum_{i=0}^m \tilde{A}_i\right\| \\ &\leq \left\|\left(\sum_{i=0}^m A_i\right)^{-1}\right\|\left(\sum_{i=0}^m \overline{\lim_{t\to\infty}}\,\sigma_i(t)\|A_i\|\right)\left\|\sum_{i=0}^m \tilde{A}_i\right\|,\end{aligned}$$

and have proved the following proposition.

Proposition 4.1 *Assume that*

(i) the matrix $\sum_{i=0}^m A_i$ *has eigenvalues only with negative real parts,*

(ii) $\sum_{i=0}^m a_{jk}^{(i)} \geq 0$ *for* $j,k = 1,2,\ldots,n$, $j \neq k$, *and*

(iii) $\displaystyle\sum_{i=0}^m \overline{\lim_{t\to\infty}}\,\sigma_i(t)\|A_i\| < \frac{1}{\|\left(\sum_{i=0}^m A_i\right)^{-1}\|\cdot\|\sum_{i=0}^m \tilde{A}_i\|}$,

then the trivial solution of (4.1) *is asymptotically stable.*

Next we consider the scalar linear delay equations with multiple delays of the form

$$\dot{x}(t) = -\sum_{i=0}^m a_i x(t - \sigma_i(t)), \qquad t \geq 0, \tag{4.3}$$

and the corresponding equation

$$\dot{y}(t) = -\left(\sum_{i=0}^{m} a_i\right) y(t), \qquad t \geq 0. \tag{4.4}$$

The scalar version of Proposition 4.1 can be stated as follows:

Proposition 4.2 *Assume that*

(i) $\sum_{i=0}^{m} a_i > 0$, *and*

(ii) $\sum_{i=0}^{m} |a_i| \overline{\lim_{t\to\infty}} \sigma_i(t) < \dfrac{\sum_{i=0}^{m} a_i}{\sum_{i=0}^{m} |a_i|}$,

then the trivial solution of (4.3) *is asymptotically stable.*

For the case when each $a_i > 0$ we have the following result.

Corollary 4.3 *Assume that* $a_i > 0$ *for* $i = 0, 1, \ldots, m$. *Then, if*

$$\sum_{i=0}^{m} a_i \overline{\lim_{t\to\infty}} \sigma_i(t) < 1, \tag{4.5}$$

then the trivial solution of (4.3) *is asymptotically stable.*

In the rest of this section we assume that $a_i > 0$ for all $i = 0, 1, \ldots, m$. In this special case, by imposing additional assumptions, we can obtain larger bound for the "average delay" in (4.5) which guarantees the asymptotic stability of the trivial solution of (4.3).

Rewrite (4.3) in the form

$$\dot{x}(t) = -\sum_{i=0}^{m} a_i x(t - \tau - (\sigma_i(t) - \tau)), \qquad t \geq 0, \tag{4.6}$$

and consider the equation

$$\dot{y}(t) = -\left(\sum_{i=0}^{m} a_i\right) y(t - \tau), \qquad t \geq 0. \tag{4.7}$$

Equation (4.7) is a single delay equation, which is asymptotically stable if and only if $\sum_{i=0}^{m} a_i > 0$ and $\tau \sum_{i=0}^{m} a_i < \pi/2$. We have assumed that each $a_i \geq 0$ therefore the first condition is satisfied, and let

$$\tau = \frac{\alpha}{e \sum_{i=0}^{m} a_i},$$

where $0 \leq \alpha \leq 1$. With this choice of τ equation (4.7) is asymptotically stable, and moreover, the fundamental solution of (4.7) is positive. We consider equation (4.6) as an equation obtained by perturbing the delay τ in (4.7) with $\eta_i(t) = \sigma_i(t) - \tau$. By Theorem

3.1 and the discussion after the theorem, the trivial solution of (4.6) (therefore the trivial solution of (4.3) as well) is asymptotically stable if (3.7) holds. Using the nonnegativeness of each a_i, and that (4.6) has the form (3.1) with a_i replaced by $-a_i$, we get that for our equation this condition is equivalent to

$$\sum_{i=0}^{m} a_i \overline{\lim_{t\to\infty}} \left| \sigma_i(t) - \frac{\alpha}{e\sum_{j=0}^{m} a_j} \right| < 1. \tag{4.8}$$

To further simplify this condition we consider special cases. It is easy to see that

$$\overline{\lim_{t\to\infty}} |\sigma_i(t) - \tau| = \begin{cases} \overline{\lim_{t\to\infty}} \sigma_i(t) - \tau, & \text{if } \sigma_i(t) \geq \tau \text{ for } t \geq T, \\ \tau - \underline{\lim_{t\to\infty}} \sigma_i(t), & \text{if } \sigma_i(t) \leq \tau \text{ for } t \geq T. \end{cases}$$

First assume that we can select $0 \leq \alpha \leq 1$ such that for some $T \geq 0$ the delays satisfy

$$\sigma_i(t) \geq \frac{\alpha}{e\sum_{j=0}^{m} a_j}, \qquad t \geq T, \quad i = 0, 1, \ldots, m. \tag{4.9}$$

Then condition (4.8) can be rewritten as

$$\sum_{i=0}^{m} a_i \overline{\lim_{t\to\infty}} \sigma_i(t) < 1 + \frac{\alpha}{e}. \tag{4.10}$$

Note that $\alpha = 0$ satisfies (4.9), therefore we can always use condition (4.10) with $\alpha = 0$, and we get the same condition as in Corollary 4.3. On the other hand, if $\underline{\lim_{t\to\infty}} \sigma_i(t) > 0$ for all $i = 0, 1, \ldots, m$, then there exists a positive α satisfying (4.9), and we get a larger bound in (4.10) than that in Corollary 4.3.

Next consider the case when there exists $0 < \alpha \leq 1$ such that for some $T \geq 0$

$$\sigma_i(t) \leq \frac{\alpha}{e\sum_{j=0}^{m} a_j}, \qquad t \geq T, \quad i = 0, 1, \ldots, m.$$

Then we also have that

$$\sigma_i(t) \leq \frac{1}{e\sum_{j=0}^{m} a_j}, \qquad t \geq T, \quad i = 0, 1, \ldots, m,$$

and it is easy to see that (4.8) is always satisfied.

We summarize our results in the next proposition.

Proposition 4.4 *Assume that $a_i \geq 0$, $i = 0, 1, \ldots, m$. Then either one of the following two conditions is sufficient for the asymptotic stability of the trivial solution of* (4.3).

(i) There exist $T \geq 0$ and $0 \leq \alpha \leq 1$ such that

(a) $\sigma_i(t) \geq \frac{\alpha}{e\sum_{j=0}^{m} a_j}, \qquad t > T, \quad i = 0, 1, \ldots, m,$ *and*

(b) $\sum_{i=0}^{m} a_i \overline{\lim}_{t\to\infty} \sigma_i(t) < 1 + \frac{\alpha}{e}$.

(ii) There exists $T \geq 0$ *such that* $\sigma_i(t) \leq \frac{1}{e\sum_{j=0}^{m} a_j}, \qquad t > T, \quad i = 0, 1, \ldots, m.$

To illustrate condition (i) in the previous proposition, consider the special case when $\lim_{t\to\infty} \sigma_i(t) = \sigma$, for $i = 0, 1, \ldots, m$. In this case condition

$$\frac{\alpha}{e} < \sigma \sum_{i=0}^{m} a_i < 1 + \frac{\alpha}{e}$$

implies condition (i).

We refer to [12], [17], [18], [19], [20], and the references therein for similar stability conditions for linear delay equations with multiple delays. Finally, we note that in all the above references the supremums of the time-delays are used, while our conditions are formulated in terms of the limit superiors of the time-delays.

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