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Identification of parameters in delay equations with state-dependent delays

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Abstract

In this paper we study parameter identification problems for a class of nonlinear delay equations with distributed state-dependent delays. We describe a numerical identification technique using an Euler-type approximation scheme and show its theoretical convergence. Illustrative numerical examples are also included.

1 Introduction

In this paper we study parameter identification problems for the nonlinear delay system with state dependent delays

$$\dot{x}(t) = f\left(t, x(t), \int_{-r}^{0} d_s\mu(s,t,x_t,\sigma)\, x(t+s), \theta\right), \qquad t \in [0,T] \tag{1.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r,0]. \tag{1.2}$$

The term

$$\Lambda(t,x_t,\sigma) \equiv \int_{-r}^{0} d_s\mu(s,t,x_t,\sigma)x(t+s) \tag{1.3}$$

describing the delay dependence is a Stieltjes-integral of the solution segment $x(t+\cdot)$ with respect to $\mu(\cdot,t,x_t,\sigma)$, which is a matrix valued function of bounded variations depending on time, t, the state of the equation, x_t, and a parameter $\sigma \in \Sigma$. Here $0 < T < \infty$ and $r > 0$ are fixed, and $x_t \;:\; [-r,0] \to \mathbf{R}^n$, $x_t(s) \equiv x(t+s)$. For simplicity of the presentation we assume that only part of the delay term and the function f, represented by the parameters $\sigma \in \Sigma$

and $\theta \in \Theta$, respectively, and the initial function, φ, are unknowns in the equation. Σ and Θ are normed linear spaces with norms $|\cdot|_\Sigma$ and $|\cdot|_\Theta$, respectively.

To give some motivation and/or justification on the particular form selected by (1.3) for the delay term, assume for example that the delayed term depends linearly on the state, i.e., has the form Lx_t, where L is a bounded linear operator on $C \equiv C([-r,0],\mathbf{R}^n)$. In this case the Riesz Representation Theorem yields (1.3) with $\mu = \mu(s)$. If $L = L(t,\sigma)$ depends on t and a parameter σ, then by the same result we get that there exists $\mu = \mu(s,t,\sigma)$ such that (1.3) holds. Therefore it seems like a natural extension of the above cases to assume the structure described by (1.3) for the state-dependent case. Moreover, representation (1.3) includes discrete and distributed, constant and time-dependent delays, and the "usual" state-dependent delays, $x(t-\tau(t,x(t),\sigma))$ or $x(t-\tau(t,x_t,\sigma))$ as well. A nice feature of this form is that it also allows delayed terms of the form

$$\Lambda(t,x_t,\sigma) = \sum_{i=1}^{\infty} A_i(t,x_t,\sigma)x(t-\tau_i(t,x_t,\sigma)) + \int_{-\tau_0}^{0} G(s,t,x_t,\sigma)x(t+s)\,ds.$$

We assume that some parameters ($\gamma \in \Gamma$) of the initial value problem (IVP) (1.1)-(1.2) are unknown, but we have measurements $(X_0, X_1, \ldots, X_l)$ at discrete time values $(t_0, t_1, \ldots, t_l)$ for the solution of the IVP. The goal is to find the parameter value, which minimizes the least squares fit-to-data criterion

$$J(\gamma) = \sum_{i=0}^{l} |x(t_i;\gamma) - X_i|^2, \qquad \gamma \in \Delta,$$

i.e., which is the best-fit parameter for the measurements. (Denote this problem by $\mathcal{P}$). Here $\Delta \subset \Gamma$ is the admissible set of parameters, Γ is the parameter space. Problem $\mathcal{P}$ has been studied by many authors, for different classes of differential equations (see e.g. [1] and the references therein), including delay equations as well ([2, 3, 4]).

All the above cited papers use the same idea to find the solution of the optimization problem $\mathcal{P}$:

Step 1) First take finite dimensional approximations of the parameters, γ^N, (i.e., $\gamma^N \in \Gamma^N \subset \Gamma$, $\dim \Gamma^N < \infty$, $\gamma^N \to \gamma$ as $N \to \infty$).

Step 2) Consider a sequence of approximate initial value problems ($\mathrm{IVP}_{M,N}$) corresponding to a discretization of IVP (1.1)-(1.2) for some fixed parameter $\gamma^N \in \Gamma^N$ with solutions $y^M(\cdot;\gamma^N)$ satisfying that $y^M(t,\gamma^N) \to x(t,\gamma)$ as $N, M \to \infty$, uniformly on compact time intervals.

Step 3) Define the least square minimization problems ($\mathcal{P}^{N,M}$) for each $N, M = 1, 2, \ldots$, i.e., find $\gamma^{N,M} \in \Gamma^N$, which minimizes the least squares fit-to-data criterion

$$J^{N,M}(\gamma^N) = \sum_{i=0}^{l} |y^M(t_i;\gamma^N) - X_i|^2, \qquad \gamma^N \in \Delta^N,$$

where $\Delta^N \subset \Gamma^N$ is the projection of Δ to Γ^N.

Step 4) Assuming that Δ is a compact subset of Γ, argue, that the sequence of solutions, $\gamma^{N,M}$ $(N, M = 1, 2, \ldots)$, of the finite dimensional minimization problems $\mathcal{P}^{N,M}$, has a convergent subsequence with limit $\bar{\gamma} \in \Gamma$.

Step 5) Show that $\bar{\gamma}$ is the solution of the minimization problem $\mathcal{P}$.

Note that Step 4) and 5) can be argued without using the particular approximation method of the initial value problem, using only compactness arguments and Step 2) (see e.g. [4]).

In Section 2 we define an Euler-type approximation scheme for IVP (1.1)-(1.2) using equations with piecewise constant arguments. The theoretical convergence of this scheme was shown in [5] assuming conditions (H1)–(H3) (see below). Note that the theoretical convergence of this scheme follows from Theorem 3 below as well. In Section 3 we show that this approximation scheme has the property required in Step 2), and in Section 4 we present numerical examples for estimating parameters of IVP (1.1)-(1.2) by applying our approximation scheme and the method described above.

For numerical approximation methods using equations with piecewise constant arguments we refer to [6, 7]. Related numerical identification schemes for neutral delay differential equations were studied in [8, 9, 10]. See also [11], where identification methods based on difference equations were studied.

We close this section by noting that in this paper we focus on the convergence of our numerical scheme to minimize the least square cost function. The underlying identifiability issues (i.e., uniqueness of the solution of problem $\mathcal{P}$) are not addressed here. (See [1, 12] on related developments for FDEs without state-dependent delays.) Although in case of certain equations identifiability or the lack of it can be argued, to the best of our knowledge, there is no comprehensive theory for identifiability in general state-dependent delay equations.

2 Notations, preliminaries

Throughout this paper $|\cdot|$ denotes a vector norm on $\mathbf{R}^n$. We denote the open and closed ball about x_0 with radius R in a Banach space X by $\mathcal{G}_X(x_0;\, R)$ and $\overline{\mathcal{G}}_X(x_0;\, R)$, respectively. If the ball is centered at the origin, we use simply $\mathcal{G}_X(R)$ and $\overline{\mathcal{G}}_X(R)$. Similarly, the open and closed neighborhood of a set $A \subset X$ is denoted by $\mathcal{G}_X(A;\, R)$ and $\overline{\mathcal{G}}_X(A;\, R)$, respectively. The supremum norm on $C \equiv C([-r,0],\mathbf{R}^n)$ is denoted by $|\cdot|_C$. We denote the space of absolutely continuous functions, $\psi\,:\,[-r,0]\to\mathbf{R}^n$, with essentially bounded derivatives by $W^{1,\infty}$. The norm in this Banach-space is defined by $|\psi|_{W^{1,\infty}} \equiv \max\{\sup_{s\in[-r,0]}|\psi(s)|,\, \operatorname{ess\,sup}_{s\in[-r,0]}|\dot{\psi}(s)|\}$.

Introduce the simplifying notation

$$\lambda(t,\psi,\sigma,\xi) \equiv \int_{-r}^{0} d_s\mu(s,t,\psi,\sigma)\xi(s). \tag{2.4}$$

With this notation we have that $\Lambda(t,\psi,\sigma) = \lambda(t,\psi,\sigma,\psi)$.

Next we summarize conditions guaranteeing well-posedness of IVP (1.1)-(1.2) (see [5] and Theorem 3 below):

(H1) (i) $f\,:\,[0,T]\times\Omega_1\times\Omega_2\times\Omega_3\to\mathbf{R}^n$ is continuous, where Ω_1 and Ω_2 are open subsets of $\mathbf{R}^n$, and Ω_3 is an open subset of Θ,

(ii) for every $M_1 > 0$ and $M_2 > 0$ there exists a constant $L_1 = L_1(M_1, M_2)$ such that for all $t \in [0, T]$, $x, \bar{x} \in \overline{\mathcal{G}}_{\mathbf{R}^n}(M_1) \cap \Omega_1$, $y, \bar{y} \in \overline{\mathcal{G}}_{\mathbf{R}^n}(M_1) \cap \Omega_2$, and $\theta, \bar{\theta} \in \overline{\mathcal{G}}_{\Theta}(M_2) \cap \Omega_3$

$$|f(t, x, y, \theta) - f(t, \bar{x}, \bar{y}, \bar{\theta})| \leq L_1 \Big(|x - \bar{x}| + |y - \bar{y}| + |\theta - \bar{\theta}|_\Theta \Big),$$

(H2) $\mu(\cdot, t, \psi, \sigma)$ is a matrix valued function of bounded variation for every $t \in [0, T]$, $\psi \in \Omega_4$, and $\sigma \in \Omega_5$, where $\Omega_4 \subset C$ is open, and $\Omega_5 \subset \Sigma$ is open such that

(i) $\|\mu\| \equiv \sup \{ |\lambda(t, \psi, \sigma, \xi)| \ : \ t \in [0, T],\ \psi \in \Omega_4,\ \sigma \in \Omega_5,\ \xi \in \mathcal{G}_C(1) \} < \infty$,

(ii) for each $\xi \in C$ the function $[0, T] \times \Omega_4 \times \Omega_5 \to \mathbf{R}^n$, $(t, \psi, \sigma) \mapsto \lambda(t, \psi, \sigma, \xi)$ is continuous,

(iii) for every $M_1 > 0$ and $M_2 > 0$ there exists a constant $L_2 = L_2(M_1, M_2)$ such that for all $\xi \in W^{1,\infty}$, $t \in [0, T]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M_1) \cap \Omega_4$, and $\sigma, \bar{\sigma} \in \overline{\mathcal{G}}_\Sigma(M_2) \cap \Omega_5$,

$$|\lambda(t, \psi, \sigma, \xi) - \lambda(t, \bar{\psi}, \bar{\sigma}, \xi)| \leq L_2 |\xi|_{W^{1,\infty}} \Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma \Big),$$

(H3) $\varphi \in W^{1,\infty}$, i.e., φ is Lipschitz-continuous.

It is easy to see that in order to have a properly posed problem, the initial function φ, the parameter σ, and the function μ have to satisfy

$$\varphi(0) \in \Omega_1, \ \ \varphi \in \Omega_4, \quad \text{and} \quad \int_{-r}^{0} d_s\mu(s, 0, \varphi, \sigma)\, \varphi(s) \in \Omega_2. \tag{2.5}$$

The respective definitions of $\Lambda(t, \psi, \sigma)$ and $\|\mu\|$ and assumption (H2) immediately imply the following lemmas, which we shall need later.

Lemma 1 *Assume (H2). Then*

$$\begin{aligned} &|\Lambda(t, \psi, \sigma) - \Lambda(t, \bar{\psi}, \bar{\sigma})| \\ &\quad \leq \Big(\|\mu\| + L_2(M_1, M_2) |\bar{\psi}|_{W^{1,\infty}} \Big) \Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma \Big), \end{aligned}$$

where $t \in [0, T]$, $\psi, \bar{\psi} \in \overline{\mathcal{G}}_C(M_1) \cap \Omega_4$, $\bar{\psi} \in W^{1,\infty}$ *and* $\sigma, \bar{\sigma} \in \overline{\mathcal{G}}_\Sigma(M_2) \cap \Omega_5$.

Proof Let M_1 and M_2 be fixed, and $L_2 = L_2(M_1, M_2)$ be the corresponding constant from assumption (H2). Let t, ψ, $\bar{\psi}$, σ and $\bar{\sigma}$ satisfy the assumptions of the lemma. Assumption (H2), the definition of $\|\mu\|$, and elementary estimates imply the inequalities

$$\begin{aligned} &|\Lambda(t, \psi, \sigma) - \Lambda(t, \bar{\psi}, \bar{\sigma})| \\ &\quad \leq |\lambda(t, \psi, \sigma, \psi) - \lambda(t, \psi, \sigma, \bar{\psi})| + |\lambda(t, \psi, \sigma, \bar{\psi}) - \lambda(t, \bar{\psi}, \bar{\sigma}, \bar{\psi})| \\ &\quad \leq \Big| \int_{-r}^{0} d_s\mu(s, t, \psi, \sigma) \big[\psi(s) - \bar{\psi}(s) \big] \Big| + L_2 |\bar{\psi}|_{W^{1,\infty}} \Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma \Big) \\ &\quad \leq \|\mu\| |\psi - \bar{\psi}|_C + L_2 |\bar{\psi}|_{W^{1,\infty}} \Big(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma \Big), \end{aligned}$$

which proves the lemma. $\square$

Lemma 2 *Assume (H2). Then $\Lambda(t,\psi,\sigma)$ is continuous on $[0,T]\times\Omega_4\times\Omega_5$.*

Proof Definition of $\|\mu\|$ and assumption (H2) (ii) together with inequalities

$$\begin{aligned}
&|\Lambda(t,\psi,\sigma)-\Lambda(\bar t,\bar\psi,\bar\sigma)|\\
&\qquad\le\ |\lambda(t,\psi,\sigma,\psi)-\lambda(t,\psi,\sigma,\bar\psi)|+|\lambda(t,\psi,\sigma,\bar\psi)-\lambda(\bar t,\bar\psi,\bar\sigma,\bar\psi)|\\
&\qquad\le\ \|\mu\||\psi-\bar\psi|_C+|\lambda(t,\psi,\sigma,\bar\psi)-\lambda(\bar t,\bar\psi,\bar\sigma,\bar\psi)|
\end{aligned}$$

yield the lemma. □

Let $h>0$. Throughout this paper we shall use the notation $[t]_h\equiv[t/h]h$, where $[\cdot]$ is the greatest integer function. For later reference we mention an elementary property of this function, namely that:

$$t-h<[t]_h\le t. \tag{2.6}$$

3 Convergence Results

Consider the delay differential equation

$$\dot x(t)=f\Big(t,x(t),\Lambda(t,x_t,\sigma),\theta\Big),\qquad t\in[0,T], \tag{3.7}$$

and the corresponding initial condition

$$x(t)=\varphi(t),\qquad t\in[-r,0], \tag{3.8}$$

where $\theta\in\Omega_3$, $\sigma\in\Omega_5$, Ω_3 and Ω_5 are open subsets of Θ and Σ, respectively, the delayed term Λ is defined by (1.3). We assume that f and Λ (i.e., μ) are given in the equation, but the parts of f and Λ represented by θ and σ, and the initial function are unknown, i.e., considered as parameters. Define the parameter space by $\Gamma\equiv C\times\Sigma\times\Theta$, with norm $|(\varphi,\sigma,\theta)|_\Gamma\equiv|\varphi|_C+|\sigma|_\Sigma+|\theta|_\Theta$, and the set of admissible parameters by $\Pi\equiv\Big\{(\varphi,\sigma,\theta)\in W^{1,\infty}\times\Omega_5\times\Omega_3\ :\ \varphi(0)\in\Omega_1,\ \varphi\in\Omega_4,\ \int_{-r}^0 d_s\mu(s,0,\varphi,\sigma)\varphi(s)\in\Omega_2\Big\}$. (See also (2.5).)

We assume that f, φ and μ satisfy (H1)–(H3) and $(\varphi,\sigma,\theta)\in\Pi$. We recall the following result concerning the well-posedness of IVP (3.7)-(3.8) from [5].

3.2 Assume that f, μ and $(\varphi,\sigma,\theta)\in\Pi$ satisfy (H1)–(H3). Then there exist constants $0<T^*\le T$, $\delta>0$ and $L_3=L_3(T^*,\varphi,\sigma,\theta,\delta)>0$, such that $\mathcal{G}_\Gamma((\varphi,\sigma,\theta);\,\delta)\subset\Pi$, and IVP (3.7)-(3.8) has a unique solution on $[0,T^*]$ for all $(\bar\varphi,\bar\sigma,\bar\theta)\in\mathcal{G}_\Gamma((\varphi,\sigma,\theta);\,\delta)$, the solution, $x(\cdot;\varphi,\sigma,\theta)$, is absolutely continuous on $[-r,T^*]$ with essentially bounded derivative, and

$$\begin{aligned}
&|x(\cdot;\varphi,\sigma,\theta)_t-x(\cdot;\bar\varphi,\bar\sigma,\bar\theta)_t|_C\\
&\qquad\le\ L_3\Big(|\varphi-\bar\varphi|_C+|\sigma-\bar\sigma|_\Sigma+|\theta-\bar\theta|_\Theta\Big),\quad t\in[0,T^*].
\end{aligned}\tag{3.9}$$

We note that this result was shown in [5] for the case $\Sigma = \Theta = \mathbf{R}^m$, (as a part of the proof of the stronger statement, where the $|\cdot|_C$ norms in (3.9) are replaced by $|\cdot|_{W^{1,\infty}}$ norms). The proof for the case when the parameters σ and θ are infinite dimensional is an obvious modification of that of the finite dimensional case, and it is omitted here.

We make the assumption for the rest of this paper that $T = T^*$, i.e., IVP (3.7)-(3.8) has a unique solution on $[0, T]$.

To follow the general identification procedure described in the introduction, we take finite dimensional approximations, $\Gamma^N \equiv \Phi^N \times \Sigma^N \times \Theta^N$, of the parameter space, Γ: Let $\Phi^N \subset C$, $\Sigma^N \subset \Sigma$ and $\Theta^N \subset \Theta$ be sequences of finite dimensional spaces, such that for each $\varphi \in C$, $\sigma \in \Sigma$, and $\theta \in \Theta$, the corresponding projections, $\varphi^N \in \Phi^N$, $\sigma^N \in \Sigma^N$, and $\theta^N \in \Theta^N$ satisfy that $|\varphi^N - \varphi|_C \to 0$, $|\sigma^N - \sigma|_\Sigma \to 0$, and $|\theta^N - \theta|_\Theta \to 0$, as $N \to \infty$. Let $\Pi^N \equiv \Pi \cap \Gamma^N$.

Next we define approximate IVPs corresponding to parameters $(\varphi^N, \sigma^N, \theta^N) \in \Pi^N$, using Euler's method. Let h be a positive constant, and define the following delay equation with piecewise constant arguments

$$\dot{y}_{h,N}(t) = f\Big([t]_h, y_{h,N}([t]_h), \Lambda([t]_h, (y_{h,N})_{[t]_h}, \sigma^N), \theta^N\Big), \qquad t \in [0, T], \tag{3.10}$$

with initial condition

$$y_{h,N}(t) = \varphi^N(t), \qquad t \in [-r, 0]. \tag{3.11}$$

Here, to emphasize that the solution corresponds to a given $h > 0$ and $(\varphi^N, \sigma^N, \theta^N)$, we denote the solution and the solution segment function of IVP (3.10)-(3.11) by $y_{h,N}(t)$ and $(y_{h,N})_t$, respectively.

By a solution of IVP (3.10)-(3.11) we mean a function $y_{h,N} : [-r, T] \to \mathbf{R}^n$, which is defined on $[-r, 0]$ by (3.11) and satisfies the following properties on $[0, T]$:

(i) the function $y_{h,N}$ is continuous on $[0, T]$,

(ii) the derivative $\dot{y}_{h,N}(t)$ exists at each point $t \in [0, T)$ with the possible exception of the points ih $(i = 0, 1, 2, \ldots)$ where finite one-sided derivatives exist,

(iii) the function $y_{h,N}$ satisfies (3.10) on each interval $[ih, (i+1)h) \cap [0, T]$ for $i = 0, 1, 2, \ldots$

It is easy to see, using the method of steps, that for each fixed $h > 0$ and $(\varphi^N, \sigma^N, \theta^N) \in \Pi^N$, IVP (3.10)-(3.11) has a unique solution. For a fixed $h > 0$ and $N > 0$ let $[0, \alpha_{h,N})$ (or $[0, T]$ if $\alpha_{h,N} = T$) be the maximal interval where the solution can be continued. Then $\alpha_{h,N} > 0$ is the largest possible number such that

$$y_{h,N}(t) \in \Omega_1, \ (y_{h,N})_t \in \Omega_4, \ \text{and } \Lambda(t, (y_{h,N})_t, \sigma^N) \in \Omega_2 \quad \text{for } t \in [0, \alpha_{h,N}).$$

The following theorem guarantees Step 2) of the identification method described in the introduction using the approximation scheme (3.10)-(3.11).

3.2 Assume that f, μ and $(\varphi, \sigma, \theta) \in \Pi$ satisfy (H1)–(H3). Fix sequences $\varphi^N \in \Phi^N$, $\sigma^N \in \Omega_5$, and $\theta^N \in \Omega_3$ such that $|\varphi^N - \varphi|_C \to 0$, $|\sigma^N - \sigma|_\Sigma \to 0$, and $|\theta^N - \theta|_\Theta \to 0$ as $N \to \infty$. Then

(i) there exist $h_0 > 0$ and $N_0 > 0$ such that for all $0 < h < h_0$ and $N > N_0$, IVP (3.10)-(3.11) has unique solution defined on $[0,T]$, i.e., $\alpha_{h,N} = T$ for $0 < h < h_0$ and $N > N_0$,

(ii) the solution, $y_{h,N}$, of IVP (3.10)-(3.11) converges uniformly on $[0,T]$ to the solution, x, of IVP (3.7)-(3.8) as $h \to 0^+$ and $N \to \infty$, i.e.,

$$\lim_{\substack{h\to 0+\\ N\to\infty}} \max_{0\le t\le T} |x(t) - y_{h,N}(t)| = 0.$$

Proof Fix $(\varphi,\sigma,\theta) \in \Pi$, and let $\delta > 0$ be the constant from Theorem 3, i.e., such that $\mathcal{G}_\Gamma((\varphi,\sigma,\theta);\,\delta) \subset \Pi$. We can (and do) assume that $(\varphi^N,\sigma^N,\theta^N) \in \mathcal{G}_\Gamma((\varphi,\sigma,\theta);\,\delta)$ for all N. Theorem 3 yields that the solution of IVP (3.7)-(3.8), $x(t)$, is absolutely continuous on $[0,T]$ with essentially bounded derivative, therefore the constant

$$M_0 \equiv \max\Big\{ \sup_{-r\le u\le T} |x(u)|,\, \operatorname*{ess\,sup}_{-r\le u\le T} |\dot x(u)| \Big\} + \delta$$

is finite. Let $M \equiv \max\{\|\mu\| M_0, M_0\}$. Then the definitions of M and M_0, and inequality $|\Lambda(t,\psi,\sigma)| \le \|\mu\||\psi|_C$ imply that x_t and $\Lambda(t,x_t,\sigma)$ remain in $\overline{\mathcal{G}}_{\mathbf{R}^n}(M)$ for $t \in [0,T]$. Let $L_1 = L_1(M, |\sigma|_\Sigma + \delta)$ be the constant given by (H1) (ii). Since $\varphi^N(0) \in \mathcal{G}_{\mathbf{R}^n}(M_0)$, continuity argument gives that for each $h > 0$ and $N > 0$ there exists $0 < \alpha^*_{h,N} \le \alpha_{h,N}$ such that $|y_{h,N}(t)| \le M_0$ for $t \in [0, \alpha^*_{h,N}]$. But then also $y_{h,N}(t)$, $\Lambda(t,(y_{h,N})_t,\sigma^N) \in \overline{\mathcal{G}}_{\mathbf{R}^n}(M)$ for $t \in [0,\alpha^*_{h,N}]$. Therefore (3.7), (3.10), assumption (H1) (ii), and standard estimates yield the following inequalities for $t \in [0,\alpha^*_{h,N}]$:

$$\begin{aligned}
&|x(t) - y_{h,N}(t)|\\
&\quad\le |\varphi(0) - \varphi^N(0)|\\
&\qquad + \int_0^t \Big| f\Big(u, x(u), \Lambda(u,x_u,\sigma),\theta\Big) - f\Big([u]_h, x(u), \Lambda(u,x_u,\sigma),\theta\Big)\Big|\,du\\
&\qquad + \int_0^t \Big| f\Big([u]_h, x(u), \Lambda(u,x_u,\sigma),\theta\Big)\\
&\qquad\qquad - f\Big([u]_h, y_{h,N}([u]_h), \Lambda([u]_h, (y_{h,N})_{[u]_h}, \sigma^N), \theta^N\Big)\Big|\,du\\
&\quad\le |\varphi - \varphi^N|_C + L_1|\theta-\theta^N|_\Theta t\\
&\qquad + \int_0^t \Big| f\Big(u, x(u), \Lambda(u,x_u,\sigma),\theta\Big) - f\Big([u]_h, x(u), \Lambda(u,x_u,\sigma),\theta\Big)\Big|\,du \qquad (3.12)\\
&\qquad + L_1 \int_0^t \Big(|x(u) - y_{h,N}([u]_h)| + \Big|\Lambda(u,x_u,\sigma) - \Lambda([u]_h, (y_{h,N})_{[u]_h}, \sigma^N)\Big|\Big)\,du.
\end{aligned}$$

Introduce the function

$$v_{h,N}(t) \equiv \max_{-r\le u\le t} |x(u) - y_{h,N}(u)|, \qquad t \in [0,\alpha^*_{h,N}].$$

The definition and the monotonicity of $v_{h,N}$ and M, (2.6), and the Mean Value Theorem imply that

$$\begin{aligned}
|x(u) - y_{h,N}([u]_h)| &\le |x([u]_h) - y_{h,N}([u]_h)| + |x(u) - x([u]_h)|\\
&\le v_{h,N}([u]_h) + M_0 h\\
&\le v_{h,N}(u) + M_0 h, \qquad u \in [-r, \alpha^*_{h,N}], \qquad (3.13)
\end{aligned}$$

and similarly

$$|x_u - (y_{h,N})_{[u]_h}|_C \le v_{h,N}(u) + M_0 h, \qquad u \in [0, \alpha^*_{h,N}]. \tag{3.14}$$

Using (3.14), Lemma 1 with $L_2 = L_2(M_0, |\sigma|_\Sigma + \delta)$, we have the following estimate for $u \in [0, \alpha^*_{h,N}]$:

$$\begin{aligned}
&\Big|\Lambda(u, x_u, \sigma) - \Lambda([u]_h, (y_{h,N})_{[u]_h}, \sigma^N)\Big| \\
&\quad \le \; |\Lambda(u, x_u, \sigma) - \Lambda([u]_h, x_u, \sigma)| + \Big|\Lambda([u]_h, x_u, \sigma) - \Lambda([u]_h, (y_{h,N})_{[u]_h}, \sigma^N)\Big| \\
&\quad \le \; |\Lambda(u, x_u, \sigma) - \Lambda([u]_h, x_u, \sigma)| \\
&\qquad + (\|\mu\| + L_2 |x_u|_{W^{1,\infty}}) \Big(|x_u - (y_{h,N})_{[u]_h}|_C + |\sigma - \sigma^N|_\Sigma\Big) \\
&\quad \le \; |\Lambda(u, x_u, \sigma) - \Lambda([u]_h, x_u, \sigma)| + (\|\mu\| + L_2 M_0) v_{h,N}(u) \\
&\qquad + (\|\mu\| + L_2 M_0)\Big(M_0 h + |\sigma - \sigma^N|_\Sigma\Big).
\end{aligned} \tag{3.15}$$

Combining (3.12), (3.13) and (3.15) we get

$$|x(t) - y_{h,N}(t)| \le g_{h,N}(t) + \int_0^t L_1(1 + \|\mu\| + L_2 M_0) v_{h,N}(u)\, du, \tag{3.16}$$

$t \in [0, \alpha^*_{h,N}]$, where

$$\begin{aligned}
g_{h,N}(t) \;\equiv\; & \int_0^t \Big| f\Big(u, x(u), \Lambda(u, x_u, \sigma), \theta\Big) - f\Big([u]_h, x(u), \Lambda(u, x_u, \sigma), \theta\Big)\Big|\, du \\
& + L_1 \int_0^t |\Lambda(u, x_u, \sigma) - \Lambda([u]_h, x_u, \sigma)|\, du \\
& + L_1 (1 + \|\mu\| + L_2 M_0) M_0 h t \\
& + L_1 \Big((\|\mu\| + L_2 M_0)|\sigma - \sigma^N|_\Sigma + |\theta - \theta^N|_\Theta\Big) t + |\varphi - \varphi^N|_C.
\end{aligned} \tag{3.17}$$

Note that $g_{h,N}(t)$ can be defined for $t \in [0, T]$. Using the monotonicity of $g_{h,N}$, (3.16) yields

$$v_{h,N}(t) \le g_{h,N}(T) + \int_0^t L_1(1 + \|\mu\| + L_2 M_0) v_{h,N}(u)\, du, \quad t \in [0, \alpha^*_{h,N}],$$

which, by Gronwall-Bellman's inequality, implies that

$$\begin{aligned}
|x(t) - y_{h,N}(t)| \;&\le\; v_{h,N}(t) \\
&\le\; g_{h,N}(T) \exp\Big(L_1(1 + L_2 M + \|\mu\|) t\Big) \\
&\le\; g_{h,N}(T) \exp\Big(L_1(1 + L_2 M + \|\mu\|) T\Big), \quad t \in [0, \alpha^*_{h,N}].
\end{aligned} \tag{3.18}$$

Using Lebesgue's Dominated Convergence Theorem for the first two integrals in (3.17), (2.6), the continuity of f and Λ guaranteed by (H1) (i) and Lemma 2, respectively, and the assumptions that $\varphi^N \to \varphi$, $\sigma^N \to \sigma$, and $\theta^N \to \theta$ as $N \to \infty$, we get that

$$g_{h,N}(T) \to 0, \qquad \text{as } h \to 0^+ \quad \text{and} \quad N \to \infty. \tag{3.19}$$

In particular, we get that there exist $h_1 > 0$ and $N_1 > 0$ such that for all $0 < h < h_1$ and $N > N_1$ it follows that $|x(t) - y_{h,N}(t)| < \delta$, whenever $y_{h,N}(t)$ is defined. But then we have that $|y_{h,N}(t)| \leq M_0$, and $\Lambda(t,(y_{h,N})_t,\sigma^N) \in \overline{\mathcal{G}}_{\mathbf{R}^n}(M)$, for all $t \in [0,T]$ for which $y_{h,N}(t)$ is defined, i.e.,

$$\alpha_{h,N} = \alpha^*_{h,N} \qquad \text{for } 0 < h < h_1, \quad N > N_1. \tag{3.20}$$

Let define $A_1 \equiv \{x(t) \;:\; t \in [0,T]\} \subset \Omega_1$, $A_2 \equiv \{\Lambda(t,x_t,\sigma) \;:\; t \in [0,T]\} \subset \Omega_2$ and $A_4 \equiv \{x_t \;:\; t \in [0,T]\} \subset \Omega_4$. The continuity of the maps $[0,T] \to \mathbf{R}^n$, $\; t \mapsto x(t)$, $t \mapsto \Lambda(t,\psi,\sigma)$, and $[0,T] \to C$, $\; t \mapsto x_t$ implies that A_1, A_2 and A_4 are compact subsets of their respective spaces, therefore there exist $\varepsilon > 0$, such that $\mathcal{G}_{\mathbf{R}^n}(A_i;\, \varepsilon) \subset \Omega_i$ $(i = 1,2)$ and $\mathcal{G}_C(A_4;\, \varepsilon) \subset \Omega_4$. Relation (3.18) and (3.19) yield that there exist $0 < h_2 \leq h_1$ and $N_2 \geq N_1$ such that $|x(t) - y_{h,N}(t)| < \varepsilon$ for $t \in [0,\alpha_{h,N}]$, $0 < h < h_2$ and $N > N_2$, hence

$$y_{h,N}(\alpha_{h,N}) \in \Omega_1, \quad (y_{h,N})_{\alpha_{h,N}} \in \Omega_4, \quad \text{for } 0 < h < h_2, \quad N > N_2. \tag{3.21}$$

Inequality (3.15), together with the continuity of Λ, (2.6), (3.18) and (3.19), and that $\sigma^N \to \sigma$ as $N \to \infty$ imply that there exist $0 < h_0 \leq h_2$ and $N_0 > N_2$ such that $|\Lambda(t,x_t,\sigma) - \Lambda([t]_h,(y_{h,N})_{[t]_h},\sigma^N)| < \varepsilon$ for $t \in [0,\alpha_{h,N}]$, $0 < h < h_0$ and $N > N_0$, therefore

$$\Lambda([\alpha_{h,N}]_h,(y_{h,N})_{[\alpha_{h,N}]_h},\sigma^N) \in \Omega_2, \qquad \text{for } 0 < h < h_0, \quad N > N_0. \tag{3.22}$$

Combining (3.21) and (3.22) we get that the solution of IVP (3.10)-(3.11) can be extended to $[0,T]$, i.e.,

$$\alpha_{h,N} = \alpha^*_{h,N} = T, \qquad \text{for } 0 < h < h_0, \quad N > N_0, \tag{3.23}$$

which proves (i) of the statement. Part (ii) follows from (3.23), and from (3.18) and (3.19). $\square$

4 Numerical examples

In this section we present applications of the identification method described in the introduction and in Section 3. Consider an identification problem corresponding to IVP (3.7)-(3.8), then we define the approximating IVPs by (3.10)-(3.11). Define the corresponding finite dimensional minimization problems, and find the solutions of them. Choose small enough h and large enough N, and use the solution of the minimization problem corresponding to this h and N as an approximation of the solution of the original identification problem.

In each example we used a nonlinear least square minimization code, based on a secant method with Dennis-Gay-Welsch update, combined with a trust region technique. See Section 10.3 in [13] for detailed description of this method.

Example 3 Consider first the scalar linear delay equation

$$\dot{x}(t) = \theta(t)x(t-\sigma(t)), \qquad t \in [0,T], \tag{4.24}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r,0], \tag{4.25}$$

where $0 \leq \sigma(t) \leq r$ for $t \in [0,T]$, and θ, σ and φ are continuous functions. IVP (4.24)-(4.25) can be written in the form IVP (3.7)-(3.8) in the following way: Define $\Sigma \equiv C([0,T];\, \mathbf{R})$ and

$\Theta \equiv C([0,T];\mathbf{R})$. Let $f(t,x,y,\theta) = \theta(t)y$, and $\mu(s,t,\psi,\sigma) = \chi_{[-\sigma(t),0]}(s)$, where $\chi_{[-\sigma(t),0]}(s)$ is the characteristic function of the interval $[-\sigma(t),0]$. It is easy to see that $\Lambda(t,\psi,\sigma) = \psi(-\sigma(t))$, and $\lambda(t,\psi,\sigma,\xi) = \xi(-\sigma(t))$. Clearly, f satisfies (H1)(i), and it is easy to check that $L_1 \equiv \max\{M_1, M_2\}$ is good in (H1) (ii). The function μ clearly satisfies (H2) (i) and (ii). To show (H2) (iii), consider

$$\begin{aligned} |\lambda(t,\psi,\sigma,\xi) - \lambda(t,\bar\psi,\bar\sigma,\xi)| &= |\xi(-\sigma(t)) - \xi(-\bar\sigma(t))| \\ &\le |\xi|_{W^{1,\infty}}|\sigma(t) - \bar\sigma(t)| \\ &\le |\xi|_{W^{1,\infty}}|\sigma - \bar\sigma|_C, \end{aligned}$$

hence (H2) (iii) follows with $L_2 = 1$ for all M_1 and M_2.

Consider a particular case of IVP (4.24)-(4.25): Let $T = 3$, $r = 3$, $\varphi(t) = t+1$, and

$$\sigma(t) = \begin{cases} -t^2 + 2t + 2, & t \in [0,2], \\ 2, & t \in [2,3]. \end{cases} \tag{4.26}$$

The solution of this IVP corresponding to $\theta(t) = (t-1)^2$ is

$$x(t) = \begin{cases} 1 - t + \frac{t^2}{2} + \frac{2\,t^3}{3} - \frac{3\,t^4}{4} + \frac{t^5}{5}, & t \in [0,2], \\ \frac{173}{45} - \frac{281\,t}{15} + \frac{1237\,t^2}{30} - \frac{4387\,t^3}{90} + \frac{101\,t^4}{3} & \\ \quad - \frac{167\,t^5}{12} + \frac{611\,t^6}{180} - \frac{9\,t^7}{20} + \frac{t^8}{40}, & t \in [2,3]. \end{cases}$$

We use this function with $t_i = 0.1i$ $(i = 0,\ldots,30)$ to generate the measurements, X_i.

The parameter space is $\Gamma = \Theta = C([0,3];\mathbf{R})$, (since θ is the only parameter in the IVP). To follow the general identification procedure, we have to assume that θ lies in a compact subset of the parameter space, which we can do by assuming some a priori estimates on θ, e.g., $\theta \in \Delta \equiv \{\theta \in \Theta : |\theta|_\Theta \le c_1,\ \operatorname{ess\,sup}_{0\le t\le 3}|\dot\theta(t)| \le c_2\}$ for some $c_1 > 0$ and $c_2 > 0$. In practice, if the parameter space is infinite dimensional and there is no natural constraint for the parameters in the equation, like in this example, we solve unconstraint minimization problems to simplify the numerical minimization process. (In our examples, of course, global minimum exists.)

We discretize the parameter space using linear splines, i.e., piecewise linear continuous functions, which are linear on the intervals $[(i-1)T/(N-1), iT/(N-1)]$ $(i = 1,\ldots,N-1)$. It is known (see e.g. [14]) that linear splines can be used to approximate continuous functions uniformly on compact time intervals. We can identify Γ^N by $\mathbf{R}^N$, as $\gamma = (a_1,\ldots,a_N)$, where a_i is the value of the spline function at the ith mesh point, $(i-1)T/(N-1)$.

Let $h > 0$ and define the approximating IVP according to IVP (3.10)-(3.11)

$$\dot y_h(t) = \theta^N([t]_h)y_h([t]_h - \sigma([t]_h)), \qquad t \in [0,3], \tag{4.27}$$

with initial condition

$$y_h(t) = t+1, \qquad t \in [-3,0]. \tag{4.28}$$

It is easy to obtain by integrating (4.27) from nh to $(n+1)h$ that

$$y_h((n+1)h) = y_h(nh) + \theta^N(nh)y_h(nh - \sigma(nh))h, \qquad n = 0,1,2,\ldots,$$

where, using (4.28), $y_h(nh-\sigma(nh)) = nh-\sigma(nh)+1$ if $nh-\sigma(nh) \le 0$, otherwise we evaluate it by interpolating between already computed solution values at mesh points.

Consider the minimization problem

$$\min J_h(\theta^N) \equiv \sum_{i=0}^{30} (y_h(t_i;\theta^N) - X_i)^2.$$

We solved this problem numerically for several N values, using $\theta^N = 0$ as the initial guess. In Figure 1 we plotted out the true θ function (solid graph) and the numerical solutions of the minimization problem corresponding to $N = 3$, 5 and 7 and $h = 0.0001$ (dashed graphs). The numerical results show good recovery of the coefficient, even for such small dimensions. In Table 1 and 2 we list the value of the cost function, $J_h(\bar{\theta}^N)$, and the maximal error, i.e., $\max_{i=0,\dots,30} |y_h(t_i;\theta^N) - X_i|$), respectively, for $N = 3$, 11, 19 and 27. Both values show decreasing pattern as we increase N and decrease h, which illustrate the convergence of the numerical identification method.

Table 1: $J_h(\bar{\theta}^N)$

h	N=3	N=11	N=19	N=27
0.1000	2.189e-02	1.132e-04	2.638e-05	2.678e-05
0.0100	2.065e-02	1.827e-06	3.123e-07	1.030e-07
0.0010	2.105e-02	1.561e-06	6.204e-08	8.192e-10
0.0001	2.109e-02	1.578e-06	5.928e-08	2.591e-11

Table 2: Maximal error

h	N=3	N=11	N=19	N=27
0.1000	0.603619	0.373733	0.496216	0.571648
0.0100	0.570814	0.023817	0.034678	0.040783
0.0010	0.568044	0.016307	0.006010	0.004769
0.0001	0.567774	0.016116	0.006304	0.002605

Example 4 In this example we consider IVP (4.24)-(4.25) again with $T = 3$, $r = 3$, $\varphi(t) = t + 1$, $\theta(t) = (t-1)^2$, and using the measurements of Example 3, we identify the delay function, (4.26). We proceed as before, we approximate σ by linear spline function σ^N, define the approximate IVP (4.27)-(4.28) (where we replace σ by σ^N and θ^N by θ), and minimize the corresponding cost function, $J_h(\sigma^N)$. Note that in this example we had to use constrained minimization to guarantee that σ^N remains nonnegative, i.e., to have a delay equation in (4.27). We present the corresponding numerical results on Figure 2 ($N = 3$, 5, 7 and $h = 0.0001$), and on Tables 3 and 4 ($N = 3$, 11, 19 and 27). The numerical findings indicate convergence to the true delay function as both $h \to 0$ and $N \to \infty$, but it is interesting to see that how the maximal error can increase for fixed h as N increases.

Table 3: $J_h(\bar{\sigma}^N)$

h	N=3	N=11	N=19	N=27
0.1000	1.095e-02	7.067e-03	3.955e-05	1.849e-05
0.0100	6.249e-03	4.964e-04	3.809e-06	1.053e-09
0.0010	5.838e-03	2.602e-05	1.932e-06	7.267e-09
0.0001	5.798e-03	2.525e-05	3.076e-08	7.901e-09

Table 4: Maximal error

h	N=3	N=11	N=19	N=27
0.1000	0.421799	2.119123	2.722342	7.429697
0.0100	0.324328	1.932437	1.993864	2.023813
0.0010	0.347840	0.105441	1.945538	0.080971
0.0001	0.352248	0.106412	0.014420	0.017938

Example 5 Our next example is a scalar delay equation with distributed constant delay

$$\begin{aligned}\dot{x}(t) &= \int_{-1}^{0}\Big((2+s)^{\sigma}x(t+s)-\frac{7}{3}t^2+\theta_1 t+\theta_2\Big)\,ds, \qquad t\in[0,3],\\ x(t) &= t^2, \qquad t\in[-1,0].\end{aligned}$$

The goal in this example is the identification of the parameters $\sigma,\theta_1,\theta_2\in\mathbf{R}$. It is easy to check, that the solution of this IVP corresponding to $\sigma=2$, $\theta_1=23/6$ and $\theta_2=-8/15$ is $x(t)=t^2$. We used this solution to generate measurements at $t_i=0.1i$, $(i=0,1,\dots,30)$. In this example the parameter space, $\Gamma=\mathbf{R}^3$, is finite dimensional, therefore there is no need for discretization, i.e., we use $\Gamma^N=\mathbf{R}^3$. The corresponding approximate IVP is

$$\dot{y}_h(t) = \int_{-1}^{0}\Big((2+s)^{\sigma}y([t]_h+s)-\frac{7}{3}[t]_h^2+\theta_1[t]_h+\theta_2\Big)\,ds, \qquad t\geq 0, \tag{4.29}$$

$$y_h(t) = t^2, \qquad t\in[-1,0]. \tag{4.30}$$

Using the fact that y_h is linear on the intervals $[ih,(i+1)h]$, we can easily obtain a difference equation for the values of the solution at mesh points.

In Table 5 we present our numerical findings, which shows convergence to the true parameter values, as $h\to 0$. Note that our numerical approximation method, and therefore the numerical identification method as well, takes a long time for "small" h, since we, in fact, compute the true values of the integral in (4.29) for a piecewise linear y_h by summing up the true value of the integral in each interval $[ih,(i+1)h]$. On the other hand, we got relatively good estimate for the true parameters using "large" h values.

Table 5:

h	$\bar{\sigma}$	$\bar{\theta}_1$	$\bar{\theta}_2$	$J_h(\bar{\sigma}, \bar{\theta}_1, \bar{\theta}_2)$	Maximal error
0.1000	2.000000	3.833333	-0.437222	16.0249849	9.611167e-02
0.0100	2.000002	3.833329	-0.523370	0.2343549	9.962903e-03
0.0010	1.999999	3.833336	-0.532335	0.0024401	9.981619e-04

Example 6 Consider the scalar state-dependent delay equation

$$\dot{x}(t) = -\frac{9}{16}x^2\Big(t - \tau(t, x(t), \sigma)\Big), \qquad t \in [0, 3], \tag{4.31}$$

$$x(t) = \varphi(t), \qquad t \in [-1, 0], \tag{4.32}$$

where

$$\varphi(t) = \begin{cases} 1/(t+1), & t \in [-0.5, 0], \\ 2, & t \in [-1, -0.5], \end{cases} \tag{4.33}$$

and $\sigma \in C([0, 3]; \mathbf{R})$, $\tau(t, u, \sigma) \equiv \min\{\sigma(t)|u|, 1\}$. It is easy to see that the solution of this IVP corresponding to $\sigma(t) = \frac{1}{4}(t+1)^2$ is $x(t) = 1/(t+1)$. Using this solution, we generated measurements at $t_i = 0.1i$, $(i = 0, 1, \ldots, 30)$.

Taking $\mu(s, t, \psi, \sigma) \equiv \chi_{[-\tau(t,\psi(0),\sigma),0]}(s)$ and $f(t, x, y, \theta) = -\frac{9}{16}y^2$, one can rewrite IVP (4.31)-(4.32) in the form IVP (1.1)-(1.2), and can check that (H1)–(H3) are satisfied.

We printed out the numerical results in Tables 6 and 7 using spline approximations with dimensions $N = 3, 11, 19$ and 27. In Figure 3 the graph of the true σ function (solid graph) and the approximate functions (dashed graphs) corresponding to $N = 3, 5, 7$ and $h = 0.0001$ are presented.

Table 6: $J_h(\bar{\sigma}^N)$

h	N=3	N=11	N=19	N=27
0.1000	3.580e-04	1.578e-07	6.713e-09	4.688e-10
0.0100	2.923e-04	4.867e-08	8.884e-10	2.241e-11
0.0010	2.864e-04	4.350e-08	9.756e-10	4.400e-13
0.0001	2.858e-04	4.305e-08	9.808e-10	2.986e-11

Example 7 Finally, consider again IVP (4.31)-(4.32), with $\sigma(t) = \frac{1}{4}(t+1)^2$, and the same measurements of the previous example, and we identify the initial function, (4.33). The numerical results are presented in Table 8 (for N = 3, 11, 19 and 27), and in Figure 4 (for N = 3, 5, 7, 9 and $h = 0.0001$).

Let $-r^* \equiv \min\{t - \tau(t, x(t), \sigma) \,:\, t \in [0, 3]\}$. Then, clearly, $r^* \leq 1$, and the actual initial interval is $[-r^*, 0]$, the values of initial function on $[-1, -r^*]$ are superfluous, i.e., are not used and needed in the equation. The difficulty of identifying initial functions in state-dependent

Table 7: Maximal error

h	N=3	N=11	N=19	N=27
0.1000	0.334270	0.156771	0.158563	0.150708
0.0100	0.360240	0.019216	0.016235	0.015785
0.0010	0.363738	0.005593	0.002634	0.002150
0.0001	0.364103	0.004317	0.001382	0.001623

delay equations is that the actual initial interval, $[-r^*, 0]$, depends on the actual solution, therefore it is not known in advance. In this example, using initial function (4.33) and $\sigma(t) = \frac{1}{4}(t+1)^2$, the true solution is $x(t) = 1/(t+1)$, therefore $r^* = 0.25$. We approximate the initial function by linear spline functions defined on $[-1, 0]$, and use $\varphi^N(t) = 0.5$ as our initial guess in the numerical minimization routine. We can observe that the solution of the corresponding finite dimensional minimization problem remains constant 0.5 in between mesh points, which are not used in the numerical approximation, i.e., from which the least square cost function is independent. We denote the first mesh point, where the numerical solution of the minimization problem is not equal to the initial guess, by $-r^N$. That is, the numerical solution is constant 0.5 on the interval $[-1, -r^N - h]$. We listed the values of r^N corresponding to $h = 0.001$ in Table 9 for several choices of N. This experiment indicates that $r^N \to r^*$.

Table 8: $J_h(\bar{\varphi}^N)$

h	N=3	N=11	N=19	N=27
0.1000	3.580e-04	1.578e-07	6.713e-09	4.688e-10
0.0100	2.923e-04	4.867e-08	8.884e-10	2.241e-11
0.0010	2.864e-04	4.350e-08	9.756e-10	4.400e-13
0.0001	2.858e-04	4.305e-08	9.808e-10	2.986e-11

Table 9:

N	3	5	9	17
$-r^N$	-1.00000	-0.50000	-0.50000	-0.37500
N	33	65	129	257
$-r^N$	-0.31250	-0.28125	-0.25781	-0.25390

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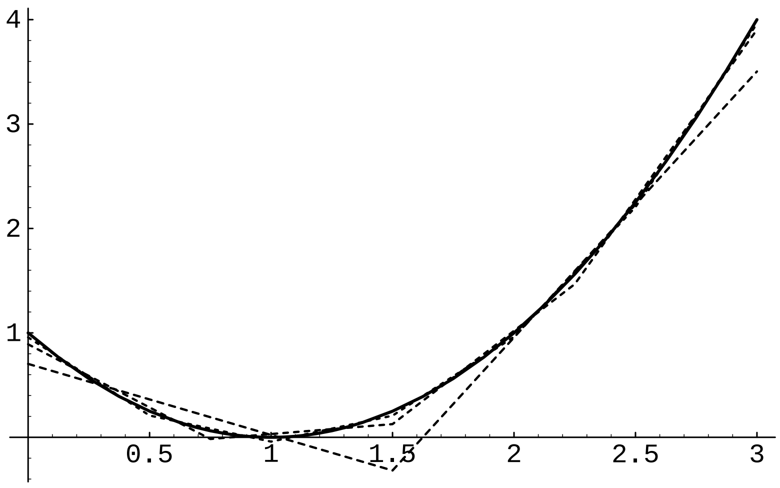

Figure 1:

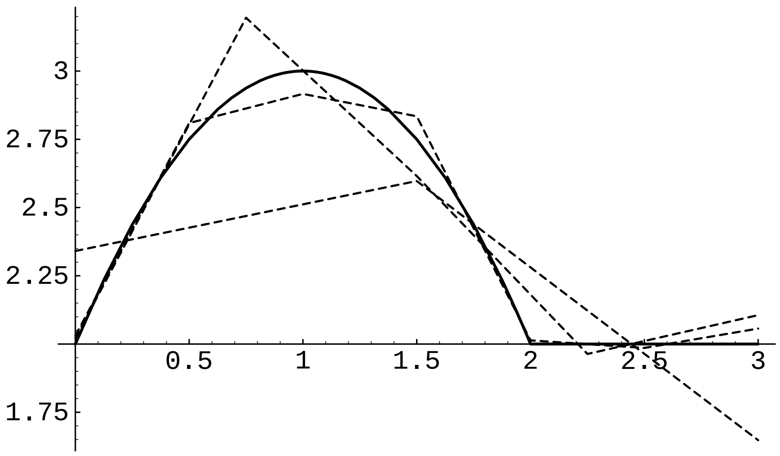

Figure 2:

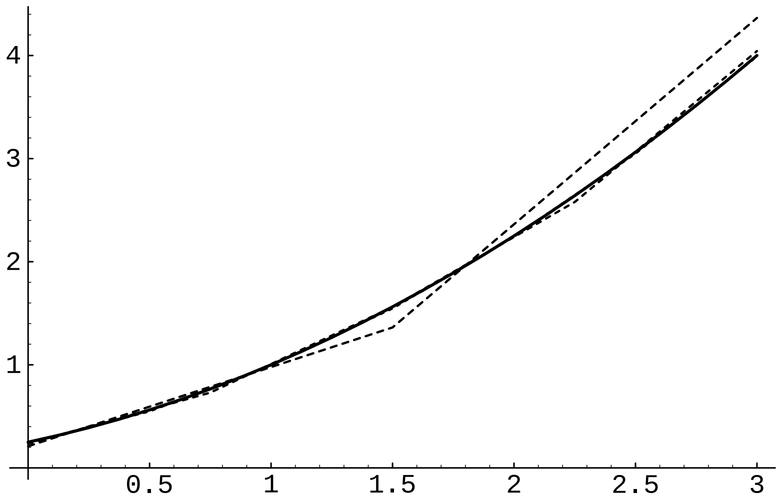

Figure 3:

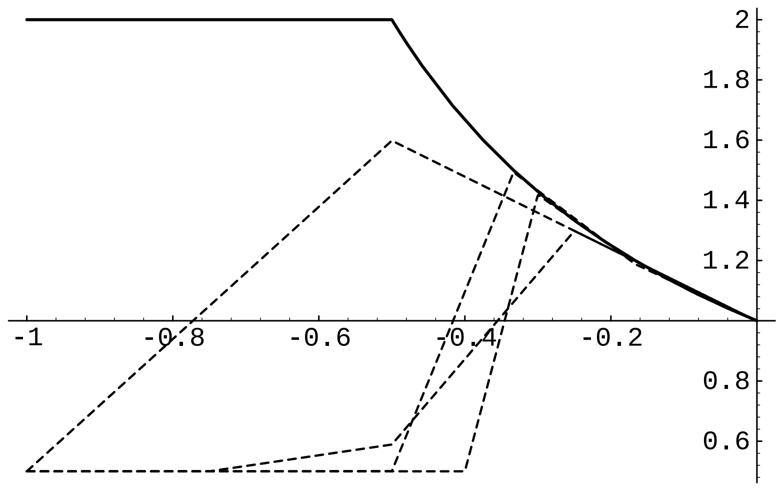

Figure 4: