Applied Numerical Mathematics, 24:2-3 (1997) 393–409.

On Existence, Uniqueness and Numerical Approximation for Neutral Equations with State-Dependent Delays

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Abstract

We consider a class of neutral functional differential equations with state-dependent delays, and discuss existence, uniqueness, and numerical approximation of solutions of corresponding initial value problems. In the sequel we make use of an Euler-type approximate method based on equations with piecewise constant arguments.

1 Introduction

In this paper we study local existence, uniqueness and numerical approximation of solutions in a class of neutral functional differential equations (NFDEs) with state-dependent delays described by

$$\frac{d}{dt}\Big(x(t)+q(t)x(t-\tau(t,x(t)))\Big) = f\Big(t,x(t),x(t-\sigma(t,x(t)))\Big). \tag{1.1}$$

This is the single delay version ($m=1$, $l=1$) of the more general equation

$$\frac{d}{dt}\Big(x(t)+\sum_{i=1}^{m} q_i(t)x(t-\tau_i(t,x(t)))\Big) = f\Big(t,x(t),x(t-\sigma_1(t,x(t))),\ldots,x(t-\sigma_l(t,x(t)))\Big). \tag{1.2}$$

We study (1.1) for simplicity of the presentation, but our results have a natural generalization for the multiple delay case (see Remark 2.17 below).

The time-dependent delay case of (1.2) has been used widely in applications (see e.g. [15]). For well-posedness of time-dependent NFDEs of the form (1.2), or, in general, for NFDEs of

This research was supported in part by the Air Force Office of Scientific Research under Grant AFOSR F49620-93-1-0280

the form $\frac{d}{dt}D(t, x_t) = f(t, x_t)$ we refer to [6] and [7]. To the best knowledge of the authors, a general theory of the state-dependent NFDE, (1.2), has not been studied before.

Another typical class of NFDEs (including state-dependent delays) can be described by

$$\dot{x}(t) = f\Big(t, x(t), x(t - \alpha(t, x(t))), \dot{x}(t - \beta(t, x(t)))\Big). \tag{1.3}$$

For corresponding well-posedness results we refer to [2], [3] and [13]. Note that (1.1) and (1.3), especially in the state-dependent case, can not be easily transformed to the other form.

In Section 2 we define an approximation scheme based on equations with piecewise constant arguments (EPCAs), and using a classical argument based on the Arsela-Ascoli's Lemma, show local existence of solutions of our equation. We also discuss uniqueness of the solution, and show convergence of the numerical method. In Section 3 we present a few numerical examples.

Note that EPCAs were used first in [4] to obtain numerical approximation schemes and to prove the convergence of the approximation method for linear delay and neutral differential equations with constant delays, and later in [5] for nonlinear delay equations with state-dependent delays. EPCA based numerical schemes were used in [8]–[11] to define numerical methods for identifying parameters in various classes of functional differential equations. We refer the interested reader to [12] and [14], and the references therein, for numerical approximation methods for NFDEs of the form (1.3).

2 Existence, Uniqueness of Solutions

Consider the vector NFDE

$$\frac{d}{dt}\Big(x(t) + q(t)x(t - \tau(t, x(t)))\Big) = f\Big(t, x(t), x(t - \sigma(t, x(t)))\Big), \qquad t \in [0, T] \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.2}$$

By a solution of the initial value problem (IVP) (2.1)-(2.2) we mean a continuous function, $x(t)$, such that $t \mapsto x(t) + q(t)x(t - \tau(t, x(t)))$ is continuously differentiable, and $x(t)$ satisfies (2.1) and (2.2).

We make the following assumptions:

(H1) $f \in C([0, T] \times \mathbf{R}^n \times \mathbf{R}^n; \mathbf{R}^n)$ is locally Lipschitz-continuous in its second and third arguments, i.e., for every $M \geq 0$ there exists $L_1 = L_1(M) \geq 0$ such that

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq L_1\Big(|x - \bar{x}| + |y - \bar{y}|\Big),$$

for $t \in [0, T]$, $x, \bar{x}, y, \bar{y} \in \mathbf{R}^n$, $|x|$, $|\bar{x}|$, $|y|$, $|\bar{y}| \leq M$,

(H2) $q \in C([0, T]; \mathbf{R})$ is Lipschitz-continuous, i.e., there exists $L_2 \geq 0$ such that

$$|q(t) - q(\bar{t})| \leq L_2|t - \bar{t}|, \qquad \text{for } t, \bar{t} \in [0, T],$$

(H3) $\tau, \sigma \in C([0,T] \times \mathbf{R}^n; \mathbf{R})$ are such that

(i) there exist $r > 0$ and $r_0 > 0$ such that

$$-r \le t - \tau(t,x) \le t - r_0 \qquad \text{and} \qquad -r \le t - \sigma(t,x) \le t, \qquad \text{for } t \in [0,T],\ x \in \mathbf{R}^n,$$

(ii) τ is locally Lipschitz-continuous in its first and second arguments, i.e., for every $M \ge 0$ there exists constants $L_3 = L_3(M) \ge 0$ and $L_4 = L_4(M) \ge 0$ such that

$$|\tau(t,x) - \tau(\bar{t},\bar{x})| \le L_3|t - \bar{t}| + L_4|x - \bar{x}|,$$

for $t, \bar{t} \in [0,T]$, $x, \bar{x} \in \mathbf{R}^n$, $|x|$, $|\bar{x}| \le M$,

(iii) σ is locally Lipschitz-continuous in its second argument, i.e., for every $M \ge 0$ there exists $L_5 = L_5(M) \ge 0$ such that

$$|\sigma(t,x) - \sigma(t,\bar{x})| \le L_5|x - \bar{x}|$$

for $t \in [0,T]$, $x, \bar{x} \in \mathbf{R}^n$, $|x|$, $|\bar{x}| \le M$,

(H4) $\varphi \in C([-r,0]; \mathbf{R}^n)$ is Lipschitz-continuous on $[-r,0]$, i.e., there exists $L_6 \ge 0$ such that

$$|\varphi(t) - \varphi(\bar{t})| \le L_6|t - \bar{t}|, \qquad \text{for } t, \bar{t} \in [0,T].$$

Here, and throughout this paper, $|\cdot|$ denotes a vector norm on $\mathbf{R}^n$, $|q|_C$ and $|\varphi|_C$ denote the respective supremum norms on $C([0,T]; \mathbf{R})$ and $C([-r,0]; \mathbf{R}^n)$.

We comment that for delay equations with state-dependent delays ($q(t) = 0$) (H1)–(H4) are standard assumptions for existence and uniqueness (see e.g. [1] or [5]).

For $h > 0$ we introduce the "greatest integer function with respect to h", $[t]_h \equiv [t/h]h$, where $[\cdot]$ is the greatest integer function. It is a piecewise constant, right continuous function satisfying

$$t - h < [t]_h \le t. \tag{2.3}$$

Following the ideas of [4] and [10], we discretize (2.1) by changing the time variable t to the piecewise constant function, $[t]_h$. Consider

$$\begin{aligned}\frac{d}{dt}\Big(y_h(t) &+ q([t]_h)y_h(t - [\tau([t]_h, y_h([t]_h - h))]_h)\Big) \\ &= f\Big([t]_h, y_h([t]_h), y_h([t]_h - [\sigma([t]_h, y_h([t]_h))]_h)\Big)\end{aligned} \tag{2.4}$$

for $t \in [0,T]$, with the initial condition

$$y_h(t) = \varphi(t), \qquad t \in [-r,0]. \tag{2.5}$$

Note that we also introduced a delayed second argument of τ in order to get an explicit recursive formula for the approximate solution (see (2.7)-(2.8) below). We will show that the solutions of IVP (2.4)-(2.5) approximate that of IVP (2.1)-(2.2) as $h \to 0+$.

The subscript h of $y_h(t)$ emphasizes that $y_h(t)$ is the solution of (2.4) corresponding to the discretization parameter h. By a solution of IVP (2.4)-(2.5) we mean a function

$y_h : [-r,T] \to \mathbf{R}^n$, which is defined on $[-r,0]$ by (2.5), such that the function $t \mapsto y_h(t) + q([t]_h)y_h(t-[\tau([t]_h, y_h([t]_h - h))]_h)$ is continuous on $[0,T]$, and its derivative exists at each point $t \in [0,T)$, with the possible exception of the points kh $(k = 0,1,2,\dots)$ where finite one-sided derivatives exist, and the function y_h satisfies (2.4) on each interval $[kh,(k+1)h) \cap [0,T]$ $(k = 0,1,2,\dots)$.

This definition yields that (2.4) is equivalent to the integral equation

$$\begin{aligned} y_h(t) + q([t]_h)y_h(t-[\tau([t]_h, y_h([t]_h - h))]_h) &= \varphi(0) + q(0)\varphi(-[\tau(0,\varphi(-h))]_h) \\ &+ \int_0^t f\Big([s]_h, y_h([s]_h), y_h([s]_h - [\sigma([s]_h, y_h([s]_h))]_h)\Big)\, ds. \end{aligned} \tag{2.6}$$

Hence applying the method of steps and using that $\varphi(t)$ is a.e. differentiable by (H4) we immediately obtain the next lemma.

Lemma 2.1 *Let $0 < h \le r_0$. Then IVP (2.4)-(2.5) has a unique, a.e. differentiable solution on $[0,T]$.*

Since the initial function is continuous but $[t]_h$ is only right continuous at mesh points kh (left-limit of $[t]_h$ exists at mesh points) we get that $y_h(t)$ is, in general, only right-continuous at mesh points, and it has jump discontinuities at mesh points. We introduce the notation $a_h(k) \equiv y_h(kh)$ and $b_h(k) \equiv \lim_{t\to kh-} y_h(t)$ for the value of the solution and its left-sided limit at mesh points, respectively. Integrating (2.4) from kh to t and taking the limit as $t \to (k+1)h+$ yields the recursive formula

$$\begin{aligned} a_h(k+1) &= a_h(k) + q(kh)a_h(k-[\tau(kh,a_h(k-1))/h]) \\ &\quad - q((k+1)h)a_h(k+1-[\tau((k+1)h,a_h(k))/h]) \\ &\quad + hf(kh,a_h(k),a_h(k-[\sigma(kh,a_h(k))/h])), \quad \text{for } k = 0,1,\dots, \end{aligned} \tag{2.7}$$

$$a_h(k) = \varphi(kh), \qquad \text{for } -r \le kh \le 0, \quad k = 0,-1,\dots . \tag{2.8}$$

This recursive formula uses past values of the solution only at mesh points. Note that $y_h(t)$, in general, is not linear on $(kh,(k+1)h)$, so the computation of $y_h(t)$ between mesh points is not convenient.

The continuity of $y_h(t)+q([t]_h)y_h(t-[\tau([t]_h,y_h([t]_h-h))]_h)$ and $\varphi(t)$ at mesh points yield

$$\begin{aligned} b_h(k+1) &= a_h(k+1) - q(kh)b_h(k+1-[\tau(kh,a_h(k-1))/h]) \\ &\quad + q((k+1)h)a_h(k+1-[\tau((k+1)h,a_h(k))/h]), \qquad k = 0,1,\dots, \end{aligned} \tag{2.9}$$

$$b_h(k) = \varphi(kh), \qquad \text{for } k = 0,-1,\dots, \quad -r \le kh \le 0. \tag{2.10}$$

Our proof of local existence will be based on the following lemmas.

Lemma 2.2 *Assume (H1)–(H4). Let $h_0 \equiv r_0/2$. Then there exist constants $M_1 > 0$, $\alpha = \alpha(M_1)$, and $M_2 = M_2(M_1)$ such that $0 < \alpha \le r_0/2$,*

$$|y_h(t)| \le M_1, \qquad t \in [-r,\alpha], \quad 0 < h \le h_0, \tag{2.11}$$

and

$$|\dot y_h(t)| \le M_2, \qquad a.e.\ t \in [-r,\alpha], \quad 0 < h \le h_0. \tag{2.12}$$

Proof Fix

$$M_1 > K \equiv |q|_C|\varphi|_C + (1+|q(0)|)|\varphi|_C + \frac{r_0}{2}\max\{|f(t,0,0)| \,:\, t\in[0,r_0/2]\}.$$

We show that we can find corresponding $\alpha=\alpha(M_1)$ and $M_2=M_2(M_1)$ which satisfy (2.11) and (2.12).

Let $w_h(t) \equiv \max_{-r\le u\le t}|y_h(u)|$, $L_1 = L_1(M_1)$ be the constant from (H1), and let $0 < \alpha_h \le r_0/2$ be the largest number such that $|y_h(t)| < M_1$ on $[-r,\alpha_h)$. Then (2.6), (2.3), and (H1) imply for $t\in[0,\alpha_h]$:

$$\begin{aligned}
|y_h(t)| &\le |q([t]_h)||y_h(t-[\tau([t]_h, y_h([t]_h-h))]_h)| + |\varphi(0)| + |q(0)||\varphi(-[\tau(0,\varphi(-h))]_h)| \\
&\quad + \int_0^t \Big|f\Big([s]_h, y_h([s]_h), y_h([s]_h - [\sigma([s]_h, y_h([s]_h))]_h)\Big) - f([s]_h,0,0)\Big|\,ds \\
&\quad + \int_0^t |f([s]_h,0,0)|\,ds \\
&\le |q|_C|y_h(t-[\tau([t]_h, y_h([t]_h-h))]_h)| + (1+|q(0)|)|\varphi|_C + \int_0^{r_0/2}|f([s]_h,0,0)|\,ds \\
&\quad + L_1\int_0^t \Big(|y_h([s]_h)| + |y_h([s]_h - [\sigma([s]_h, y_h([s]_h))]_h)|\Big)ds.
\end{aligned}$$

For $0<h\le h_0$ and $t\in[0,r_0/2]$ condition (H3) (i) and (2.3) imply

$$t-[\tau([t]_h, y_h([t]_h-h))]_h \le t-[r_0]_h \le t-r_0+h \le t-r_0/2 \le 0. \tag{2.13}$$

Therefore the monotonicity of w_h yields

$$w_h(t) \le K + 2L_1\int_0^t w_h(s)\,ds, \qquad t\in[0,\alpha_h],$$

and hence

$$|y_h(t)| \le w_h(t) \le Ke^{2L_1\alpha_h}, \qquad t\in[0,\alpha_h],$$

so (2.11) holds with $\alpha = \min\{\log(M_1/K)/(2L_1), r_0/2\}$.

Since $q([t]_h)$ and $\tau([t]_h, y_h([t]_h-h))$ are constant on each interval $(kh,(k+1)h)$, and by (H4) the initial function, $\varphi(t)$, is a.e. differentiable, it follows that $y_h(t)$ is a.e. differentiable, and

$$\begin{aligned}
\dot y_h(t) &= -q([t]_h)\dot\varphi(t-[\tau([t]_h, y_h([t]_h-h))]_h) \\
&\quad + f\Big([t]_h, y_h([t]_h), y_h([t]_h-[\sigma([t]_h, y_h([t]_h))]_h)\Big), \qquad \text{a.e. } \quad t\in[0,\alpha].
\end{aligned} \tag{2.14}$$

Therefore, we have that

$$M_2 = \max\Big\{|q|_C L_6 + \max\{|f(t,x,y)| \,:\, t\in[0,T],\ |x|,|y|\le M_1\}, L_6\Big\} \tag{2.15}$$

satisfies (2.12). ∎

Remark 2.3 It is easy to see that if f is globally Lipschitz-continuous in its second and third arguments, then $\alpha = r_0/2$ can be used in Lemma 2.2.

The next result shows that the jumps of the solutions of IVP (2.4)-(2.5) at mesh points go to 0 as $h \to 0+$.

Lemma 2.4 *Assume (H1)–(H4). Let h_0, α, M_1 and M_2 be defined by Lemma 2.2, $L_4 = L_4(M_1)$ be the constant from (H3) (ii), and assume that $|q|_C L_4 L_6 < 1$. Then there exists a constant $M_3 \geq 0$ such that*

$$|a_h(k) - b_h(k)| \leq M_3 h, \qquad \text{for } 0 < h \leq h_0, \quad k = 1, 2, \ldots, [\alpha/h]. \tag{2.16}$$

Proof It follows from (2.9) that

$$\begin{aligned} |a_h(k+1) - b_h(k+1)| &= |q((k+1)h)a_h(k+1-[\tau((k+1)h, a_h(k))/h]) \\ &\qquad - q(kh)b_h(k+1-[\tau(kh, a_h(k-1))/h])| \\ &\leq |q((k+1)h) - q(kh)||a_h(k+1-[\tau((k+1)h, a_h(k))/h])| \\ &\quad + |q(kh)||a_h(k+1-[\tau((k+1)h, a_h(k))/h]) \\ &\qquad - b_h(k+1-[\tau(kh, a_h(k-1))/h])|. \end{aligned}$$

Let $L_3 = L_3(M_1)$ and $L_4 = L_4(M_1)$ be the constants from (H3) (ii). Then (2.13), (2.8), (2.10), (2.3), (H2)–(H4) imply for $k < [\alpha/h]$:

$$\begin{aligned} &|a_h(k+1) - b_h(k+1)| \\ &\quad \leq |q((k+1)h) - q(kh)||\varphi((k+1)h - [\tau((k+1)h, a_h(k))]_h)| \\ &\qquad + |q|_C|\varphi((k+1)h - [\tau((k+1)h, a_h(k))]_h) - \varphi((k+1)h - [\tau(kh, a_h(k-1))]_h)| \\ &\quad \leq |q((k+1)h) - q(kh)||\varphi|_C + |q|_C L_6|[\tau((k+1)h, a_h(k))]_h - [\tau(kh, a_h(k-1))]_h| \\ &\quad \leq L_2 h|\varphi|_C + |q|_C L_6(2h + L_3 h + L_4|a_h(k) - a_h(k-1)|) \\ &\quad \leq L_2 h|\varphi|_C + |q|_C L_6(2h + L_3 h + L_4|b_h(k) - a_h(k-1)|) + |q|_C L_4 L_6|a_h(k) - b_h(k)|. \end{aligned}$$

Hence, noting that $|b_h(k) - a_h(k-1)| \leq M_2 h$ by (2.12), it follows that

$$|a_h(k+1) - b_h(k+1)| \leq L_2 h|\varphi|_C + |q|_C L_6(2h + L_3 h + L_4 M_2 h) + |q|_C L_4 L_6|a_h(k) - b_h(k)|.$$

Therefore the assumed condition $|q|_C L_4 L_6 < 1$ yields the statement of the lemma with

$$M_3 = (L_2|\varphi|_C + |q|_C L_6(2 + L_3 + L_4 M_2))/(1 - |q|_C L_4 L_6). \tag{2.17}$$

■

Remark 2.5 If q and τ are constant functions, then $M_3 = 0$, i.e., y_h is continuous.

Lemma 2.6 *Assume (H1)–(H4). Let h_0, α, M_1, M_2 and M_3 be defined by Lemma 2.2 and 2.4, $L_4 = L_4(M_1)$ be the constant from (H3) (ii), and assume that $|q|_C L_4 L_6 < 1$. Then there exist a sequence $\{h_k\}$ and a function $x \in C([-r, \alpha]; \mathbf{R}^n)$ such that $h_k \to 0+$ as $k \to \infty$, and*

$$\sup_{-r \leq t \leq \alpha} |y_{h_k}(t) - x(t)| \to 0, \qquad \text{as } k \to \infty.$$

Proof For $0 < h \le h_0$ define the function $z_h \in C([-r,\alpha];\mathbf{R}^n)$ by

$$z_h(t) \equiv \begin{cases} a_h(k)\frac{(k+1)h-t}{h} + a_h(k+1)\frac{t-kh}{h}, & t \in [kh,(k+1)h),\ 0 \le k \le [\alpha]_h, \\ \varphi(t), & t \in [-r,0]. \end{cases} \tag{2.18}$$

The function z_h is the linear spline interpolation of y_h on $[0,\alpha]$ using the mesh points kh. Fix $t \in [kh,(k+1)h)$, and let $\nu \equiv (t-kh)/h$. Then (2.11) yields

$$|z_h(t)| \le |a_h(k)|\nu + |a_h(k+1)|(1-\nu) \le M_1, \qquad 0 < h \le h_0. \tag{2.19}$$

The definition of z_h and (2.12) and (2.16) imply

$$\begin{aligned} |\dot{z}_h(t)| &= \left|\frac{a_h(k+1)-a_h(k)}{h}\right| \\ &\le \left|\frac{b_h(k+1)-a_h(k)}{h}\right| + \left|\frac{a_h(k+1)-b_h(k+1)}{h}\right| \\ &\le M_2 + M_3. \end{aligned} \tag{2.20}$$

Relations (2.19) and (2.20) show that $\{z_h : 0 < h \le h_0\}$ is a family of uniformly bounded and equicontinuous functions, therefore, by Arsela-Ascoli's Lemma, there exist a sequence $\{h_k\}$ and a function $x \in C([-r,\alpha];\mathbf{R}^n)$ such that $h_k \to 0+$ as $k \to \infty$, and

$$\sup_{-r \le t \le \alpha} |z_{h_k}(t) - x(t)| \to 0, \qquad \text{as } k \to \infty.$$

Finally, the inequalities

$$\begin{aligned} |y_{h_k}(t) - x(t)| &\le |y_{h_k}(t) - z_{h_k}(t)| + |z_{h_k}(t) - x(t)| \\ &\le |y_{h_k}(t) - a_{h_k}(k)|\nu + |y_{h_k}(t) - b_{h_k}(k+1)|(1-\nu) \\ &\quad + |a_{h_k}(k+1) - b_{h_k}(k+1)|(1-\nu) + |z_{h_k}(t) - x(t)| \\ &\le (M_2+M_3)h_k + |z_{h_k}(t) - x(t)| \end{aligned}$$

establish the lemma. ■

Remark 2.7 It follows from the proof of the previous lemma that

$$|z_h(t) - y_h(t)| \le (M_2+M_3)h, \qquad t \in [-r,\alpha], \quad 0 < h \le h_0.$$

Lemma 2.8 *Suppose that the assumptions of Lemma 2.6 hold. Then*

$$|y_h(t) - y_h(\bar{t})| \le (M_2+M_3)|t-\bar{t}|, \qquad t,\bar{t} \in [-r,\alpha], \quad 0 < h \le h_0. \tag{2.21}$$

Proof Assume first that $0 \le t \le \bar{t} \le \alpha$ are such that $t \in [kh,(k+1)h)$ and $\bar{t} \in [mh,(m+1)h)$. Then (2.12) and (2.16) imply

$$\begin{aligned} &|y_h(t) - y_h(\bar{t})| \\ &\le |y_h(t) - b_h(k+1)| + \sum_{i=k+1}^{m} |a_h(i) - b_h(i)| + \sum_{i=k+1}^{m-1} |b_h(i+1) - a_h(i)| + |y_h(\bar{t}) - a_h(m)| \\ &\le M_2(t-(k+1)h) + (m-k)M_3h + (m-k-1)M_2 + M_2(\bar{t}-mh) \\ &\le (M_2+M_3)|t-\bar{t}|. \end{aligned}$$

For $t, \bar{t} \in [-r, 0]$ the inequality $L_6 \leq M_2$ yields (2.21). Finally, for $-r \leq t \leq 0 \leq \bar{t} \leq \alpha$ the inequalities

$$|y_h(t) - y_h(\bar{t})| \leq |y_h(t) - y_h(0)| + |y_h(0) - y_h(\bar{t})| \leq L_6(-t) + (M_2 + M_3)\bar{t} \leq (M_2 + M_3)|t - \bar{t}|$$

conclude the proof of the lemma. ■

Now we are ready to prove existence and uniqueness of solutions of IVP (2.1)-(2.2).

Theorem 2.9 *Assume (H1)–(H4). Let h_0, α, M_1, M_2 and M_3 be defined by Lemma 2.2 and 2.4, $L_4 = L_4(M_1)$ be the constant from (H3) (ii), and assume that*

$$|q|_C L_4 L_6 < 1. \tag{2.22}$$

Then IVP (2.1)-(2.2) has a Lipschitz-continuous solution, $x(t)$, on $[-r, \alpha]$.

Proof For every $0 < h \leq h_0$ consider IVP (2.4)-(2.5), and let y_h be the corresponding solution. Lemma 2.6 yields the existence of a sequence $\{h_k\}$ and a function $x \in C([-r, \alpha]; \mathbf{R}^n)$ such that $h_k \to 0+$ and $y_{h_k}(t)$ converges to $x(t)$ as $k \to \infty$, uniformly on $[-r, \alpha]$. Clearly, $x(t) = \varphi(t)$ on $[-r, 0]$, and $|x(t)| \leq M_1$ for $t \in [-r, \alpha]$.

We need to show that $x(t)$ satisfies the following integral equation for $t \in [0, \alpha]$:

$$x(t) + q(t)x(t - \tau(t, x(t))) = \varphi(0) + q(0)\varphi(-\tau(0, \varphi(0))) + \int_0^t f(s, x(s), x(s - \sigma(t, x(s))))\, ds. \tag{2.23}$$

Using (2.6), $y_{h_k}(t)$ satisfies the integral equation

$$\begin{aligned} y_{h_k}(t) + q([t]_{h_k}) y_{h_k}(t - [\tau([t]_{h_k}, y_{h_k}([t]_{h_k} - h_k))]_{h_k}) &= \varphi(0) + q(0)\varphi(-[\tau(0, \varphi(-h_k))]_{h_k}) \\ &+ \int_0^t f\Big([s]_{h_k}, y_{h_k}([s]_{h_k}), y_{h_k}([s]_{h_k} - [\sigma([s]_{h_k}, y_{h_k}([s]_{h_k}))]_{h_k})\Big)\, ds. \end{aligned} \tag{2.24}$$

The continuity of q, φ and τ, and (2.3) yield that

$$q([t]_{h_k}) \to q(t), \qquad \text{and} \qquad [\tau(0, \varphi(-h_k))]_{h_k} \to \tau(0, \varphi(0)), \quad \text{as } k \to \infty. \tag{2.25}$$

Next we show that

$$y_{h_k}(t - [\tau([t]_{h_k}, y_{h_k}([t]_{h_k} - h_k))]_{h_k}) \to x(t - \tau(t, x(t))) \qquad \text{as } k \to \infty. \tag{2.26}$$

Let $L_4 = L_4(M_1)$ be the constant from (H3), and consider the estimates

$$\begin{aligned} &|[\tau([t]_{h_k}, y_{h_k}([t]_{h_k} - h_k))]_{h_k} - \tau(t, x(t))| \\ &\leq |[\tau([t]_{h_k}, y_{h_k}([t]_{h_k} - h_k))]_{h_k} - \tau([t]_{h_k}, y_{h_k}([t]_{h_k} - h_k))| \\ &\quad + |\tau([t]_{h_k}, y_{h_k}([t]_{h_k} - h_k)) - \tau([t]_{h_k}, x(t))| + |\tau([t]_{h_k}, x(t)) - \tau(t, x(t))| \\ &\leq h_k + L_4|y_{h_k}([t]_{h_k} - h_k) - x(t)| + |\tau([t]_{h_k}, x(t)) - \tau(t, x(t))| \\ &\leq h_k + L_4|y_{h_k}([t]_{h_k} - h_k) - y_{h_k}(t)| + L_4|y_{h_k}(t) - x(t)| + |\tau([t]_{h_k}, x(t)) - \tau(t, x(t))| \\ &\leq h_k + 2L_4 M_2 h_k + L_4|y_{h_k}(t) - x(t)| + |\tau([t]_{h_k}, x(t)) - \tau(t, x(t))| \\ &\to 0, \qquad \text{as } k \to \infty. \end{aligned}$$

Hence, using Lemma 2.8,

$$\begin{aligned}
|x(t-\tau(t,x(t))) - y_{h_k}(t-[\tau([t]_{h_k}, y_{h_k}([t]_{h_k}-h_k))]_{h_k}))| \\
\leq\ & |x(t-\tau(t,x(t))) - y_{h_k}(t-\tau(t,x(t)))| \\
& + |y_{h_k}(t-\tau(t,x(t))) - y_{h_k}(t-[\tau([t]_{h_k}, y_{h_k}([t]_{h_k}-h_k))]_{h_k}))| \\
\leq\ & |x(t-\tau(t,x(t))) - y_{h_k}(t-\tau(t,x(t)))| \\
& + (M_2+M_3)|\tau(t,x(t)) - [\tau([t]_{h_k}, y_{h_k}([t]_{h_k}-h_k))]_{h_k})|
\end{aligned}$$

implies (2.26). Similarly, $y_{h_k}([t]_h - [\sigma([t]_{h_k}, y_{h_k}([t]_{h_k}))]_{h_k}) \to x(t-\sigma(t,x(t)))$ as $k\to\infty$. Therefore Lebesgue Dominant Convergence Theorem, the continuity of f, the estimate

$$\begin{aligned}
&\Big| f\Big([s]_{h_k}, y_{h_k}([s]_{h_k}), y_{h_k}([s]_{h_k} - [\sigma([s]_{h_k}, y_{h_k}([s]_{h_k}))]_{h_k})\Big)\Big| \\
&\qquad \leq \max\{|f(u,v,w)| \,:\, u\in[0,\alpha], |v|,|w|\leq M_1\}, \qquad s\in[0,\alpha],
\end{aligned}$$

together with (2.25) and (2.24) imply (2.23). Consequently $x(t)$ is a solution of IVP (2.1)-(2.2).

To show that $x(t)$ is Lipschitz-continuous, fix $-r\leq t<\bar{t}\leq\alpha$, and $\varepsilon>0$. Let k be such that $\sup\{|y_{h_k}(u)-x(u)| \,:\, u\in[-r,\alpha]\}<\varepsilon|t-\bar{t}|/2$. Then Lemma 2.8 yields

$$\begin{aligned}
|x(t)-x(\bar{t})| &\leq |x(t)-y_{h_k}(t)| + |y_{h_k}(t)-y_{h_k}(\bar{t})| + |y_{h_k}(\bar{t})-x(\bar{t})| \\
&\leq (M_2+M_3+\varepsilon)|t-\bar{t}|.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, it follows that $x(t)$ is Lipschitz-continuous with Lipschitz-constant M_2+M_3. ■

Remark 2.10 The proof of Theorem 2.9 yields that the solution obtained by the theorem satisfies

$$|\dot{x}(t)|\leq M_2+M_3, \qquad \text{for a.e. } t\in[-r,\alpha],$$

where M_2 and M_3 are defined by (2.15) and (2.17), respectively.

Next we state a slightly generalized version of Lemma 3.2 from [4].

Lemma 2.11 *Let $a>0$, $b\geq 0$, $\alpha\geq 0$, $\beta>0$, $\gamma\equiv\max\{\alpha,\beta\}$, and $g:[0,T]\to[0,\infty)$ be continuous and nondecreasing. Let $u:[-\gamma,T]\to[0,\infty)$ be continuous except at finite many points $0<t_1<t_2<\ldots<t_m\leq T$, where finite one-sided limits exist, and satisfy the inequality*

$$u(t)\leq g(t)+bu(t-\beta)+a\int_0^t u(s-\alpha)\,ds, \qquad t\in[0,T].$$

Then $u(t)\leq d(t)e^{ct}$ for $t\in[0,T]$, where c is the unique positive solution of $cbe^{-c\beta}+ae^{-c\alpha}=c$, and

$$d(t)\equiv\max\left\{\frac{g(t)}{1-be^{-c\beta}},\ \max_{-\gamma\leq s\leq 0} e^{-cs}u(s)\right\}, \qquad t\in[0,T].$$

This result was stated and proved in [4] for the case when $u(t)$ is continuous. The proof for this case is an obvious modification of that of the continuous case, and therefore we will not include the proof here.

The uniqueness of the solution follows from the following theorem.

Theorem 2.12 *Assume (H1)–(H4), and let $x(t)$ be a Lipschitz-continuous solution of IVP (2.1)-(2.2) on $[-r,\alpha]$. Let $M_1^* \equiv \max\{|x(t)| : t \in [-r,\alpha]\} + \varepsilon$ for some $\varepsilon > 0$, $L_4 = L_4(M_1^*)$ be the corresponding constant from (H3) (ii), and $M_2^* \equiv \operatorname{ess\,sup}\{|\dot{x}(t)| : t \in [-r,\alpha]\}$. If*

$$|q|_C L_4 M_2^* < 1, \tag{2.27}$$

then $x(t)$ is the unique solution of IVP (2.1)-(2.2) on $[-r,\alpha]$, and

$$\lim_{h\to 0+} \sup_{0\le t\le\alpha} |x(t) - y_h(t)| = 0, \tag{2.28}$$

where y_h is the solution of the initial value problem (2.4)-(2.5).

Proof For $h > 0$ let $0 < \alpha_h \le \alpha$ be the largest number such that $|y_h(t)| < M_1^*$ for $[0,\alpha_h)$. Such α_h exists since $M_1^* > |\varphi|_C$.

Clearly, the uniqueness of the solution follows if we prove (2.28). To show (2.28) we subtract (2.6) from (2.23):

$$\begin{aligned} x(t) - y_h(t) &= -q(t)x(t-\tau(t,x(t))) + q([t]_h)y_h(t - [\tau([t]_h, y_h([t]_h - h))]_h) \\ &\quad + q(0)\varphi(-\tau(0,\varphi(0))) - q(0)\varphi(-[\tau(0,\varphi(-h))]_h) \\ &\quad + \int_0^t f\Big(s, x(s), x(s-\sigma(s,x(s)))\Big)\,ds \\ &\quad - \int_0^t f\Big([s]_h, y_h([s]_h), y_h([s]_h - [\sigma([s]_h, y_h([s]_h))]_h)\Big)\,ds. \end{aligned}$$

Let $L_1 = L_1(M_1^*)$ be the constant from (H1). Then (H1) and (H2) yield for $t \in [0,\alpha_h]$:

$$\begin{aligned} &|x(t) - y_h(t)| \\ &\quad\le |q(t) - q([t]_h)||y_h(t - [\tau([t]_h, y_h([t]_h - h))]_h)| \\ &\qquad + |q(t)||x(t-\tau(t,x(t))) - y_h(t - [\tau([t]_h, y_h([t]_h - h))]_h)| \\ &\qquad + |q(0)||\varphi(-\tau(0,\varphi(0))) - \varphi(-[\tau(0,\varphi(-h))]_h)| \\ &\qquad + \int_0^t \Big| f\Big(s, x(s), x(s-\sigma(s,x(s)))\Big) - f\Big([s]_h, x(s), x(s-\sigma(s,x(s)))\Big)\Big|\,ds \\ &\qquad + \int_0^t \Big| f\Big([s]_h, x(s), x(s-\sigma(s,x(s)))\Big) \\ &\qquad\qquad - f\Big([s]_h, y_h([s]_h), y_h([s]_h - [\sigma([s]_h, y_h([s]_h))]_h)\Big)\Big|\,ds. \\ &\quad\le L_2 M_1^* h + |q|_C |x(t-\tau(t,x(t))) - y_h(t - [\tau([t]_h, y_h([t]_h - h))]_h)| \\ &\qquad + |q|_C L_6 |\tau(0,\varphi(0)) - [\tau(0,\varphi(-h))]_h| \\ &\qquad + \int_0^t \Big| f\Big(s, x(s), x(s-\sigma(s,x(s)))\Big) - f\Big([s]_h, x(s), x(s-\sigma(s,x(s)))\Big)\Big|\,ds \\ &\qquad + L_1 \int_0^t \Big(|x(s) - y_h([s]_h)| + |x(s-\sigma(s,x(s)))) - y_h([s]_h - [\sigma([s]_h, y_h([s]_h))]_h))|\Big)\,ds. \end{aligned} \tag{2.29}$$

Let $w_h(t) \equiv \max_{-r\le u\le t}|x(u)-y_h(u)|$, and $L_3 = L_3(M_1^*)$ be the constant from (H3) (ii). Then (2.13) and (H3) (ii) yield

$$\begin{aligned}
&|x(t-\tau(t,x(t))) - y_h(t-[\tau([t]_h, y_h([t]_h-h))]_h))| \\
&\le |x(t-[\tau([t]_h, y_h([t]_h-h))]_h) - y_h(t-[\tau([t]_h, y_h([t]_h-h))]_h)| \\
&\quad + |x(t-\tau(t,x(t))) - x(t-[\tau([t]_h, y_h([t]_h-h))]_h)| \\
&\le w_h(t-[\tau([t]_h, y_h([t]_h-h))]_h) + M_2^*|\tau(t,x(t)) - [\tau([t]_h, y_h([t]_h-h))]_h| \\
&\le w_h(t-r_0/2) + M_2^*(h + |\tau(t,x(t)) - \tau([t]_h, y_h([t]_h-h))|) \\
&\le w_h(t-r_0/2) + M_2^*(h + L_3 h + L_4|x(t) - y_h([t]_h-h)|) \\
&\le w_h(t-r_0/2) + M_2^*(h + L_3 h + L_4 w_h([t]_h-h) + 2L_4M_2^*h). \qquad (2.30)
\end{aligned}$$

Similarly, for $t\in[0,\alpha_h]$:

$$\begin{aligned}
&|x(t-\sigma(t,x(t)))) - y_h([t]_h - [\sigma([t]_h, y_h([t]_h))]_h))| \\
&\le w_h([t]_h - \sigma([t]_h, y_h([t]_h))) + M_2^*(2h + |\sigma(t,x(t)) - \sigma([t]_h, x(t))| + L_5 w_h([t]_h) + L_5 M_2^* h). \qquad (2.31)
\end{aligned}$$

Combining (2.29), (2.30) and (2.31), and the inequalities $|x(t)-y_h([t]_h)| \le M_2^*h + w_h([t]_h)$, $w_h([t]_h-h) \le w_h([t]_h) \le w_h(t)$, and $w_h([t]_h - \sigma([t]_h, y_h([t]_h))) \le w_h(t)$, we get for $t\in[0,\alpha_h]$:

$$|x(t)-y_h(t)| \le g_h(t) + |q|_C L_4 M_2^* w_h(t) + |q|_C w_h(t-r_0/2) + L_1(2+L_5M_2^*)\int_0^t w_h(s)\,ds,$$

and therefore

$$(1-|q|_C L_4 M_2^*) w_h(t) \le g_h(t) + |q|_C w_h(t-r_0/2) + L_1(2+L_5M_2^*)\int_0^t w_h(s)\,ds, \qquad t\in[0,\alpha_h], \qquad (2.32)$$

where

$$\begin{aligned}
g_h(t) \equiv\ & L_2 M_1^* h + |q|_C M_2^*(1 + L_3 + L_4 M_2^*)h + |q|_C L_6(1+L_6)h \\
&+ M_2^*(3 + L_5 M_2^*)\alpha h + L_1 M_2^* \int_0^\alpha |\sigma(s,x(s)) - \sigma([s]_h, x(s))|\,ds \\
&+ \int_0^\alpha \Big| f\Big(s, x(s), x(s-\sigma(s,x(s)))\Big) - f\Big([s]_h, x(s), x(s-\sigma(s,x(s)))\Big)\Big|\,ds.
\end{aligned}$$

Note that $g_h(t)$ is defined on $[0,\alpha]$. Lemma 2.11, (2.32) and $w_h(t)=0$ for $t\in[-r,0]$ yield that $y_h(t) \le w_h(t) \le d_h(t)e^{\lambda\alpha}$ for $t\in[0,\alpha_h]$, where λ is the unique positive solution of

$$\lambda|q|_C e^{-\lambda r_0/2} + L_1(2+L_5M_2^*) = (1-|q|_C L_4 M_2^*)\lambda,$$

and

$$d_h(t) \equiv \frac{g_h(t)}{1-|q|_C L_4 M_2^* - |q|_C e^{-\lambda r_0/2}}.$$

We introduce the following notations for $h\ge 0$, $M\ge 0$:

$$\begin{aligned}
\omega_\sigma(h,M) &\equiv \sup\{|\sigma(s,x)-\sigma(\bar s,x)| : |s-\bar s|\le h,\ s,\bar s\in[0,\alpha],\ x\in\mathbf{R}^n,\ |x|\le M\}, \qquad (2.33)\\
\omega_f(h,M) &\equiv \sup\{|f(s,x,y)-f(\bar s,x,y)| : |s-\bar s|\le h, s,\bar s\in[0,\alpha],\ x,y\in\mathbf{R}^n, |x|,|y|\le M\}.
\end{aligned}$$

Using these notations, the definition of $g_h(t)$ implies

$$\begin{aligned} g_h(t) &\leq L_2 M_1^* h + |q|_C M_2^*(1 + L_3 + L_4 M_2^*)h + |q|_C L_6(1 + L_6)h \\ &\quad + M_2^*(3 + L_5 M_2^*)\alpha h + L_1 M_2^* \omega_\sigma(h, M_1^*)\alpha + \omega_f(h, M_1^*)\alpha. \end{aligned} \tag{2.34}$$

The continuity of σ and f yield that $\omega_\sigma(h, M_1^*) \to 0$ and $\omega_f(h, M_1^*) \to 0$ as $h \to 0+$. Hence $g_h(t) \to 0$ as $h \to 0+$, and therefore $d_h(t) \to 0$ as $h \to 0+$ uniformly in $t \in [0, \alpha]$. In particular, for some $h_0 > 0$ it follows that $\sup_{0\leq t\leq \alpha_h} |x(t) - y_h(t)| < \varepsilon$, hence $|y_h(t)| < M_1^*$ on $[0, \alpha_h]$, therefore $\alpha_h = \alpha$ can be used for $0 < h < h_0$, and the theorem follows. ■

Remark 2.13 If in addition to (H1)–(H4) we assume that σ and f are locally Lipschitz-continuous in their first arguments with Lipschitz-constants K_1 and K_2, respectively, then $\omega_\sigma(h, M_1^*) \leq K_1 h$, and $\omega_f(h, M_1^*) \leq K_2 h$. Therefore in this case the convergence in (2.28) is linear in h, i.e., there exists a constant K_0 such that $|x(t) - y_h(t)| \leq K_0 h$ for $t \in [0, \alpha]$.

Remark 2.14 Theorem 2.12 and Remark 2.7 yield that under the assumption of Theorem 2.12,

$$\lim_{h\to 0+} \sup_{0\leq t\leq \alpha} |x(t) - z_h(t)| = 0,$$

where z_h is defined by (2.18). Therefore in practice it is convenient to approximate IVP (2.1)-(2.2) using the scheme generated by $z_h(t)$: compute the sequence $a(k)$ using the recursive definition (2.7)-(2.8), and then the approximate solution is the linear interpolate of these values.

Remark 2.15 If $\tau(t, x)$ is independent of x, then $L_4 = 0$ can be used in (H3) (ii), therefore the conditions (2.22) and (2.27) in Theorem 2.9 and 2.12, respectively, are automatically satisfied.

Remark 2.16 Let M_1, M_2 and M_3 be the constants from Lemma 2.2 and 2.4. Remark 2.10 implies that condition (2.27) of Theorem 2.12 can be replaced by $|q|_C L_4(M_1)(M_2 + M_3) < 1$. Note that if a solution of IVP (2.1)-(2.2) is known, then (2.27) is usually a weaker condition, since M_1 and $M_2 + M_3$ could be large estimates for $\max |x(t)|$ and $\operatorname{ess\,sup} |\dot{x}(t)|$.

Remark 2.17 Theorem 2.9 and 2.12 can be generalized for equations of the form (1.2): Assume that each q_i and τ_i satisfy (H2) and (H3) (with Lipschitz-constants $L_4^{(i)}$ in (H3) (ii)). Then (H1)–(H4) and the conditions $L_6 \sum_{i=1}^m |q_i|_C L_4^{(i)} < 1$, $M_2^* \sum_{i=1}^m |q_i|_C L_4^{(i)} < 1$ imply the statements of Theorem 2.9 and 2.12, respectively. The proofs of there results are immediate consequences of the results given for the IVP (2.1)-(2.2).

3 Examples

Example 3.1 First we present an example of a state-dependent NFDE which has two solutions. Consider

$$\frac{d}{dt}\Big(x(t) + qx(t - \tau(t, x(t)))\Big) = 1 \tag{3.1}$$

with initial condition

$$x(t) = \frac{1}{q}t + 1, \qquad t \in [-10, 0], \tag{3.2}$$

where

$$\tau(t, x) \equiv \begin{cases} 10, & |x| > 10, \\ |x|, & 0.1 \le |x| \le 10, \\ 0.1, & |x| < 0.1. \end{cases}$$

Then clearly (H1)–(H4) are satisfied, in particular, $r = 10$, $r_0 = 0.1$, τ satisfies (H3) (ii) with $L_4 = 1$, and φ satisfies (H4) with $L_6 = 1/q$. It is easy to see that $x_1(t) = t + 1$ and $x_2(t) = t + 1 - t^2$ are solutions of IVP (3.1)-(3.2) for all $t \in [0, 9]$ and $t \in [0, 0.5]$, respectively. x_1 is Lipschitz-continuous, and $M_2^* = \operatorname{ess\,sup}|\dot{x}_1(t)| = \max\{1, 1/q\}$. Therefore $|q|_C L_4 M_2^* = \max\{1, 1/q\}$, i.e., condition (2.27) of Theorem 2.12 is not satisfied.

Example 3.2 Consider the state-dependent NFDE

$$\begin{aligned} &\frac{d}{dt}\Big(x(t) + (0.5t^2 - t - 0.5)x(t - \tau(t, x(t)))\Big) \\ &\qquad = 0.0003tx(t) - 0.0255x(t - |x(t)|) + (0.5088t - 1.4895)x(t) + 2.99t, \quad t \ge 0, \\ &x(t) = t^2, \qquad t \in [-50, 0], \end{aligned}$$

where

$$\tau(t, x) = \min\Big\{0.5 + 0.5t + 0.01|x|,\ 50\Big\}.$$

It is easy to check that $x(t) = t^2$ is a solution of this IVP for $t \in [0, 49]$ (more precisely, until $0.5 + 0.5t + 0.01t^2 \le 50$). The exact initial interval in this case is $[-0.5, 0]$, since $\min\{t - \tau(t, t^2)\} = -0.5$. If we consider the interval $[0, 5]$, then we can see that $|q|_C = 7$ and $M_2^* = \operatorname{ess\,sup}\{|\dot{x}(t)| : t \in [-0.5, 5]\} = 10$. $L_4 = 0.01$, therefore (2.27) holds, hence $x(t) = t^2$ is the unique solution of the IVP, and Theorem 2.12 yields theoretical convergence of our approximation method. Table 1 contains our numerical findings for different values of the discretization parameter, h. The approximate solution converges linearly to the true solution. Note that in this case we can observe convergence of the approximate solution on the interval $[0, 7]$, i.e., on a larger interval than guaranteed by Theorem 2.12.

Example 3.3 Consider the simple state-dependent neutral difference equation

$$\frac{d}{dt}\Big(x(t) + qx(t - \tau(t, x(t)))\Big) = 0, \qquad t \ge 0, \tag{3.3}$$

$$x(t) = t + 1, \qquad t \in [-3, 0], \tag{3.4}$$

Table 1:

t	$h = 0.0100$		$h = 0.0010$		$h = 0.00010$		$h = 0.00001$	
	$y_h(t)$	error	$y_h(t)$	error	$y_h(t)$	error	$y_h(t)$	error
1.0	0.9953075	4.693e-03	0.9995417	4.583e-04	0.9999546	4.543e-05	0.9999955	4.532e-06
2.0	3.9924221	7.578e-03	3.9993110	6.890e-04	3.9999312	6.883e-05	3.9999945	5.550e-06
3.0	8.9627737	3.723e-02	8.9962909	3.709e-03	8.9996242	3.758e-04	8.9999627	3.727e-05
4.0	15.8677646	1.322e-01	15.9877990	1.220e-02	15.9988234	1.177e-03	15.9998782	1.218e-04
5.0	24.6906086	3.094e-01	24.9686883	3.131e-02	24.9965592	3.441e-03	24.9996401	3.599e-04
6.0	35.3139144	6.861e-01	35.8793144	1.207e-01	35.9878106	1.219e-02	35.9993075	6.925e-04
7.0	39.3937123	9.606e+00	48.2541641	7.458e-01	48.9256982	7.430e-02	48.9925757	7.424e-03

where

$$\tau(t,x) = \min\{t + x^2 + 1,\, 6\}.$$

Suppose that IVP (3.3)-(3.4) has a solution, $x(t)$. Since $0 - \tau(0, x(0)) = -\tau(0,1) = -2$, it follows that there exists $\alpha^* > 0$ such that $t - \tau(t, x(t)) \le 0$ and $t + (x(t))^2 + 1 < 6$ for $t \in [0, \alpha^*]$. Then IVP (3.3)-(3.4) is equivalent to the quadratic functional equation

$$-q(x(t))^2 + x(t) + q - 1 = 0, \qquad t \in [0, \alpha^*], \qquad x(0) = 1,$$

which has a unique solution, $x(t) = 1$, for all q. In fact, $x(t) = 1$ is a solution of the IVP for $t \in [0, 4]$. Along this solution we have $t - \tau(t, x(t)) = -2$, i.e., $r = 2$ can be used in (H3) (i). Compute $M_1^* = \max\{|x(t)| \,:\, t \in [-r, \alpha^*]\} + \varepsilon = 1 + \varepsilon$, and $M_2^* = \operatorname{ess\,sup}\{|\dot{x}(t)| \,:\, t \in [-r, \alpha^*]\} = 1$. We have $L_4(M) = 2M$. Therefore for this equation condition (2.27) is $|q| L_4(M_1^*) M_2^* = 2|q| + 2|q|\varepsilon < 1$. Theorem 2.12 yields that our numerical scheme converges for $|q| < 0.5$. Numerical experiments show that, in fact, $|q| = 0.5$ is a critical parameter value. In Table 2 we print out numerical solutions corresponding to $q = -0.1$, $q = 0.1$ and $q = 0.4$ and several discretization constants. We can see linear convergence to the true solution. In Table 3 we print out the first 60 term of the approximate sequence for $q = 0.55$ and $h = 0.01$, 0.001 and 0.0001. We can observe rapidly increasing error in each cases. Note that the delay function becomes the constant 6 after the 18th, 36th and 58th terms, respectively, and then the dynamics of the approximating equation is changed.

This example also illustrates that, of course, conditions (2.22) and (2.27) in Theorem 2.9 and 2.12 are only sufficient conditions. An IVP can have a solution or a unique solution even if these conditions do not hold.

Example 3.4 Finally, consider

$$\frac{d}{dt}\Big(x(t) + \frac{1}{2}x(t - \tau(t, x(t)))\Big) = 1, \qquad t \ge 0, \tag{3.5}$$

$$x(t) = 1, \qquad t \in [-15, 0], \tag{3.6}$$

where

$$\tau(t,x) = \begin{cases} 15, & |x| > 10, \\ 2|x| - 5, & 4 < |x| \le 10, \\ \frac{1}{2}|x| + 1, & |x| \le 4. \end{cases}$$

Table 2:

q	t	$h = 0.0100$ $y_h(t)$	error	$h = 0.0010$ $y_h(t)$	error	$h = 0.0001$ $y_h(t)$	error
-0.1	1.0	0.9999000	1.00e-04	0.9999900	1.00e-05	0.9999990	1.00e-06
	2.0	0.9999000	1.00e-04	0.9999900	1.00e-05	0.9999990	1.00e-06
	3.0	0.9999000	1.00e-04	0.9999900	1.00e-05	0.9999990	1.00e-06
0.1	1.0	1.0020000	2.00e-03	1.0002000	2.00e-04	1.0000200	2.00e-05
	2.0	1.0020000	2.00e-03	1.0002000	2.00e-04	1.0000200	2.00e-05
	3.0	1.0020000	2.00e-03	1.0002000	2.00e-04	1.0000200	2.00e-05
0.4	1.0	1.0240000	2.40e-02	1.0024000	2.40e-03	1.0002400	2.40e-04
	2.0	1.0240000	2.40e-02	1.0024000	2.40e-03	1.0002400	2.40e-04
	3.0	1.0240000	2.40e-02	1.0024000	2.40e-03	1.0002400	2.40e-04

Table 3:

$h = 0.0100$ t	$y_h(t)$	error	$h = 0.0010$ t	$y_h(t)$	error	$h = 0.0001$ t	$y_h(t)$	error
0.1	1.170500	0.170500	0.01	1.014850	0.014850	0.001	1.001430	0.001430
0.2	3.101000	2.101000	0.02	1.058300	0.058300	0.002	1.005170	0.005170
0.3	3.046000	2.046000	0.03	1.292600	0.292600	0.003	1.015455	0.015455
0.4	2.991000	1.991000	0.04	3.179100	2.179100	0.004	1.047190	0.047190
0.5	2.936000	1.936000	0.05	3.173600	2.173600	0.005	1.193105	0.193105
0.6	2.881000	1.881000	0.06	3.168100	2.168100	0.006	3.196810	2.196810

Clearly, (H1)–(H4) are satisfied. Since $\tau(0, x(0)) = \tau(0, 1) = 1.5 > 0$, there exists $\alpha^* > 0$ such that $t - \tau(t, x(t)) \leq 0$ for $t \in [0, \alpha^*]$. But then for $t \in [0, \alpha^*]$, the IVP is equivalent to $\dot{x}(t) = 1$, $x(0) = 1$, therefore its unique solution is $x(t) = t + 1$. One can verify that $\alpha^* = 3$ for this solution. Since $\dot{x}(3) = 1 > 0$, $x(t) > x(3) = 4$ for $t > 3$, close enough to 3. But then, for t such that $x(t) > 4$ and $0 < t - \tau(t, x(t)) < 3$, the equation is equivalent to $\frac{d}{dt}(x(t) + \frac{1}{2}(t - 2x(t) + 6) = 1$, which has no solution. Shifting the initial time back to zero, we get that finding the continuation of the solution of IVP (3.5)-(3.6) for $t \geq 3$ is equivalent to solving (3.5) with the initial condition

$$x(t) = \begin{cases} t + 4, & t \in [-3, 0] \\ 1, & t \in [-15, -3). \end{cases} \tag{3.7}$$

Condition (2.22) in Theorem 2.9 becomes $|q|_C L_4 L_6 = \frac{1}{2} \cdot 2 \cdot 1 = 1$ for IVP (3.5)-(3.7), and, as we have seen, this IVP has no solution.

This example illustrates that continuation of solutions for the class of IVP (2.1)-(2.2) under our hypotheses may not exhibit the "usual" properties of that of ODEs or state-independent NFDEs. A maximal solution can exist on a closed finite interval, and it can have a finite limit at the end of its time-domain.

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