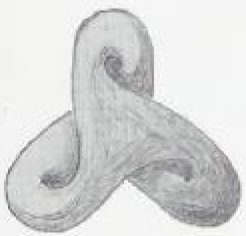
The Mathematical Gazette






Knitting the projective plane

Volume 89
Number 516
November 2005

THE MATHEMATICAL ASSOCIATION £17.00

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Typeset by Bill Richardson

Printed in Great Britain by J. W. Arrowsmith Ltd
ISSN 0025-5572

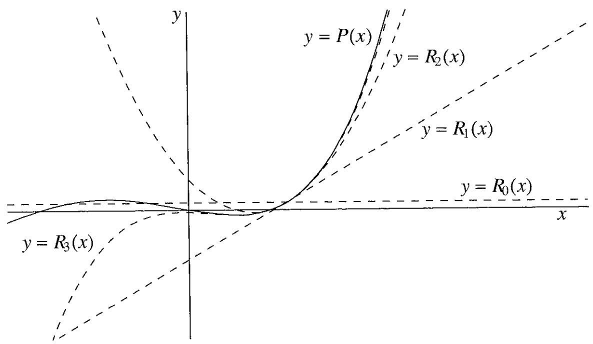

FIGURE 1

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89.66 A short remark on difference equations

I read with interest Hirschhorn's note [1] on the recurrence relations

$$\begin{cases} a_0 = a, & a_{n+1} = a_n - b_n \\ b_0 = b, & b_{n+1} = b_n - c_n \\ c_0 = c, & c_{n+1} = c_n - d_n \\ d_0 = d, & d_{n+1} = d_n - a_n \end{cases} \qquad (1)$$

and the 'lengthy calculation' (*sic*) of its explicit formula.

In this note we present the simple (and general but natural) method for solving higher dimensional linear recurrence relations:

$$\mathbf{x}_0 = \mathbf{a}_0, \qquad \mathbf{x}_{n+1} = A\mathbf{x}_n, \qquad n \geqslant 0$$

where $\mathbf{x}_n \in \mathbb{R}^k\ (n \in \mathbb{N})$ and $A \in \mathbb{R}^{k \times k}$ ($k \in \mathbb{N}$ is any fixed number).

The solution clearly is $\mathbf{x}_n = A^n\mathbf{x}_0$ where the hard task is to determine the powers of the matrix $A \in \mathbb{R}^{k \times k}$.

In our example

$$\mathbf{x}_n = \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \qquad (2)$$

We present the general method using this example. This method may be well known, but we found it only in [2].

First calculate the eigenvalues and eigenvectors of A:

$$\lambda_1 = 2, \quad h_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \lambda_2 = 1 + i, \quad h_2 = \begin{bmatrix} -i \\ -1 \\ i \\ 1 \end{bmatrix},$$

$$\lambda_3 = 1 - i, \quad h_3 = \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}, \qquad \lambda_4 = 0, \quad h_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

For determining the powers of A we use the fact that A is similar to a diagonal matrix B (since its eigenvalues are all different) where

$$A = TBT^{-1}$$

and
$$T = \begin{bmatrix} 1 & -i & i & 1 \\ -1 & -1 & -1 & 1 \\ 1 & i & -i & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

(the eigenvectors are the rows) and

$$B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1-i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(the eigenvalues are in the diagonal) .

So

$$\mathbf{x}_n = A^n\mathbf{x}_0 = TB^nT^{-1}\mathbf{x}_0 =$$

$$\begin{bmatrix} 1 & -i & i & 1 \\ -1 & -1 & -1 & 1 \\ 1 & i & -i & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^n & 0 & 0 & 0 \\ 0 & (1+i)^n & 0 & 0 \\ 0 & 0 & (1-i)^n & 0 \\ 0 & 0 & 0 & 0^n \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4}i & -\frac{1}{4} & -\frac{1}{4}i & \frac{1}{4} \\ -\frac{1}{4}i & -\frac{1}{4} & \frac{1}{4}i & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \mathbf{x}_0.$$

References

1. M. D. Hirschhorn, Some remarks on difference equations, *Math. Gaz.* **87** (July 2003) pp. 291-296.
2. I. Szalkai, Discrete mathematics and foundations of computer science, Lecture Notes, University of Veszprém, 2000 (in Hungarian).

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